QUASI-HEREDITARY ALGEBRAS AND THE CATEGORY OF MODULES WITH STANDARD FILTRATION

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Abstract. This paper is a survey on some of the most basic results in the theory of quasi-hereditary algebras. In the last section we briefly discuss a recent development.

This survey covers some of the basic results in the theory of quasi-hereditary algebras. It corresponds to the first half of the talk given by the author at the CIMPA Research School on *"Algebraic and Geometric Aspects of Representation Theory"*, Curitiba, Brazil, February 25 to March 9 2013.

Quasi-hereditary algebras appear in the representation theory of algebraic groups and semi-simple Lie algebras [Par], and recently also in algebraic geometry [HP]. Quasi-hereditary algebras were originally defined by Cline, Parshall and Scott [CPS], the definition first appearing in print in [Sco]. The general ideas were anticipated by other authors in earlier works like [Nic] and [BGG]. In these notes we follow more closely the approach of Dlab and Ringel [DR1]. The presentation borrows from the survey papers [Par] and [DR2]. The interested reader should consult these sources for a treatment of more advanced topics, including the important concept of characteristic tilting module which is not treated here.

In the last section we briefly discuss a recent development in the theory.

1. Standard and costandard modules

Let $\Bbbk$ be a field and let B be a finite-dimensional $\Bbbk$-algebra. Throughout J denotes the Jacobson radical of the algebra B. The category of left B-modules is denoted by $\mathsf{Mod}\, B$ and the full subcategory of finitely generated B-modules is denoted by $\mathsf{mod}\, B$. Fix an ordering on a complete set of non-isomorphic simple B-modules $S_1, \ldots, S_r$. Let $P_1, \ldots, P_r$ denote the corresponding indecomposable projective modules and $I_1, \ldots, I_r$ denote the corresponding indecomposable injective modules.

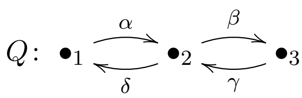
Definition 1.1. For each $1 \leq i \leq r$, define the *standard module* Δ_i to be the largest quotient of the projective module P_i having no simple composition factors S_j with $j > i$. Dually, define the *costandard module* ∇_i to be the largest submodule of the injective module I_i having no simple composition factors S_j with $j > i$. Let

$$\Delta = \bigoplus_{i=1}^{r} \Delta_i$$

and $\nabla = \bigoplus_{i=1}^{r} \nabla_i$. (Note: In some papers Δ and ∇ are used to denote the sets of standard and costandard modules respectively rather than their respective direct sums.)

The standard and costandard modules depend on the chosen ordering of the simple modules, a different choice of ordering may give different standard and costandard modules.

Example 1.2. Let B be the path algebra $B = \Bbbk Q/I$, where Q is the quiver

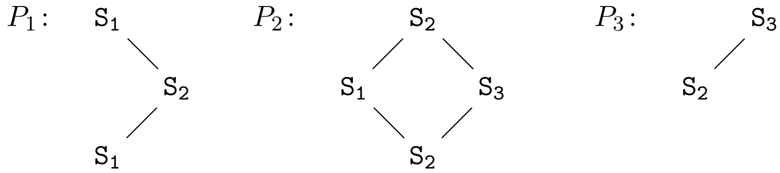


and I is the ideal

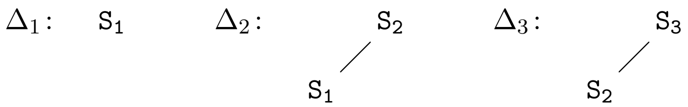
$$I = \langle \beta\alpha, \beta\gamma, \delta\gamma, \alpha\delta - \gamma\beta \rangle.$$

In path algebra examples we let the ordering of simple modules coincide with the indexing of vertices in the quiver.

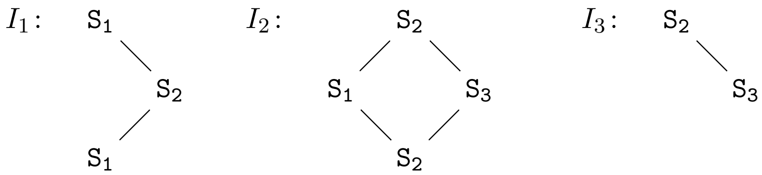
The projective modules are:



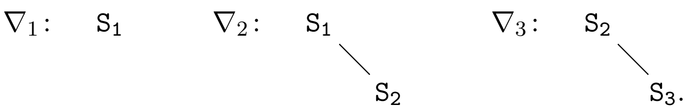
The standard modules are:



The injective modules are:



The costandard modules are:



The duality $D = \mathrm{Hom}_{\Bbbk}(-, \Bbbk)\colon \mathsf{mod}\, B \to \mathsf{mod}\, B^{\mathrm{op}}$ sends costandard B-modules to standard modules over the opposite algebra B^{op}. By this duality, for any statement about standard modules, there is a corresponding statement about costandard modules.

Without any further assumptions, we have the following vanishing result.

Lemma 1.3. $\mathrm{Ext}^1_B(\Delta, \nabla) = 0$.

