INTERIOR-POINT METHODS FOR $P_*(\kappa)$-LINEAR COMPLEMENTARITY PROBLEM BASED ON GENERALIZED TRIGONOMETRIC BARRIER FUNCTION

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Abstract: Recently, M. Bouafoa, et al. [3] investigated a new kernel function which differs from the self-regular kernel functions. The kernel function has a trigonometric Barrier Term. In this paper we generalize the analysis presented in the above paper for $P_*(\kappa)$ Linear Complementarity Problems (LCPs). It is shown that the iteration bound for primal-dual large-update and small-update interior-point methods based on this function is as good as the currently best known iteration bounds for these type methods. The analysis for LCPs deviates significantly from the analysis for linear optimization. Several new tools and techniques are derived in this paper.

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1. Introduction

In this paper we consider the following linear complementarity problem:

$$
\begin{aligned}
s &= Mx + q, \\
xs &= 0, \\
x, s &\geq 0,
\end{aligned}
\tag{1}
$$

where $M \in \mathbf{R}^{n \times n}$ is a $P_*(\kappa)$ matrix and q, x, s are vectors of $\mathbf{R}^n$, and xs denotes the componentwise product (Hadamard product) of vectors x and s. Linear complementarity problems have many applications in mathematical programming and equilibrium problems. Indeed, it is known that by exploiting the first-order optimality conditions of the optimization problem, any differentiable convex quadratic program can be formulated into a monotone linear complementarity problem, i.e. $P_*(0)$ LCP, and vice versa [18]. Variational inequality problems are widely used in the study of equilibrium in economics, transportation planning, and game theory, and have a close connection to the $LCPs$. The reader can refer to Section 5.9 in [6] for the basic theory, algorithms, and applications.

The primal-dual IPM for linear optimization (LO) problems was first introduced in [11, 14] and extended to various class of problems, e.g. [4, 16]. Kojima et al. [11] and Monteiro et al. [14] first proved the polynomial computational complexity of the algorithm for LO problem independently, and since then many other algorithms have been developed based on the primal-dual strategy. Kojima et al. [12] proved the existence of the central path for any $P_*(\kappa)$ LCP, generalized the primal-dual interior-point algorithm in [11] to $P_*(\kappa)$ LCP and proved the same complexity results. Miao [13] extended the Mizuno-Todd-Ye predictor-corrector method to $P_*(\kappa)$ $LCPs$. His algorithm uses the l_2-neighborhood of the central path and has $O\left((1+\kappa)\sqrt{n}L\right)$ iteration complexity. Illés and Nagy [10] give a version of the Mizuno-Todd-Ye predictor-corrector interior point algorithm for the $P_*(\kappa)$ LCP and show that the complexity of the algorithm is $O\left((1+\kappa)^{\frac{3}{2}}\sqrt{n}L\right)$. They choose τ and τ' neighborhood parameters in such a way that at each iteration a predictor step is followed by one corrector step. For larger value of κ the values of τ and τ' decrease fast, therefore the constant in the complexity results is increasing.

Most of the polynomial-time interior point algorithms for LO are based on the use of the logarithmic barrier function [11, 17]. Peng et al. [16] introduced self-regular barrier functions for primal-dual interior-point methods ($IPMs$) for LO, semidefinite optimization (SDO), second order cone optimization (SOCO)

and also extended to $P_*(\kappa)$ $LCPs$. Recently in [2, 7] the authors proposed a new primal-dual IPM for LO based on a new class of kernel functions which are not logarithmic and not necessarily self-regular barrier functions.

In this paper we propose a new large-update primal-dual IPM which generalizes the results obtained in [7] to $P_*(\kappa)$ $LCPs$. We use a new search direction based on kernel functions which are neither logarithmic nor self-regular barrier. The new analysis which is derived in this paper is different from the one used in early papers [10, 12, 13, 16]. Furthermore, our analysis provides a simpler way to analyze primal dual $IPMs$.

We use the following notational conventions. Throughout the paper, $\|\cdot\|$ denotes the 2-norm of a vector. The nonnegative orthant and positive orthant are denoted as $\mathbf{R}_+^n$ and $\mathbf{R}_{++}^n$, respectively. If $z \in \mathbf{R}_+^n$ and $f : \mathbf{R}_+ \to \mathbf{R}_+$, then $f(z)$ denotes the vector in $\mathbf{R}_+^n$ whose i-th component is $f(z_i)$, with $1 \le i \le n$. Finally, for $x \in \mathbf{R}^n$, $X = \operatorname{diag}(x)$ is the diagonal matrix from vector x, and $J = \{1, 2, ..., n\}$ is the index set.

This paper is organized as follows. In Section 2 we recall basic concepts and the notion of the central path. In Section 3 we review known results relevant for the development of the analysis. Section 4 contains new results to compute the feasible step size and the study of the amount of decrease of the proximity function during an inner iteration. Section 5 combiners the results from Section 3 and the derived results in Section 4 to show the bound for the total number of iterations of the algorithm. Finally, concluding remarks are given in Section 6.

2. Preliminaries

In this section we introduce the definition of $P_*(\kappa)$ matrix and its proprieties, [12].

Definition 1. Let Y be an open convex subset of $\mathbf{R}^n$ and $\kappa \ge 0$. A matrix $M \in \mathbf{R}^{n \times n}$ is called a $P_*(\kappa)$-matrix on Y if and only if

$$(1 + 4\kappa) \sum_{i \in J_+(x)} x_i (Mx)_i + \sum_{i \in J_-(x)} x_i (Mx)_i \ge 0,$$

for all $x \in Y$, where

$$J_+(x) = \{i \in J : x_i (Mx)_i \ge 0\} \text{ and } J_-(x) = \{i \in J : x_i (Mx)_i < 0\}.$$

Further, M is called a P_*-matrix if it is a $P_*(\kappa)$-matrix for some $\kappa \ge 0$.

Note that the class of P_*-matrices is the union of all $P_*(\kappa)$-matrices for $\kappa \geq 0$, and contains the class of positive semi-definite matrices, i.e. symmetric matrices M satisfying $\sum_{i \in J} x_i(Mx)_i \geq 0$ for all $x \in \mathbf{R}^n$, by choosing $\kappa = 0$. The class of P_* matrices also contains matrices with all positive principal minors. In the following we recall some results which are essential in our analysis.