Proof. Suppose $\mathrm{Ext}^1_B(\Delta_i, \nabla_j) \neq 0$ for some $1 \leq i, j \leq r$. By applying the functor $\mathrm{Hom}_B(-, \nabla_j)$ to the exact sequence

$$0 \to \Omega(\Delta_i) \to P_i \to \Delta_i \to 0,$$

we get a long-exact sequence

$$0 \to \mathrm{Hom}_B(\Delta_i, \nabla_j) \to \mathrm{Hom}_B(P_i, \nabla_j) \to \mathrm{Hom}_B(\Omega(\Delta_i), \nabla_j) \to \mathrm{Ext}^1_B(\Delta_i, \nabla_j) \to 0.$$

Since $\mathrm{Ext}^1_B(\Delta_i, \nabla_j) \neq 0$, we have $\mathrm{Hom}_B(\Omega(\Delta_i), \nabla_j) \neq 0$. Therefore the top of $\Omega(\Delta_i)$ must have a composition factor S_l with $l \leq j$. By the definition of Δ_i, the composition factors of the top of $\Omega(\Delta_i)$ must have index greater than i. So $i < l \leq j$. By duality, from the exact sequence

$$0 \to \nabla_j \to I_j \to \Omega^{-1}(\nabla_j) \to 0,$$

we obtain that the socle of $\Omega^{-1}(\nabla_j)$ must have a composition factor S_v with $j < v \leq i$. Since $i < j$ and $j < i$, we reach a contradiction. In conclusion $\mathrm{Ext}^1_B(\Delta, \nabla) = 0$. □

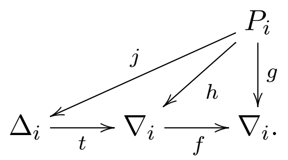
Let M be a finitely generated B-module. We say that M *admits a Δ-filtration* if there is a filtration $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_t = B$ where the subfactors M_j/M_{j-1} are standard modules for all and $1 \leq j \leq t$. If M admits a Δ-filtration, then it follows from Lemma 1.3 and induction on the length of the Δ-filtration that $\mathrm{Ext}^1_B(M, \nabla) = 0$.

Definition 1.4. We say that B is a *quasi-hereditary algebra* if (i) B admits a Δ-filtration, and (ii) $\mathrm{End}_B(\Delta_i)$ is a division ring for all $1 \leq i \leq r$.

The lack of left-right symmetry in this definition is only apparent. We first deal with the second condition.

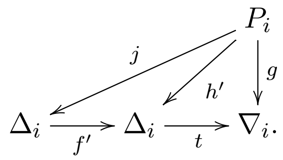
Lemma 1.5. *For each $1 \leq i \leq r$, the algebra $\mathrm{End}_B(\Delta_i)$ is a division ring if and only if $\mathrm{End}_B(\nabla_i)$ is a division ring.*

Proof. Suppose $\mathrm{End}_B(\nabla_i)$ is not a division ring and let $f\colon \nabla_i \to \nabla_i$ be a non-zero non-isomorphism. Let $g\colon P_i \to \nabla_i$ be a map with $\mathrm{im}\, g = \mathrm{soc}(\nabla_i)$. Since $\mathrm{im}\, g \subseteq \mathrm{im}\, f$ and P_i is projective, there is a map $h\colon P_i \to \nabla_i$ with $g = fh$. All composition factors of ∇_i have index less than or equal to i, so the map h factors through the surjection $j\colon P_i \to \Delta_i$.



Since $\mathrm{soc}(\nabla_i) \subseteq \ker f$, the image of h is strictly larger than $\mathrm{soc}(\nabla_i)$. Therefore the image of $t\colon \Delta_i \to \nabla_i$ is strictly larger than $\mathrm{soc}(\nabla_i)$.

Since $\operatorname{im} g \subseteq \operatorname{im} t$ and P_i is projective, there is a map $h' \colon P_i \to \Delta_i$ with $g = th'$. The map h' factors through the surjection $j \colon P_i \to \Delta_i$.



The image of t is strictly larger than $\operatorname{soc}(\nabla_i)$, so h' is not an epimorphism. Therefore $f' \colon \Delta_i \to \Delta_i$ is a non-zero non-isomorphism, which means $\operatorname{End}_B(\Delta_i)$ is not a division ring. So if $\operatorname{End}_B(\Delta_i)$ is a division ring, then $\operatorname{End}_B(\nabla_i)$ is a division ring. By duality, the statement of the lemma follows. □

If $\operatorname{End}_B(\Delta_i)$ is a division ring, then the composition factors of $J\Delta_i$ have index strictly less than i, and, as a consequence of the lemma, all composition factors of $\nabla_i / \operatorname{soc}(\nabla_i)$ have index strictly less than i.

In the definition of quasi-hereditary algebras, the existence of filtrations can be replaced by a self-dual statement, namely the vanishing of $\operatorname{Ext}^2_B(\Delta, \nabla)$.

Theorem 1.6. *The algebra B is quasi-hereditary if and only if*

(i) $\operatorname{Ext}^2_B(\Delta, \nabla) = 0$, *and*
(ii) $\operatorname{End}_B(\Delta_i)$ *is a division ring for all* $1 \leq i \leq r$.

Proof. Suppose B is quasi-hereditary. Then each indecomposable projective module P_i admits a Δ-filtration. Since P_i has a simple top, the top quotient in the filtration must be Δ_i. Therefore the kernel $\Omega(\Delta_i)$ of the morphism $P_i \to \Delta_i$ admits a Δ-filtration. So $\operatorname{Ext}^1_B(\Omega(\Delta_i), \nabla) = 0$ by the remark following Lemma 1.3. By dimension shift

$$\operatorname{Ext}^2_B(\Delta_i, \nabla) \simeq \operatorname{Ext}^1_B(\Omega(\Delta_i), \nabla) = 0.$$

Since this is true for all $1 \leq i \leq r$, we get $\operatorname{Ext}^2_B(\Delta, \nabla) = 0$.

For the converse, assume $\operatorname{End}_B(\Delta_i)$ is a division ring for all $1 \leq i \leq r$.

Let M be a finitely generated B-module. Under the hypothesis $\operatorname{Ext}^2_B(\Delta, \nabla) = 0$, we claim that M admits a Δ-filtration if and only if $\operatorname{Ext}^1_B(M, \nabla) = 0$.

If M admits a Δ-filtration, then $\operatorname{Ext}^1_B(M, \nabla) = 0$ as already remarked.

Assume $\operatorname{Ext}^1_B(M, \nabla) = 0$. Let l be the smallest index such that S_l is a composition factor of $\operatorname{top}(M)$. Suppose $j \leq l$. There is a long-exact sequence

$$\begin{aligned} 0 &\to \operatorname{Hom}_B(M, S_j) \to \operatorname{Hom}_B(M, \nabla_j) \to \operatorname{Hom}_B(M, \nabla_j / \operatorname{soc}(\nabla_j)) \\ &\to \operatorname{Ext}^1_B(M, S_j) \to \operatorname{Ext}^1_B(M, \nabla_j) \to \dots \end{aligned}$$

Since all composition factors of $\nabla_j / \operatorname{soc}(\nabla_j)$ have index strictly less than j, we have $\operatorname{Hom}_B(M, \nabla_j / \operatorname{soc}(\nabla_j)) = 0$; Since $\operatorname{Hom}_B(M, \nabla_j / \operatorname{soc}(\nabla_j)) = 0$ and $\operatorname{Ext}^1_B(M, \nabla_j) = 0$, we have $\operatorname{Ext}^1_B(M, S_j) = 0$. There is a long-exact sequence

$$0 \to \operatorname{Hom}_B(M, J\Delta_l) \to \operatorname{Hom}_B(M, \Delta_l) \to \operatorname{Hom}_B(M, S_l) \to \operatorname{Ext}^1_B(M, J\Delta_l) \to \dots$$

Since all composition factors of $J\Delta_l$ have index less than l, we have $\mathrm{Hom}_B(M, J\Delta_l) = 0$ and also $\mathrm{Ext}^1_B(M, J\Delta_l) = 0$ by induction on the length of $J\Delta_l$. Any morphism $M \to S_l$ will therefore factor through the map $\Delta_l \to S_l$. So there exists a surjective morphism from M to Δ_l. Let $f\colon M \to \Delta_l$ denote such a morphism with kernel M'. There is a long-exact sequence

$$\begin{aligned}0 &\to \mathrm{Hom}_B(\Delta_l, \nabla) \to \mathrm{Hom}_B(M, \nabla) \to \mathrm{Hom}_B(M', \nabla)\\ &\to \mathrm{Ext}^1_B(\Delta_l, \nabla) \to \mathrm{Ext}^1_B(M, \nabla) \to \mathrm{Ext}^1_B(M', \nabla) \to \mathrm{Ext}^2_B(\Delta_l, \nabla) \to \dots\end{aligned}$$

By assumption $\mathrm{Ext}^1_B(M, \nabla) = 0$ and $\mathrm{Ext}^2_B(\Delta_l, \nabla) = 0$, so $\mathrm{Ext}^1_B(M', \nabla) = 0$. The claim follows by induction on the length of M.

We have $\mathrm{Ext}^1_B(B, \nabla) = 0$, so if $\mathrm{Ext}^2_B(\Delta, \nabla) = 0$, then B admits a Δ-filtration and therefore B is quasi-hereditary. □

Corollary 1.7. *The algebra B is quasi-hereditary if and only if B^{op} is quasi-hereditary.*

Proof. Condition (i) in Theorem 1.6 is a self-dual statement. By Lemma 1.5, condition (ii) in Theorem 1.6 holds if and only if the dual statement holds. □

Since the conditions only depend on the module category, Theorem 1.6 also shows that the property of being quasi-hereditary is Morita invariant.

2. Homological properties

Throughout this section B denotes some quasi-hereditary algebra.

Theorem 2.1. $\mathrm{Ext}^n_B(\Delta, \nabla) = 0$ *for all* $n > 0$.

Proof. We know $\mathrm{Ext}^1_B(\Delta, \nabla) = 0$ by Lemma 1.3. Assume as induction hypothesis that $\mathrm{Ext}^k_B(\Delta, \nabla) = 0$ for a given $k > 0$. For any $1 \le i \le r$, since $\Omega(\Delta_i)$ admits a Δ-filtration, we have $\mathrm{Ext}^k_B(\Omega(\Delta_i), \nabla) = 0$. By dimension shift

$$\mathrm{Ext}^{k+1}_B(\Delta_i, \nabla) \simeq \mathrm{Ext}^k_B(\Omega(\Delta_i), \nabla) = 0.$$

Therefore $\mathrm{Ext}^{k+1}_B(\Delta, \nabla) = 0$. The statement of the theorem follows. □

Theorem 2.2. $\mathrm{Hom}_B(\Delta_i, \Delta_j) = 0$ *for all* $1 \le j < i \le r$.
$\mathrm{Ext}^n_B(\Delta_i, \Delta_j) = 0$ *for all* $n > 0$ *and* $1 \le j \le i \le r$.