Proposition 2. [Lemma 4.1 in [12]] *If $M \in \mathbf{R}^{n \times n}$ is a P_* matrix, then*

$$M' = \begin{pmatrix} -M & I \\ S & X \end{pmatrix}$$

is a nonsingular matrix for any positive diagonal matrices $X, S \in \mathbf{R}^{n \times n}$.

We use the following corollary of Proposition 2 to prove that the modified Newton system has a unique solution.

Corollary 3. *Let $M \in \mathbf{R}^{n \times n}$ be a P_* matrix and $x, s \in \mathbf{R}^n_{++}$. Then for all $a \in \mathbf{R}^n$ the system*

$$
\begin{aligned}
-M \triangle x + \triangle s &= 0, \\
S \triangle x + X \triangle s &= a,
\end{aligned}
$$

has a unique solution $(\triangle x, \triangle s)$.

The basic idea of primal-dual interior-point methods is to replace the second equation in (1) by the nonlinear equation $xs = \mu e$, where e is the all-one vector, and $\mu > 0$. Thus we have the following parameterized system:

$$
\begin{aligned}
s &= Mx + q, \\
xs &= \mu e, \\
x &\geq 0, \quad s \geq 0,
\end{aligned}
\tag{2}
$$

where $\mu > 0$. We assume that there exists strictly positive x and s that satisfy (1).

Since M is a $P_*(\kappa)$ matrix and (1) is strictly feasible, then the parameterized system (2) has a unique solution $(x(\mu), s(\mu))$ for each $\mu > 0$. $(x(\mu), s(\mu))$ is called μ-center of (2), the set of μ-centers ($\mu > 0$) defines a homotopy path, which is called the *central path* of (2). If $\mu \to 0$ the limit of the central path exists. This limit satisfies the complementarity condition, and belongs to the solution set of (1), [12].

Let (x, s) be a strictly feasible point and $\mu > 0$. We define the vector

$$v := \sqrt{\frac{xs}{\mu}}. \tag{3}$$

Note that the pair (x, s) coincides with the μ-center $(x(\mu), s(\mu))$ if and only if $v = e$.

Let $\Psi(v)$ be a smooth, strictly convex function defined for all $v > 0$, which is minimal at $v = e$, with $\Psi(e) = 0$. Following [1, 2, 5, 8, 16] we define search directions Δx, Δs by

$$\begin{aligned} -M\Delta x + \Delta s &= 0, \\ s\Delta x + x\Delta s &= -\mu v \nabla \Psi(v). \end{aligned} \tag{4}$$

Since M is a P_* matrix, the system (4) uniquely defines $(\Delta x, \Delta s)$ for any $x > 0$ and $s > 0$. Note that $\Delta x = 0$, $\Delta s = 0$, if and only if $v = e$, because the right-hand sides in (4) vanish if and only if $\nabla \Psi(v) = 0$, and this occurs if and only if $v = e$.

Let (x, s) be a strictly feasible point. We define the vector p by

$$p := \sqrt{\frac{x}{s}}. \tag{5}$$

Introducing the following notations

$$\bar{M} := PMP \text{ and } P := \text{diag}(p), V := \text{diag}(v) \text{ where } v = \sqrt{\frac{xs}{\mu}},$$

and

$$d_x := \frac{v\Delta x}{x}, \qquad d_s := \frac{v\Delta s}{s}, \tag{6}$$

the system (4) can be reformulated as

$$\begin{aligned} -\bar{M}d_x + d_s &= 0, \\ d_x + d_s &= -\nabla \Psi(v). \end{aligned} \tag{7}$$

From the solution d_x and d_s, the vectors Δx and Δs can be computed from (6).

Note that the vectors d_x and d_s are not orthogonal. So our analysis in this paper will deviate significantly from the analysis used for LO in [7].

The algorithm considered in this paper is described in Figure 1.

Generic Primal-Dual Algorithm for LCP

Input:
   A proximity function $\Psi(v)$;
   a threshold parameter $\tau > 0$;
   an accuracy parameter $\epsilon > 0$;
   a fixed barrier update parameter θ, $0 < \theta < 1$;
begin
   $x := x^0$; $s := s^0$; $\mu := \mu^0$;
   while $n\mu \geq \epsilon$ **do**
   begin

   $\mu := (1 - \theta)\mu$;
   while $\Psi(v) > \tau$ **do**
   begin
      Solve $(\Delta x, \Delta s)$ from (4)
      $x := x + \alpha\Delta x$;
      $s := s + \alpha\Delta s$;
      $v := \sqrt{\frac{xs}{\mu}}$;
   end
   end
end

Figure 1: The generic primal-dual interior-point algorithm for LCP

The inner while loop in the algorithm is called *inner iteration* and the outer while loop *outer iteration.* So each outer iteration consists of an update of the barrier parameter and a sequence of one or more inner iterations. We assume that (1) is strictly feasible, and the starting point (x^0, s^0) is strictly feasible for (1). Choose τ and $v^0 = \sqrt{\frac{x^0 s^0}{\mu^0}}$ initial strictly feasible point such that $\Psi(v^0) \leq \tau$ where τ is threshold value in Figure 1. We then decrease μ to $\mu := (1-\theta)\mu$, for some $\theta \in (0, 1)$. In general this will increase the value of $\Psi(v)$ above τ. To get this value smaller again, and coming closer to the current μ-center, we solve the scaled search directions from (7), and unscaled these directions by using (4). By choosing an appropriate step size α, we move along the search direction,

and construct a new pair (x_+, s_+) with

$$x_+ = x + \alpha \triangle x \quad s_+ = s + \alpha \triangle s. \tag{8}$$

If necessary, we repeat the procedure until we find iterates such that $\Psi(v)$ no longer exceed the threshold value τ, which means that the iterates are in a small enough neighborhood of $(x(\mu), s(\mu))$. Then μ is again reduced by the factor $1 - \theta$ and we apply the same procedure targeting at the new μ-centers. This process is repeated until μ is small enough, i.e. until $n\mu \leq \epsilon$. At this stage we have found an ϵ-solution of (1). Just as in the LO case, the parameters τ, θ, and the step size α should be chosen in such a way that the algorithm is 'optimized' in the sense that the number of iterations required by the algorithm is as small as possible. Obviously, the resulting iteration bound will depend on the kernel function underlying the algorithm, and our main task becomes to find a kernel function that minimizes the iteration bound.