Proof. The composition factors of Δ_j have index strictly less than j, so $\mathrm{Hom}_B(\Delta_i, \Delta_j) = 0$ for all $1 \le j < i \le r$.

Let $1 \le i \le r$. For any $j \le i$, we have $\mathrm{Hom}_B(\Delta_i, \nabla_j/\operatorname{soc}(\nabla_j)) = 0$ since all composition factors of $\nabla_j/\operatorname{soc}(\nabla_j)$ have index strictly less than j. Consider the following part of the relevant long-exact sequence.

$$\dots \to \mathrm{Hom}_B(\Delta_i, \nabla_j/\operatorname{soc}(\nabla_j)) \to \mathrm{Ext}^1_B(\Delta_i, S_j) \to \mathrm{Ext}^1_B(\Delta_i, \nabla_j) \to \dots$$

Since $\mathrm{Hom}_B(\Delta_i, \nabla_j/\operatorname{soc}(\nabla_j)) = 0$ and $\mathrm{Ext}^1_B(\Delta_i, \nabla_j) = 0$, we have $\mathrm{Ext}^1_B(\Delta_i, S_j) = 0$ whenever $j \le i$.

Assume as induction hypothesis that $\operatorname{Ext}_B^k(\Delta_i, S_j) = 0$ for all $j \leq i$. Since all composition factors of $\nabla_j/\operatorname{soc}(\nabla_j)$ have index strictly less than j, it follows that $\operatorname{Ext}_B^k(\Delta_i, \nabla_j/\operatorname{soc}(\nabla_j)) = 0$ for all $j \leq i$. Consider the following part of the long-exact sequence.

$$\ldots \to \operatorname{Ext}_B^k(\Delta_i, \nabla_j) \to \operatorname{Ext}_B^k(\Delta_i, \nabla_j/\operatorname{soc}(\nabla_j)) \to \operatorname{Ext}_B^{k+1}(\Delta_i, S_j) \to \operatorname{Ext}_B^{k+1}(\Delta_i, \nabla_j) \to \ldots$$

Here the end terms are zero, so

$$\operatorname{Ext}_B^{k+1}(\Delta_i, S_j) \simeq \operatorname{Ext}_B^k(\Delta_i, \nabla_j/\operatorname{soc}(\nabla_j)) = 0$$

for all $j \leq i$. Hence $\operatorname{Ext}_B^n(\Delta_i, S_j) = 0$ for all $n > 0$ and $1 \leq j \leq i \leq r$.

Since all composition factors of Δ_j have index j or less, it follows by induction on the length of Δ_j that $\operatorname{Ext}_B^n(\Delta_i, \Delta_j) = 0$ for all $n > 0$ and $1 \leq j \leq i \leq r$. □

One consequence of $\operatorname{Ext}_B^1(\Delta_i, \Delta_j) = 0$ for all $j \leq i$ is that if a module M admits a Δ-filtration, then the Δ-factors can always be chosen in non-increasing order, meaning that the standard modules with highest index appear at the bottom of the filtration while he standard modules with smallest index appear at the top of the filtration.

We now look at bounds for homological dimensions. For more on projective dimensions in exact sequences, see section 1 of [Mad].

Lemma 2.3. *Let R be a ring and $0 \to L \to M \to N \to 0$ an exact sequence of R-modules. Then*

(a) $\operatorname{pd}_R M \leq \max\{\operatorname{pd}_R L, \operatorname{pd}_R N\}$.
(b) *If* $\operatorname{pd}_R M < \max\{\operatorname{pd}_R L, \operatorname{pd}_R N\}$, *then* $\operatorname{pd}_R L + 1 = \operatorname{pd}_R N$.

Proof. (a) The exact sequence of functors

$$\operatorname{Ext}_R^n(N, -) \to \operatorname{Ext}_R^n(M, -) \to \operatorname{Ext}_R^n(L, -)$$

shows that $\operatorname{pd}_R M \geq n$ implies $[\operatorname{pd}_R L \geq n$ or $\operatorname{pd}_R N \geq n]$. Therefore $\operatorname{pd}_R M \leq \max\{\operatorname{pd}_R L, \operatorname{pd}_R N\}$.

(b) Assume $\operatorname{pd}_R M = n < \infty$. Then there is an epimorphism of functors $\operatorname{Ext}_R^n(L, -) \to \operatorname{Ext}_R^{n+1}(N, -) \to 0$ and isomorphisms $\operatorname{Ext}_R^m(L, -) \simeq \operatorname{Ext}_R^{m+1}(N, -)$ for $m > n$. So if $\operatorname{pd}_R N > n$, then $\operatorname{pd}_R L = \operatorname{pd}_R N - 1$. Similarly if $\operatorname{pd}_R L > n$, then $\operatorname{pd}_R N = \operatorname{pd}_R L + 1$. □

Theorem 2.4. $\operatorname{pd}_B(\Delta_i) \leq r - i$.

Proof. The standard module Δ_r is projective, so $\operatorname{pd}_B(\Delta_r) = \operatorname{pd}_B(P_r) = 0$. Let $1 \leq n < r$ and assume $\operatorname{pd}_B(\Delta_i) \leq r - i$ whenever $i > n$. Consider the exact sequence

$$0 \to \Omega(\Delta_n) \to P_n \to \Delta_n \to 0.$$

The quotients in the Δ-filtration of $\Omega(\Delta_n)$ have index strictly greater than n, so by repeated use of Lemma 2.3(a) we get $\operatorname{pd}_B(\Omega(\Delta_n)) \leq r-n-1$. Hence $\operatorname{pd}_B(\Delta_n) = \operatorname{pd}_B(\Omega(\Delta_n))+1 \leq r-n$. The theorem follows by induction. □

Theorem 2.5. $\operatorname{gldim} B \leq 2r - 2$.

Proof. The standard module Δ_1 is simple, so $\mathrm{pd}_B(S_1) = \mathrm{pd}_B(\Delta_1) \leq r-1 = r+1-2$ by the previous theorem. Let $1 < n \leq r$ and assume $\mathrm{pd}_B(S_i) \leq r+i-2$ whenever $i < n$. Consider the exact sequence

$$0 \to J\Delta_n \to \Delta_n \to S_n \to 0.$$

The composition factors of $J\Delta_n$ have index strictly less than n, so by repeated use of Lemma 2.3(a) we get $\mathrm{pd}_B(J\Delta_n) \leq r+n-3$. Also by Lemma 2.3(a) we have

$$\mathrm{pd}_B(\Delta_n) \leq \max\{\mathrm{pd}_B(J\Delta_n), \mathrm{pd}_B(S_n)\}.$$

If $\mathrm{pd}_B(\Delta_n) = \max\{\mathrm{pd}_B(J\Delta_n), \mathrm{pd}_B(S_n)\}$, then by Theorem 2.4

$$\mathrm{pd}_B(S_n) \leq \mathrm{pd}_B(\Delta_n) \leq r-n.$$

If $\mathrm{pd}_B(\Delta_i) < \max\{\mathrm{pd}_B(J\Delta_n), \mathrm{pd}_B(S_n)\}$, then by Lemma 2.3(b)

$$\mathrm{pd}_B(S_n) = \mathrm{pd}_B(J\Delta_n) + 1 \leq r+n-2.$$

In both cases $\mathrm{pd}_B(S_n) \leq r+n-2$. By induction $\mathrm{pd}_B(S_i) \leq r+i-2$ for $1 \leq i \leq r$.

Since $\operatorname{gldim} B = \max\{\mathrm{pd}_B(S_i) \mid 1 \leq i \leq r\}$, we get

$$\operatorname{gldim} B \leq r+r-2 = 2r-2.$$

□

Lemma 2.6. *Let A be a finite-dimensional $\Bbbk$-algebra and $e \in A$ an idempotent. Suppose the two-sided ideal AeA is projective as a left A-module. Let $\bar{A}$ denote the algebra $\bar{A} = A/AeA$. Then*

$$\mathrm{Ext}^n_{\bar{A}}(M,N) \simeq \mathrm{Ext}^n_A(M,N)$$

for all $n \geq 0$ and all $\bar{A}$-modules M, N.

Proof. There is a full embedding of module categories $\mathsf{Mod}\,\bar{A} \to \mathsf{Mod}\,A$ that we exploit throughout the proof. The result for $n = 0$ is immediate, we have

$$\mathrm{Hom}_{\bar{A}}(M,N) \simeq \mathrm{Hom}_A(M,N)$$

for all $\bar{A}$-modules M, N.

Let N be an $\bar{A}$-module. For any $f \in \mathrm{Hom}_A(AeA, N)$ and $aea' \in AeA$, we have $f(aea') = ae \cdot f(ea') = 0$, so $\mathrm{Hom}_A(AeA, N) = 0$. Using that AeA is projective, we see from the long-exact sequence obtained by applying $\mathrm{Hom}_A(-,N)$ to the exact sequence $0 \to AeA \to A \to A/AeA \to 0$ that

$$\mathrm{Ext}^n_A(A/AeA, N) = 0$$

for all $n > 0$. It follows that $\mathrm{Ext}^n_A(\bar{P}, N) = 0$ for any projective $\bar{A}$-module $\bar{P}$ and $n > 0$.

Let M be an $\bar{A}$-module and $0 \to \Omega_{\bar{A}}(M) \to P_{\bar{A}}(M) \to M \to 0$ an exact sequence with $P_{\bar{A}}(M)$ projective as $\bar{A}$-module. There is a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\operatorname{Hom}_{\bar{A}}(P_{\bar{A}}(M),N) & \longrightarrow & \operatorname{Hom}_{\bar{A}}(\Omega_{\bar{A}}M,N) & \longrightarrow & \operatorname{Ext}^1_{\bar{A}}(M,N) & \longrightarrow & 0 \\
\downarrow \wr & & \downarrow \wr & & \downarrow & & \\
\operatorname{Hom}_{A}(P_{\bar{A}}(M),N) & \longrightarrow & \operatorname{Hom}_{A}(\Omega_{\bar{A}}M,N) & \longrightarrow & \operatorname{Ext}^1_{A}(M,N) & \longrightarrow & 0.
\end{array}$$

Since the two left-most downward arrows are isomorphisms, we get

$$\operatorname{Ext}^1_{\bar{A}}(M,N) \simeq \operatorname{Ext}^1_{A}(M,N)$$

for all $\bar{A}$-modules M, N.