Figure 1 gives some examples of kernel functions that have been analyzed in [8] with the complexity results for the corresponding algorithms for large-update methods. For small-update interior-point methods the complexity results obtained in [8] is as good as the currently best known iteration bounds for these type methods methods namely: $O\left((1 + 2\kappa)\sqrt{n}\log\frac{n}{\epsilon}\right)$.

The aim of this paper is to investigate a new kernel function studied first in linear optimization case in [3], namely

$$\psi(t) = \frac{t^2 - 1}{2} + \frac{4}{p\pi}\left(\tan^p\left(h(t)\right) - 1\right), \tag{9}$$

with $h(t) = \frac{\pi}{2t+2}$, and $p \geq 2$ and to show that the interior-point methods for linear complementarity based on these function have favorable complexity results.

Note that the growth term of our kernel function is quadratic as all kernel functions in Table 1. However, this function (9) deviates from all other kernel functions [8] since its barrier term is trigonometric as $\frac{4}{p\pi}\left(\tan^p\left(h(t)\right) - 1\right)$. In order to study the new kernel function, several new arguments had to be developed for the analysis.

3. Properties of the New Proximity Function

This section is started by technical lemma, and then some properties of the new kernel function introduced in this paper are derived.

i	kernel functions ψ_i	Large-update
1	$\frac{t^2-1}{2} - \log t$	$O\left((1+2\kappa)\, n \log \frac{n}{\epsilon}\right)$
2	$\frac{t^2-1}{2} + \frac{t^{1-q}-1}{q(q-1)} - \frac{q-1}{q}(t-1)$	$O\left((1+2\kappa)\, qn^{\frac{q+1}{2q}} \log \frac{n}{\epsilon}\right)$
3	$\frac{1}{2}\left(t - \frac{1}{t}\right)^2$	$O\left((1+2\kappa)\, n^{\frac{2}{3}} \log \frac{n}{\epsilon}\right)$
4	$\frac{t^2-1}{2} + e^{\frac{1}{t}-1} - 1$	$O\left((1+2\kappa)\, \sqrt{n}\,\log^2 n \log \frac{n}{\epsilon}\right)$
5	$\frac{t^2-1}{2} - \int_1^t e^{\frac{1}{\xi}-1} d\xi$	$O\left((1+2\kappa)\, \sqrt{n}\,\log^2 n \log \frac{n}{\epsilon}\right)$
6	$\frac{t^2-1}{2} + \frac{t^{1-q}-1}{q-1},\ \ q>1$	$O\left((1+2\kappa)\, qn^{\frac{q+1}{2q}} \log \frac{n}{\epsilon}\right)$
7	$t - 1 + \frac{t^{1-q}-1}{q-1},\ \ \ q>1$	$O\left((1+2\kappa)\, qn \log \frac{n}{\epsilon}\right)$
8	$\frac{t^2-1}{2} + \frac{6}{\pi}\tan\left(\frac{\pi(1-t)}{4t+2}\right)$	$O\left((1+2\kappa)\, n^{\frac{3}{4}} \log \frac{n}{\epsilon}\right)$

Table 1: Examples of kernel functions studied in early paper [8] with complexity results for large-update.

3.1. Some Technical Results

The first three derivatives of ψ are given by

$$\psi'(t) \ = \ t + \frac{4h'(t)}{\pi}\sec^2\left(h(t)\right)\left(\tan^{p-1}\left(h(t)\right)\right), \tag{10}$$

$$\psi''(t) \ = \ 1 + \frac{4}{\pi}\sec^2\left(h(t)\right) g(t), \tag{11}$$

$$\psi'''(t) \ = \ \frac{4}{\pi}\sec^2\left(h(t)\right)\left(k(t)h'(t)^3 + r(t)h''(t)h'(t)\right)h'''(t) \tag{12}$$

$$+ \ \left(\tan^{p-1}\left(h(t)\right)h'''(t)\right), \tag{13}$$

with

$$g(t) \ := \ \left((p-1)\tan^{p-2}\left(h(t)\right) + (p+1)\tan^p\left(h(t)\right)\right)h'(t)^2$$
$$+ \ h''(t)\tan^{p-1}\left(h(t)\right),$$

$$k(t) \ := \ (p-1)(p-2)\tan^{p-3}\left(h(t)\right) + 2p^2 \tan^{p-1}\left(h(t)\right)$$
$$+ \ (p+1)(p+2)\tan^{p+1}\left(h(t)\right),$$

and

$$r(t) := 3\,(p-1)\tan^{p-2}(h(t)) + 3\,(p+1)\tan^p(h(t)), \tag{14}$$

and the first three derivatives of h are given by

$$h'(t) = \frac{-\pi}{2\,(t+1)^2}; \quad h''(t) = \frac{\pi}{(t+1)^3}; \quad h'''(t) = \frac{-3\pi}{(t+1)^4}.$$

The next lemma serves to prove that the new kernel function (9) is eligible.

Lemma 4 (Lemma 3.2 in [3]). *Let ψ be as defined in (9) and $t > 0$. Then,*

$$\psi''(t) > 1, \tag{15-a}$$
$$t\psi''(t) + \psi'(t) > 0, \tag{15-b}$$
$$t\psi''(t) - \psi'(t) > 0, \tag{15-c}$$
$$\text{and} \quad \psi'''(t) < 0. \tag{15-d}$$

It follows that $\psi(1) = \psi'(1) = 0$ and $\psi''(t) \geq 0$, proving that ψ is defined by $\psi''(t)$,

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)\, d\zeta d\xi. \tag{16}$$

The second property (15-b) in Lemma 4 is related to Definition 2.1.1 and Lemma 2.1.2 in [16]. This property is equivalent to convexity of the composed function $z \mapsto \psi(e^z)$ and this holds if and only if $\psi(\sqrt{t_1 t_2}) \leq \frac{1}{2}(\psi(t_1) + \psi(t_2))$ for any $t_1, t_2 \geq 0$. Following [1], we therefore say that ψ is exponentially convex, or shortly, *e*-convex, whenever $t > 0$.