We proceed by induction. Assume there is an $n \geq 1$ such that $\operatorname{Ext}^n_{\bar{A}}(M,N) \simeq \operatorname{Ext}^n_{A}(M,N)$ for all $\bar{A}$-modules M, N. There is a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \operatorname{Ext}^n_{\bar{A}}(\Omega_{\bar{A}}M,N) & \xrightarrow{\sim} & \operatorname{Ext}^{n+1}_{\bar{A}}(M,N) & \longrightarrow & 0 \\
 & & \downarrow \wr & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Ext}^n_{A}(\Omega_{\bar{A}}M,N) & \xrightarrow{\sim} & \operatorname{Ext}^{n+1}_{A}(M,N) & \longrightarrow & 0.
\end{array}$$

By the induction hypothesis, the left downward arrow is an isomorphism. It follows that $\operatorname{Ext}^{n+1}_{\bar{A}}(M,N) \simeq \operatorname{Ext}^{n+1}_{A}(M,N)$ for all $\bar{A}$-modules M, N. The lemma follows by induction. □

Let $\bar{B}$ denote the algebra $\bar{B} = B/BeB$, where e is a primitive idempotent corresponding to the simple B-module with highest index S_r. The algebra $\bar{B}$ has standard modules $\bar{\Delta}_i = \Delta_i$ and costandard modules $\bar{\nabla}_i = \nabla_i$ for $1 \leq i \leq r-1$.

Theorem 2.7. *The algebra $\bar{B}$ is quasi-hereditary.*

Proof. Let $1_B = e_1 + \ldots + e_s$ be a decomposition of the identity into primitive orthogonal idempotents. Since B is quasi-hereditary, every finitely generated projective B-module P admits a Δ-filtration. By the comment following Theorem 2.2, any non-zero morphism $g\colon \Delta_r = Be \to P$ must be an inclusion.

The two-sided ideal BeB considered as a left B-module has a decomposition

$$BeB \simeq \bigoplus_{i=1}^{s} BeBe_i.$$

For each primitive idempotent f, we have a further decomposition

$$BeBf \simeq \bigoplus_{j=1}^{t} Bq_j,$$

where $\{q_1, \ldots, q_t\}$ is a $\Bbbk$-basis for eBf. For each $1 \leq j \leq t$, there is a surjection $h_j\colon Be \to Bq_j$ given by right multiplication by q_j. The composition $Be \xrightarrow{h_j} Bq_j \hookrightarrow Bf$ is an inclusion, so each h_j must be an isomorphism.

Therefore BeB as a left B-module is a direct sum of copies of Be and hence it is projective.

Let $\bar{\Delta} = \bigoplus_{i=1}^{r-1} \Delta_i$. From Lemma 2.6 we get

$$\mathrm{Ext}^2_{\bar{B}}(\bar{\Delta}, \bar{\nabla}) \simeq \mathrm{Ext}^2_{B}(\bar{\Delta}, \bar{\nabla}) = 0.$$

For $1 \leq i \leq r-1$ we also get

$$\mathrm{End}_{\bar{B}}(\bar{\Delta}_i) \simeq \mathrm{End}_{B}(\bar{\Delta}_i) \simeq \mathrm{End}_{B}(\Delta_i),$$

which is a division ring. By Theorem 1.6, the algebra $\bar{B}$ is quasi-hereditary. □

3. Examples of quasi-hereditary algebras

The question whether an algebra is quasi-hereditary or not might depend on the chosen ordering of simple modules. Directed algebras are quasi-hereditary in (at least) two different ways, with simple standard modules or with projective standard modules. An algebra B is called *directed* if there is an ordering on a complete set of non-isomorphic indecomposable projective modules $P_1, \ldots, P_r$ such that $\mathrm{Hom}_B(P_i, P_j) = 0$ whenever $1 \leq i < j \leq r$ and $\mathrm{End}_B(P_i)$ is a division ring for all $1 \leq i \leq r$. Examples of directed algebras are path algebras $\Bbbk Q/I$ with Q a directed quiver. Let B be a directed algebra. If the same ordering $S_1, \ldots, S_r$ is used for the simple modules, then $\Delta_i = S_i$ for all $1 \leq i \leq r$ and the algebra is quasi-hereditary. If the opposite ordering is used, then $\Delta_i = P_i$ for all $1 \leq i \leq r$ and the algebra is quasi-hereditary.

Semi-simple algebras are quasi-hereditary for any ordering of simple modules. Hereditary algebras also have this property.

Theorem 3.1. *Hereditary algebras are quasi-hereditary for any ordering of simple modules.*

Proof. Let B be an hereditary algebra and fix an ordering of the simple modules. If P is a non-zero finitely generated projective module, then there is a surjection $f\colon P \to \Delta_i$ onto a standard module Δ_i. If $\ker f = 0$, then P admits a trivial Δ-filtration. Since B is hereditary, the kernel $\ker f$ is either zero or a non-zero projective module of shorter length than P. By induction on the length of P, every finitely generated projective B-module P admits a Δ-filtration. In particular B admits a Δ-filtration.

Let $h\colon \Delta_i \to \Delta_i$ be a non-zero non-isomorphism for some $1 \leq i \leq r$. It lifts to a non-zero non-isomorphism $\tilde{h}\colon P_i \to P_i$. Since B is hereditary, such a morphism cannot exist. Therefore $\mathrm{End}_B(\Delta_i)$ is a division ring for all $1 \leq i \leq r$. This proves that B is quasi-hereditary. □

So all algebras B with $\mathrm{gldim}\, B \leq 1$ are quasi-hereditary. Global dimension two algebras are also quasi-hereditary, but in this case we have to be careful with the choice of ordering.