Lemma 5. *Let ψ be as defined in (9), one has*
$$\psi(t) < \frac{1}{2}\psi''(1)\,(t-1)^2, \quad \text{if } t > 1.$$

Proof. By Taylor's theorem and $\psi(1) = \psi'(1) = 0$, we obtain

$$\psi(t) = \frac{1}{2}\psi''(1)\,(t-1)^2 + \frac{1}{6}\psi'''(\xi)\,(\xi-1)^3,$$

where $1 < \xi < t$ if $t > 1$. Since $\psi'''(\xi) < 0$, the lemma follows. $\square$

Lemma 6. *Let ψ be as defined in (9), one has*

$$t\psi'(t) \geq \psi(t), \quad \text{if } t \geq 1.$$

Proof. Defining $g(t) := t\psi'(t) - \psi(t)$ one has $g(1) = 0$ and $g'(t) = t\psi''(t) \geq 0$. Hence $g(t) \geq 0$ and the lemma follows. $\square$

Following [2], we now introduce a norm-based proximity measure $\delta(v)$, according to

$$\delta(v) := \tfrac{1}{2}\|\psi'(v)\| = \frac{1}{2}\sqrt{\sum_{i=1}^{n} \psi'(v_i)^2} = \frac{1}{2}\,\|d_x + d_s\|\,, \tag{17}$$

in terms of $\Psi(v)$. Since $\Psi(v)$ is strictly convex and attains its minimal value zero at $v = e$, we have

$$\Psi(v) = 0 \quad \Leftrightarrow \quad \delta(v) = 0 \quad \Leftrightarrow \quad v = e.$$

3.2. Relations between Proximity Measure and Norm-Based Proximity Measure

For the analysis of the algorithm in Section 4 we need to establish relations between $\Psi(v)$ and $\delta(v)$. A curial observation is that the inverse function of $\psi(t)$, for $t \geq 1$, plays an important role in this relation.

The next theorem, which is one of main results in [2], gives a lower bound on $\delta(v)$ in term of $\Psi(v)$. This is due to the fact that $\psi(t)$ satisfies (15-d). The theorem is a special case of Theorem 4.9 in [2], and is therefore stated without proof.

We denote by $\varrho : [0, \infty) \to [1, \infty)$ and $\rho : [0, \infty) \to (0, 1]$ the inverse functions of $\psi(t)$ for $t \geq 1$, and $-\tfrac{1}{2}\psi'(t)$ for $t \leq 1$, respectively. In other words,

$$s = \psi(t) \quad \Leftrightarrow \quad t = \varrho(s), \quad t \geq 1, \tag{18}$$

$$s = -\tfrac{1}{2}\psi'(t) \quad \Leftrightarrow \quad t = \rho(s), \quad t \leq 1. \tag{19}$$

Theorem 7 (Theorem 4.9 in [2]). *Let ϱ be as defined in (18). One has*

$$\delta(v) \geq \tfrac{1}{2}\psi'\left(\varrho\left(\Psi(v)\right)\right).$$

Corollary 8. *Let ϱ be as defined in (18). Thus we have*

$$\delta(v) \geq \frac{\Psi(v)}{2\varrho\left(\Psi(v)\right)}.$$

Proof. Using Theorem 7, i.e., $\delta(v) \geq \tfrac{1}{2}\psi'(\varrho(\Psi(v)))$, we obtain from Lemma 6,

$$\delta(v) \geq \frac{\psi\left(\varrho(\Psi(v))\right)}{2\varrho\left(\Psi(v)\right)} = \frac{\Psi(v)}{2\varrho\left(\Psi(v)\right)}.$$

This proves the corollary. $\qquad\square$

Theorem 9. *If* $\Psi(v) \geq 1$, *then*

$$\delta(v) \geq \frac{1}{6}\Psi^{\frac{1}{2}}. \tag{20}$$

Proof. The inverse function of $\psi(t)$ for $t \in [1, \infty)$ is obtained by solving t from

$$\psi(t) = \frac{t^2 - 1}{2} + \frac{4}{p\pi}\left(\tan^p\left(\frac{\pi}{2t+2}\right) - 1\right) = s, \quad t \geq 1.$$

We derive an upper bound for t, as this suffices for our goal. One has from (16) and $\psi''(t) \geq 1$,

$$s = \psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)\,d\zeta d\xi \geq \int_1^t \int_1^\xi d\zeta d\xi = \frac{1}{2}(t-1)^2,$$

which implies

$$t = \varrho\left(s\right) \leq 1 + \sqrt{2s}. \tag{21}$$

Assuming $s \geq 1$, we get $t = \varrho\left(s\right) \leq \sqrt{s} + \sqrt{2s} \leq 3s^{\frac{1}{2}}$. Omitting the argument v, and assuming $\Psi(v) \geq 1$, we have $\varrho(\Psi(v)) \leq 3\Psi(v)^{\frac{1}{2}}$. Now, using Corollary 8, we have

$$\delta(v) \geq \frac{\Psi(v)}{2\varrho\left(\Psi(v)\right)} \geq \frac{1}{6}\Psi(v)^{\frac{1}{2}}.$$

This proves the lemma. $\qquad\square$

Note that if $\Psi(v) \geq 1$, substitution in (20) gives

$$\delta(v) \geq \frac{1}{6}. \tag{22}$$

3.3. Growth Behavior of the Barrier Function

Note that at the start of each outer iteration of the algorithm, just before the update of μ with the factor $1 - \theta$, we have $\Psi(v) \leq \tau$. Due to the update of μ the vector v is divided by the factor $\sqrt{1 - \theta}$, with $0 < \theta < 1$, which in general leads to an increase in the value of $\Psi(v)$. Then, during the subsequent inner iterations, $\Psi(v)$ decreases until it passes the threshold τ again. Hence,

during the course of the algorithm the largest values of $\Psi(v)$ occur just after the updates of μ. In this section we derive an estimate for the effect of a μ-update on the value of $\Psi(v)$. We start with an important theorem which is valid for all kernel functions $\psi(t)$ that are strictly convex (15-a), and satisfies (15-c).

Theorem 10 (Theorem 3.2 in [2]). *Let $\varrho : [0, \infty) \to [1, \infty)$ be the inverse function of ψ on $[0, \infty)$. Then for any positive vector v and any $\beta > 1$ we have:*

$$\Psi(\beta v) \leq n\psi\left(\beta\varrho\left(\frac{\Psi(v)}{n}\right)\right). \tag{23}$$

Corollary 11. *Let $0 < \theta < 1$ and $v_+ = \frac{v}{\sqrt{1-\theta}}$. Then*

$$\Psi(v_+) \leq n\psi\left(\frac{\varrho\left(\frac{\Psi(v)}{n}\right)}{\sqrt{1-\theta}}\right). \tag{24}$$

Proof. Substitution of $\beta = \frac{1}{\sqrt{1-\theta}}$ into (23), the corollary is proved. $\square$

Suppose that the barrier update parameter θ and threshold value τ are given. According to the algorithm, at the start of each outer iteration we have $\Psi(v) \leq \tau$. By Theorem 10, after each μ-update the growth of $\Psi(v)$ is limited by (24). Therefore we define

$$L = L(n, \theta, \tau) := n\psi\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right). \tag{25}$$

Obviously, L is an upper bound of $\Psi(v_+)$, the value of $\Psi(v)$ after the μ-update.

4. Analysis of the Algorithm

In this section, we show how to compute a feasible step size α of a Newton step with the decrease of the barrier function. Since d_x and d_s, are not orthogonal the analysis in this paper is different from that of LO case. After a damped step, with step size α, using (3) and (6) we have

$$x_+ = x + \alpha\Delta x = \frac{x}{v}(v + \alpha d_x), \quad s_+ = s + \alpha\Delta s = \frac{s}{v}(v + \alpha d_s).$$

Thus we obtain

$$v_+^2 = \frac{x_+ s_+}{\mu} = (v + \alpha d_x)(v + \alpha d_s). \tag{26}$$

In the sequel we use the following notation:

$$\nu := \min_{i \in J} v_i, \quad \delta := \delta(v), \quad \sigma_+ := \sum_{i \in J_+} d_{x_i} d_{s_i}, \quad \sigma_- := -\sum_{i \in J_-} d_{x_i} d_{s_i}. \qquad (27)$$

Since M is a $P_*(\kappa)$ matrix, we have

$$(1 + 4\kappa) \sum_{i \in J_+} \Delta x_i (M \Delta x)_i + \sum_{i \in J_-} \Delta x_i (M \Delta s)_i \geq 0,$$

where $J_+ = \{i \in J : \Delta x_i (M \Delta x)_i \geq 0\}$, $J_- = J - J_+$. Using the first equation in (4) we have for $\Delta x \in \mathbf{R}^n$, $M \Delta x = \Delta s$, and

$$(1 + 4\kappa) \sum_{i \in J_+} \Delta x_i \Delta s_i + \sum_{i \in J_-} \Delta x_i \Delta s_i \geq 0.$$

From (6) it follows that $d_x d_s = \frac{v^2 \Delta x \Delta s}{xs} = \frac{\Delta x \Delta s}{\mu}$ with $\mu > 0$, and

$$(1 + 4\kappa) \sum_{i \in J_+} d_{x_i} d_{s_i} + \sum_{i \in J_-} d_{x_i} d_{s_i} = (1 + 4\kappa) \sigma_+ - \sigma_- \geq 0. \qquad (28)$$

The next lemma gives an upper bound of σ_+ and σ_-.

Lemma 12. *One has*

$$\sigma_+ \leq \delta^2, \quad \text{and} \quad \sigma_- \leq (1 + 4\kappa) \delta^2.$$

Proof. By the definition of σ_+, σ_- and δ, we have

$$\sigma_+ = \sum_{i \in J_+} d_{x_i} d_{s_i} \leq \frac{1}{4} \sum_{i \in J_+} (d_{x_i} + d_{s_i})^2 \leq \frac{1}{4} \sum_{i \in J} (d_{x_i} + d_{s_i})^2$$

$$= \frac{1}{4} \|d_{x_i} + d_{s_i}\|^2 = \delta^2.$$

Since M is a $P_*(\kappa)$ matrix, using (28), we get

$$(1 + 4\kappa) \sigma_+ - \sigma_- \geq 0.$$

Thus

$$\sigma_- \leq (1 + 4\kappa) \sigma_+ \leq (1 + 4\kappa) \delta^2.$$

This proves the lemma. $\qquad \square$

The following lemma gives an upper bound for $\|d_x\|$ and $\|d_s\|$.

Lemma 13. *One has*

$$\sum_{i=1}^{n} \left(d_{x_i}^2 + d_{s_i}^2\right) \le 4\left(1 + 2\kappa\right)\delta^2, \quad \|d_x\| \le 2\sqrt{1 + 2\kappa}\ \delta,$$

$$and \quad \|d_s\| \le 2\sqrt{1 + 2\kappa}\ \delta.$$

Proof. From the definitions (17) and (27), we have

$$\delta = \frac{1}{2}\|d_x + d_s\|, \quad \text{and} \quad \sum_{j \in J} d_{x_i} d_{s_i} = \sigma_+ - \sigma_-,$$

then

$$2\delta = \|d_x + d_s\| = \sqrt{\sum_{i=1}^{n}\left(d_{x_i} + d_{s_i}\right)^2} = \sqrt{\sum_{i=1}^{n}\left(d_{x_i}^2 + d_{s_i}^2\right) + 2\left(\sigma_+ - \sigma_-\right)}.$$