Theorem 3.2. *Algebras with global dimension two are quasi-hereditary.*

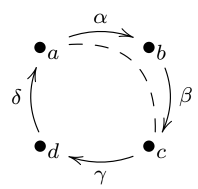
Proof. Let B be an algebra of global dimension two. Without loss of generality we can assume B is basic. The crucial property of global dimension two algebras is that the kernel of a map between projective modules is projective. Let $1_B = f_1 + \ldots + f_r$ be a decomposition of the identity into primitive orthogonal idempotents. Choose a primitive idempotent e from this decomposition such that the projective module Be has minimal Loewy length. Since Be has minimal Loewy length, any non-zero morphism $g\colon Be \to P$ with P projective must be an inclusion. In particular, any non-zero morphism $Be \to Be$ must be an isomorphism. Therefore $\mathrm{End}_B(Be)$ is a division ring.

Since any non-zero morphism $g\colon Be \to P$ is an inclusion, in the same way as in the proof of Theorem 2.7 it follows that BeB is projective as a left B-module. More precisely BeB as a left B-module is a direct sum of copies of Be.

To prove that B is quasi-hereditary we use an inductive argument. If $r = 1$, then B is local of finite global dimension, hence semi-simple and therefore quasi-hereditary. If $r > 1$, choose the simple module corresponding to the primitive idempotent e to be maximal in the ordering. Then $\Delta_r = Be$ is a projective standard module and BeB admits a standard filtration. Consider the basic algebra $\bar{B} = B/BeB$. It has $r - 1$ primitive orthogonal idempotents. From Lemma 2.6 it follows that $\mathrm{gldim}\,\bar{B} \leq 2$. The full embedding $\mathsf{mod}\,\bar{B} \to \mathsf{mod}\,B$ sends standard $\bar{B}$-modules to standard B-modules. Assume $\bar{B}$ is quasi-hereditary. Since both $\bar{B}$ and BeB admit Δ-filtrations, also B admits a Δ-filtration. By assumption $\mathrm{End}_B(\Delta_i)$ is a division ring for all $1 \leq i \leq r-1$. Also $\mathrm{End}_B(\Delta_r) = \mathrm{End}_B(Be)$ is a division ring. So $\bar{B}$ being quasi-hereditary implies that B is quasi-hereditary. By induction on r, the algebra B is quasi-hereditary. □

In the next example we consider a global dimension two path algebra with a quiver that is not directed.

Example 3.3. Let B be the path algebra $B = \Bbbk Q/I$, where Q is the quiver

$Q\colon$ 

and I is the ideal

$$I = \langle \beta\alpha \rangle.$$

This algebra has global dimension two. The indecomposable projective module with shortest Loewy length is P_a, so we let $S_4 = S_a$.

The algebra $\bar{B} = B/Be_aB$ is isomorphic to the path algebra of the quiver

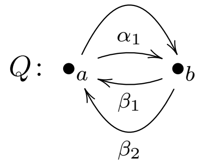
$$\bar{Q}\colon\ \bullet_b \xrightarrow{\beta} \bullet_c \xrightarrow{\gamma} \bullet_d$$

Since $\bar{B}$ is hereditary, any ordering will do, so let for instance $S_3 = S_d$, $S_2 = S_c$, $S_1 = S_b$. With this ordering B is quasi-hereditary.

As a consequence of the previous theorem, if B is an algebra of finite representation type with indecomposable modules $M_1, \ldots, M_s$, then the *Auslander algebra* $[\mathrm{End}_B(\oplus_{i=1}^s M_i)]^{\mathrm{op}}$ is quasi-hereditary. For a proof that Auslander algebras have global dimension at most two, see [Aus].

By Theorem 2.5, quasi-hereditary algebras have finite global dimension. Not all algebras with finite global dimension are quasi-hereditary, as the following example shows. This example first appeared in [Gre].

Example 3.4. Let B be the path algebra $B = \Bbbk Q/I$, where Q is the quiver



and I is the ideal

$$I = \langle \beta_1\alpha_1, \beta_2\alpha_1, \beta_2\alpha_2, \alpha_2\beta_1 \rangle.$$

This algebra has global dimension four. The bound $\mathrm{gldim}\, B \leq 2r - 2 = 2$ from Theorem 2.5 is not satisfied, so B is not quasi-hereditary.

As an illustration that quasi-hereditary algebras are ubiquitous, we record the following important theorem by Iyama [Iya].

Theorem 3.5. *Let B be a finite dimensional $\Bbbk$-algebra and let M be a finitely generated B-module. Then there exists a finitely generated B-module X such that* $[\mathrm{End}_B(M \oplus X)]^{\mathrm{op}}$ *is quasi-hereditary.*

4. The category $\mathcal{F}(\Delta)$

Suppose B is a quasi-hereditary algebra. Let $\mathcal{F}(\Delta)$ denote the full subcategory of $\mathsf{mod}\, B$ consisting of the B-modules which admit a Δ-filtration.

The category $\mathcal{F}(\Delta)$ is a *resolving* subcategory of $\mathsf{mod}\, B$, that is, (i) it contains the indecomposable projective B-modules, (ii) it is closed under extensions and direct summands, and (iii) it is closed under kernels of epimorphisms. The most convenient way to prove this fact is to use the description of $\mathcal{F}(\Delta)$ given in the proof of Theorem 1.6; a finitely generated B-module M is in $\mathcal{F}(\Delta)$ if and only if $\mathrm{Ext}^1_B(M, \nabla) = 0$. Condition (i) is then obvious. Also, condition (ii) clearly holds. To prove condition (iii), let $0 \to L \to M \to N \to 0$ be an exact sequence of finitely generated B-modules with M and N in $\mathcal{F}(\Delta)$. In the long-exact sequence

$$\ldots \to \mathrm{Ext}^1_B(M, \nabla) \to \mathrm{Ext}^1_B(L, \nabla) \to \mathrm{Ext}^2_B(N, \nabla) \to \ldots$$

we have $\mathrm{Ext}^1_B(M, \nabla) = 0$ by assumption. Since N is in $\mathcal{F}(\Delta)$, it follows from Theorem 2.1 that $\mathrm{Ext}^2_B(N, \nabla) = 0$. Therefore $\mathrm{Ext}^1_B(L, \nabla) = 0$, and N is in $\mathcal{F}(\Delta)$. So $\mathcal{F}(\Delta)$ is closed under kernels of epimorphisms.

The category $\mathcal{F}(\Delta)$ is not an abelian subcategory of $\mathsf{mod}\, B$, unless all standard modules are simple. The category $\mathcal{F}(\Delta)$ does have its own kernels and cokernels, but this is a subject we will not go into here.

In the next section we come back to a more subtle property of $\mathcal{F}(\Delta)$; it has Auslander-Reiten sequences.

5. Box characterization of quasi-hereditary algebras

One way to read Theorem 2.2 is that a quasi-hereditary structure on an algebra imposes a certain directedness to the algebra and its module category. This theme can be taken much further, and a characterization of quasi-hereditary algebras in terms of directed boxes was given by Ovsienko. The following theorem recently appeared in [KKO]. We do not define all the terms here, only remark that the exact structure on $\mathcal{F}(\Delta)$ is the one inherited from $\mathsf{mod}\, B$. For a gentle introduction to the theory of boxes, see [Bur].

Theorem 5.1 ([KKO], Theorem 1.1)**.** *A finite-dimensional algebra B is quasi-hereditary if and only if it is Morita equivalent to the right Burt-Butler algebra $R_{\mathcal{A}}$ of a directed box $\mathcal{A} = (A, V)$.*

Moreover, there is an equivalence of exact categories $\mathsf{mod}\, \mathcal{A} \to \mathcal{F}(\Delta)$.

We have the following corollary, previously obtained by Ringel by other means [Rin].

Corollary 5.2 ([KKO], Theorem 10.6)**.** *The category $\mathcal{F}(\Delta)$ has Auslander-Reiten sequences.*

References

[Aus] M. Auslander, *Representation theory of Artin algebras II*, Comm. Algebra 1 (1974), 269–310.

[BGG] I. N. Bernšteĭn, I. M. Gel'fand, S. I. Gel'fand, *A certain category of $\mathfrak{g}$-modules*, (Russian) Funkcional. Anal. i Priložen. 10 (1976), no. 2, 1–8.

[Bur] W. L. Burt, *Almost split sequences and BOCSES*, unpublished manuscipt.

[CPS] E. Cline, B. Parshall, L. Scott, *Finite dimensional algebras and highest weight categories*, J. Reine Angew. Math. 391 (1988), 85–99.

[DR1] V. Dlab, C. M. Ringel, *Quasi-hereditary algebras*, Illinois J. Math. 33 (1989), no. 2, 280–291.

[DR2] V. Dlab, C. M. Ringel, *The module theoretical approach to quasi-hereditary algebras*, Representations of algebras and related topics (Kyoto, 1990), 200–224, London Math. Soc. Lecture Note Ser., 168, Cambridge Univ. Press, Cambridge, 1992.

[Gre] E. L. Green, *Remarks on projective resolutions*, Representation theory, II (Proc. Second Internat. Conf., Carleton Univ., Ottawa, Ont., 1979), 259–279, Lecture Notes in Math., 832, Springer, Berlin, 1980.

[HP] L. Hille, M. Perling, *Tilting Bundles on Rational Surfaces and Quasi-Hereditary Algebras*, Annales de l'institut Fourier, 64 (2014), no. 2, 625–644.

[Iya] O. Iyama, *Finiteness of representation dimension*, Proc. Amer. Math. Soc. 131 (2003), no. 4, 1011–1014.

[KKO] S. Koenig, J. Külshammer, S. Ovsienko, *Quasi-hereditary algebras, exact Borel subalgebras, A-infinity-categories and boxes*, Adv. Math. 262 (2014), 546–592.

[Mad] D. Madsen, *Projective dimensions and almost split sequences*, J. Algebra 271 (2004), no. 2, 652–672.

[Nic] W. R. Nico, *An improved upper bound for global dimension of semigroup algebras*, Proc. Amer. Math. Soc. 35 (1972), 34–36.

[Par] B. Parshall, *Some finite-dimensional algebras arising in group theory*, Algebras and modules, I (Trondheim, 1996), 107–156, CMS Conf. Proc., 23, Amer. Math. Soc., Providence, RI, 1998.

[Rin] C. M. Ringel, *The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences*, Math. Z. 208 (1991), no. 2, 209–223.

[Sco] L. L. Scott, *Simulating algebraic geometry with algebra. I. The algebraic theory of derived categories*, The Arcata Conference on Representations of Finite Groups (Arcata, Calif., 1986), 271–281, Proc. Sympos. Pure Math., 47, Part 2, Amer. Math. Soc., Providence, RI, 1987.

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