Using (28), and Lemma 12, we get

$$
\begin{aligned}
2\delta &\ge \sqrt{\sum_{i=1}^{n}\left(d_{x_i}^2 + d_{s_i}^2\right) + 2\left(\frac{1}{1 + 4\kappa}\sigma_- - \sigma_-\right)} \\
&= \sqrt{\sum_{i=1}^{n}\left(d_{x_i}^2 + d_{s_i}^2\right) - \frac{8\kappa}{1 + 4\kappa}\sigma_-}.
\end{aligned}
$$

Then, we get

$$4\delta^2 + \frac{8\kappa}{1 + 4\kappa}\sigma_- \ge \sum_{i=1}^{n}\left(d_{x_i}^2 + d_{s_i}^2\right).$$

Using again Lemma 12, we have

$$4\left(1 + 2\kappa\right)\delta^2 \ge 4\delta^2 + \frac{8\kappa}{1 + 4\kappa}\sigma_- \ge \sum_{i=1}^{n}\left(d_{x_i}^2 + d_{s_i}^2\right).$$

Thus

$$\|d_x\| \le \sqrt{\sum_{i=1}^{n}\left(d_{x_i}^2 + d_{s_i}^2\right)} \le 2\sqrt{1 + 2\kappa}\ \delta.$$

Using the same argument we can prove that

$$\|d_s\| \le 2\sqrt{1 + 2\kappa}\ \delta.$$

Thus the lemma follows. □

Our aim is to find an upper bound for

$$f(\alpha) := \Psi\left(v_+\right) - \Psi\left(v\right) := \Psi\left(\sqrt{(v + \alpha d_x)(v + \alpha d_s)}\right) - \Psi\left(v\right),$$

where $\Psi : \mathbf{R}^n \to \mathbf{R}$ is given by

$$\Psi(v) = \sum_{i=1}^{n} \psi(v_i). \tag{29}$$

To do this, the next four technical lemmas are needed. It is clear that $f(\alpha)$ is not necessarily convex in α. To simplify the analysis we use a convex upper bound for $f(\alpha)$. Such a bound is obtained by using that $\psi(t)$ satisfies the condition (15-b). Hence, $\psi(t)$ is e-convex. This implies

$$\Psi\left(v_+\right) = \Psi\left(\sqrt{(v + \alpha d_x)(v + \alpha d_s)}\right) \leq \tfrac{1}{2}\left[\Psi\left(v + \alpha d_x\right) + \Psi\left(v + \alpha d_s\right)\right].$$

Thus we have $f(\alpha) \leq f_1(\alpha)$, where

$$f_1(\alpha) := \tfrac{1}{2}\left[\Psi\left(v + \alpha d_x\right) + \Psi\left(v + \alpha d_s\right)\right] - \Psi\left(v\right)$$

is a convex function of α, since $\Psi(v)$ is convex. Obviously, $f(0) = f_1(0) = 0$. Taking the derivative of $f_1(\alpha)$ to α, we get

$$f_1'(\alpha) = \tfrac{1}{2}\sum_{i=1}^{n}\left(\psi'\left(v_i + \alpha d_{xi}\right) d_{xi} + \psi'\left(v_i + \alpha d_{si}\right) d_{si}\right).$$

This gives, using last equation in (7) and (17),

$$f_1'(0) = \tfrac{1}{2}\nabla\Psi(v)^T\left(d_x + d_s\right) = -\tfrac{1}{2}\nabla\Psi(v)^T\nabla\Psi(v) = -2\delta(v)^2. \tag{30}$$

Differentiating once more, we obtain

$$f_1''(\alpha) = \tfrac{1}{2}\sum_{i=1}^{n}\left(\psi''\left(v_i + \alpha d_{xi}\right) d_{xi}^2 + \psi''\left(v_i + \alpha d_{si}\right) d_{si}^2\right). \tag{31}$$

From this stage on we can apply word-by-word the same arguments as in [8] to obtain the following results that are therefore stated without proof.

The following lemma gives an upper bound of $f_1(\alpha)$ in terms of δ and $\psi''(t)$.

Lemma 14 (Lemma 4.3 in [8]). *One has*

$$f_1''(\alpha) \leq 2\left(1 + 2\kappa\right)\delta^2\,\psi''\left(\nu - 2\alpha\sqrt{1 + 2\kappa}\,\delta\right).$$

Putting

$$\delta_\kappa := \sqrt{1 + 2\kappa} \ \delta, \tag{32}$$

we have

$$f_1''(\alpha) \le 2\delta_\kappa^2 \ \psi'' \left(\nu - 2\alpha\delta_\kappa \right), \tag{33}$$

Since $f_1(\alpha)$ is convex, we will have $f_1'(\alpha) \le 0$ for all α less than or equal to the value where $f_1(\alpha)$ is minimal, and vice versa. In this respect the next result is important.

Lemma 15 (Lemma 4.4 in [8]). *One has $f_1'(\alpha) \le 0$ if α satisfies the inequality*

$$-\psi' \left(\nu - 2\alpha\delta_\kappa \right) + \psi' \left(\nu \right) \le \frac{2\delta_\kappa}{(1 + 2\kappa)}. \tag{34}$$

The next lemma uses the inverse function $\rho : [0, \infty) \to (0, 1]$ of $-\frac{1}{2}\psi'(t)$ for $t \in (0, 1]$, as defined in (19).

Lemma 16 (Lemma 4.5 in [8]). *The largest value of the step size α satisfying (33) is given by*

$$\bar{\alpha} := \frac{1}{2\delta_\kappa} \left[\rho \left(\delta \right) - \rho \left(\frac{1 + \sqrt{1 + 2\kappa}}{1 + 2\kappa} \delta_\kappa \right) \right]. \tag{35}$$

Moreover

$$\bar{\alpha} \ge \frac{1}{(1 + 2\kappa) \ \psi'' \left(\rho \left(\frac{1 + \sqrt{1 + 2\kappa}}{1 + 2\kappa} \delta_\kappa \right) \right)}. \tag{36}$$

For future use we define

$$\widetilde{\alpha} := \frac{1}{(1 + 2\kappa) \ \psi'' \left(\rho \left(\frac{1 + \sqrt{1 + 2\kappa}}{1 + 2\kappa} \ \delta_\kappa \right) \right)}, \tag{37}$$

as the default step size. By Lemma 16 this step $\widetilde{\alpha}$ satisfies (34). By (36) we have $\bar{\alpha} \ge \tilde{\alpha}$. We recall without proof the following lemma from [15].

Lemma 17 (Lemma 3.12 in [15]). *Let $h(t)$ be a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$ and let $h(t)$ attain its (global) minimum at $t^* > 0$. If $h''(t)$ is increasing for $t \in [0, t^*]$ then*

$$h(t) \le \frac{th'(0)}{2}, \quad 0 \le t \le t^*.$$

Lemma 18 (Lemma 10 in [9]). *If the step size α satisfies (34) then*

$$f(\alpha) \leq -\alpha \delta^2. \tag{38}$$

Theorem 19. *Let ρ be defined in (19) and $\widetilde{\alpha}$ in (37). Then*

$$f(\widetilde{\alpha}) \leq -\frac{\delta^2}{(1 + 2\kappa)\, \psi'' \left(\rho \left(\frac{1+\sqrt{1+2\kappa}}{\sqrt{1+2\kappa}}\, \delta \right) \right)} \leq -\frac{\delta^{\frac{p}{1+p}}}{1320 p\, (1 + 2\kappa)}. \tag{39}$$

Proof. By combining (36) in Lemma 16 and results in Lemma 18, using also (32). Thus the first inequality in (39) follows.

To obtain the inverse function $t = \rho(s)$ of $-\frac{1}{2}\psi'(t)$ for $t \in (0, 1]$, we need to solve t from the equation

$$-\left(t + \frac{4h'(t)}{\pi} \sec^2(h(t)) \left(\tan^{p-1}(h(t)) \right) \right)$$
$$= \left(-t + \frac{4h'(t)}{\pi} \csc^2(h(t)) \left(\tan^{p+1}(h(t)) \right) \right) = 2s.$$

This implies,

$$\csc^2(h(t)) \left(\tan^{p+1}(h(t)) \right) = \frac{-\pi}{4h'(t)} (2s + t).$$

For $t \leq 1$, we get $\frac{2\pi(t+1)^2}{4\pi} (2s + t) \leq 2 (2s + 1)$. Hence, putting

$$t = \rho \left(\frac{1 + \sqrt{1 + 2\kappa}}{\sqrt{1 + 2\kappa}}\, \delta \right),$$

which is equivalent to $\frac{2\left(1+\sqrt{1+2\kappa}\right)}{\sqrt{1+2\kappa}}\delta = -\psi'(t)$. Using that $\frac{1+\sqrt{1+2\kappa}}{\sqrt{1+2\kappa}} \leq 2$ for all $\kappa \geq 0$, and $\sin^2(h(t)) \leq 1$ we get

$$\tan(h(t)) \leq (8\delta + 2)^{\frac{1}{1+p}}. \tag{40}$$

Since $\sec^2(h(t)) = 1 + \tan^2(h(t))$, By (40), thus we have

$$\tan^2(h(t)) \leq (8\delta + 2)^{\frac{2}{1+p}}, \quad \tan^{p-2}(h(t)) \leq (8\delta + 2)^{\frac{p-2}{1+p}},$$

$$\tan^{p-1}(h(t)) \leq (8\delta + 2)^{\frac{p-1}{1+p}} \text{ and } \tan^p(h(t)) \leq (8\delta + 2)^{\frac{p}{1+p}}.$$

Since $h''(t) = \frac{8\pi}{8(t+1)^3} \le \frac{3\pi}{4}$, and $h'(t)^2 = \frac{4\pi^2}{16(2t+1)^4} \le \frac{\pi^2}{4}$ for all $0 \le t \le 1$, and using also $(8\delta + 2) \ge 1$ this implies

$$\psi''(t) \le \left(1 + \frac{4}{\pi}2\left(2p\frac{\pi^2}{4} + \pi\right)\right)(8\delta + 2)^{\frac{p+2}{1+p}} = (9 + 4p\pi)(8\delta + 2)^{\frac{p+2}{1+p}}.$$

By (37), thus we have

$$\begin{aligned}
\widetilde{\alpha} &= \frac{1}{(1 + 2\kappa)\,\psi''\left(\rho\left(\frac{1+\sqrt{1+2\kappa}}{\sqrt{1+2\kappa}}\,\delta\right)\right)} \\
&\ge \frac{1}{(1 + 2\kappa)(9 + 4p\pi)(8\delta + 2)^{\frac{p+2}{1+p}}}.
\end{aligned}$$

Also using (22) (i.e., $6\delta \ge 1$) and $p \ge 2$ we get,

$$\begin{aligned}
\widetilde{\alpha} &\ge \frac{1}{(1 + 2\kappa)(9 + 4p\pi)(8\delta + 12\delta)^{\frac{p+2}{1+p}}} \\
&= \frac{1}{(1 + 2\kappa)(9 + 4p\pi)(20\delta)^{\frac{p+2}{1+p}}} \ge \frac{1}{1320p\,(1 + 2\kappa)\,\delta^{\frac{p+2}{1+p}}}.
\end{aligned}$$

Hence

$$f(\widetilde{\alpha}) \le -\frac{\delta^2}{1320p\,(1 + 2\kappa)\,\delta^{\frac{p+2}{1+p}}} = -\frac{\delta^{\frac{p}{1+p}}}{1320p\,(1 + 2\kappa)}.$$

Thus the theorem follows.   □
Substitution in (20) gives

$$\begin{aligned}
f(\tilde{\alpha}) \le -\frac{\delta^{\frac{p}{1+p}}}{1320p\,(1 + 2\kappa)} &\le -\frac{\Psi^{\frac{p}{2(1+p)}}}{1320p\,(6)^{\frac{p}{1+p}}\,(1 + 2\kappa)} \\
&\le -\frac{\Psi^{\frac{p}{2(1+p)}}}{7920p\,(1 + 2\kappa)}.
\end{aligned}$$

5. Iteration Complexity

In this section we derive the complexity bounds for large-update methods and small-update methods.

5.1. Upper Bound for the Total Number of Iterations

Let K denote the number of inner iterations. An upper bound for the total number of iterations is obtained by multiplying (the upper bound for) the number K by the number of barrier parameter updates, which is bounded above by $\frac{1}{\theta} \log \frac{n}{\epsilon}$ (cf. [17] Lemma II.17, page 116).

Lemma 20 (Proposition 2.2 in [15]). *Let $t_0, t_1, \cdots, t_K$ be a sequence of positive numbers such that*

$$t_{k+1} \le t_k - \kappa t_k^{1-\gamma}, \qquad k = 0, 1, \cdots, K-1,$$

where $\kappa > 0$ and $0 < \gamma \le 1$. Then $K \le \left\lfloor \frac{t_0^\gamma}{\kappa \gamma} \right\rfloor$.

Lemma 21. *If K denotes the number of inner iterations, we have*

$$K \le 7920p \left(1 + 2\kappa\right) \Psi_0^{\frac{2+p}{2(1+p)}}.$$

Proof. The definition of K implies $\Psi_{K-1} > \tau$ and $\Psi_K \le \tau$ and

$$\Psi_{k+1} \le \Psi_k - \kappa \left(\Psi_k\right)^{1-\gamma}, \qquad k = 0, 1, \cdots, K-1,$$

with $\kappa = \frac{1}{7920p(1+2\kappa)}$ and $\gamma = \frac{2+p}{2(1+p)}$. Application of Lemma 20, with $t_k = \Psi_k$ yields the desired inequality. $\qquad\square$

Using $\psi_0 \le L$, where the number L is as given in (25), and Lemma 21 we obtain the following upper bound on the total number of iterations:

$$\frac{7920p \left(1 + 2\kappa\right) L^{\frac{2+p}{2(1+p)}}}{\theta} \log \frac{n}{\epsilon}. \tag{41}$$

5.2. Large-Update

We just established that (41) is an upper bound for the total number of iterations, using

$$\psi(t) = \frac{t^2 - 1}{2} + \frac{4}{p\pi} \left(\tan^p \left(\frac{\pi}{2t + 2}\right) - 1\right), \quad \text{for} \quad t \ge 1, \quad p \ge 2$$

and (21), by substitution in (25) we obtain

$$
L \leq n \frac{\left(\frac{\varrho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}}\right)^2 - 1}{2} \quad \leq \quad \frac{n}{2\left(1-\theta\right)}\left(\theta + 2\sqrt{2\frac{\tau}{n}} + \frac{2\tau}{n}\right)
$$

$$
= \quad \frac{\left(\theta n + 2\sqrt{2\tau n} + 2\tau\right)}{2\left(1-\theta\right)}.
$$

Using (41), thus the total number of iterations is bounded above by

$$
\frac{K}{\theta}\log\frac{n}{\epsilon} \leq \frac{7920p\left(1+2\kappa\right)}{\theta\left(2\left(1-\theta\right)^{\frac{2+p}{2(1+p)}}\right)}\left(\theta n + 2\sqrt{2\tau n} + 2\tau\right)^{\frac{2+p}{2(1+p)}}\log\frac{n}{\epsilon}.
$$

A large-update methods uses $\tau = O(n)$ and $\theta = \Theta(1)$. The right-hand side expression is then $O\left(p\left(1+2\kappa\right)n^{\frac{2+p}{2(1+p)}}\log\frac{n}{\epsilon}\right)$, as easily may be verified.

5.3. Small-Update Methods

For small-update methods one has $\tau = O(1)$ and $\theta = \Theta\left(\frac{1}{\sqrt{n}}\right)$. Using Lemma 5, with $\psi''(1) = \frac{p\pi+8}{4}$, we then obtain

$$
L \leq \frac{n\left(p\pi+8\right)}{8}\left(\frac{\rho\left(\frac{\tau}{n}\right)}{\sqrt{1-\theta}} - 1\right)^2.
$$

Using (21), then

$$
L \leq \frac{n\left(p\pi+8\right)}{8}\left(\frac{1+\sqrt{\frac{2\tau}{n}}}{\sqrt{1-\theta}} - 1\right)^2.
$$

Using $1 - \sqrt{1-\theta} = \frac{\theta}{1+\sqrt{1-\theta}} \leq \theta$, this leads to

$$
L \leq \frac{\left(p\pi+8\right)}{8\left(1-\theta\right)}\left(\theta\sqrt{n} + \sqrt{2\tau}\right)^2.
$$

We conclude that the total number of iterations is bounded above by

$$
\frac{K}{\theta}\log\frac{n}{\epsilon} \leq \frac{7920\left(1+2\kappa\right)\left(p\pi+8\right)^{\frac{2+p}{2(1+p)}}}{\theta\left(8\left(1-\theta\right)\right)^{\frac{2+p}{2(1+p)}}}\left(\theta\sqrt{n} + \sqrt{2\tau}\right)^{\frac{2+p}{1+p}}\log\frac{n}{\epsilon}.
$$

Thus the right-hand side expression is then $O\left(\left(1+2\kappa\right)\sqrt{n}\log\frac{n}{\epsilon}\right)$.

6. Concluding Remarks

In this paper we extended the results obtained for kernel-function-based IPMs in [3] for LO to $P_*(\kappa)$ linear complementarity problems. The observation that the vectors d_x and d_s are not in general orthogonal implies that the analysis in [3, 7] does not hold. The analysis in this paper is new and different from the one using for LO. Several new tools and techniques are derived in this paper. The proposed function has a trigonometric barrier term but the function is not logarithmic and not self-regular. For this parametric kernel function, we have shown that the best result of iteration bounds for large-update methods and small-update can be achieved, namely $O\left((1 + 2\kappa) \log n \sqrt{n} \log \frac{n}{\epsilon}\right)$, for large-update and $O\left((1 + 2\kappa) \sqrt{n} \log \frac{n}{\epsilon}\right)$ for small-update methods.

The resulting iteration bounds for $P_*(\kappa)$ linear complementarity problems depend on the parameter κ. For $\kappa = 0$, the iteration bounds are the same as the bounds that were obtained in [3] for linear optimization. In our future study, we intend to generalize the primal-dual IPMs to general nonlinear symmetric cone optimization based on this parametric kernel function.

Acknowledgments

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