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The balance model for teaching linear equations: a systematic literature review

Mara Otten[1*], Marja Van den Heuvel-Panhuizen[1,2,3] and Michiel Veldhuis[1,4]

Abstract

This paper reports a systematic literature review of the balance model, an often-used aid to teach linear equations. The purpose of the review was to report *why* such a model is used, *what* types of models are used, and *when* they are used. In total, 34 peer-reviewed journal articles were analyzed, resulting in a comprehensive overview of described rationales for using the balance model, its appearances, situations in which it was used, and the gained learning outcomes. Some trends appeared about how rationales, appearances, situations, and learning outcomes are related. However, a clear pattern could not be identified. Our study shows that this seemingly simple model actually is a rather complex didactic tool of which in-depth knowledge is lacking. Further systematic research is needed for making informed instructional decisions on when and how balance models can be used effectively for teaching linear equation solving.

Keywords: Algebra, Teaching linear equations, Balance model

Introduction

A substantial component of learning algebra is learning to solve algebraic equations. Within the algebra curriculum, solving linear equations is one of the foundational topics in which students make the transition from reasoning with numbers to reasoning with unknowns (e.g., Filloy & Rojano, 1989). Similarly, early algebra has been described as a "shift from thinking about relations among particular numbers and measures towards thinking about relations among sets of numbers and measures, from computing numerical answers to describing and representing relations among variables" (Carraher, Schliemann, & Schwartz, 2007, p. 266). Solving linear equations as a basic skill (Ballheim, 1999) is a substantial part of the middle school mathematics program (Huntley & Terrell, 2014). However, many students do not achieve mastery of this basic skill and experience difficulties in learning the concepts and skills related to solving equations (e.g., Kieran, 2007).

Solving linear equations means that the values of the unknown quantities have to be found based on the equality of two given mathematical expressions—one on each side of the equal sign. The essence of an equation is that these mathematical expressions represent the same value (Alibali, 1999), which makes *equality* a key concept in solving linear equations (e.g., Bush & Karp, 2013) and understanding equality one of the main conceptual demands associated with equation solving (Kieran, 1997; Kieran, Pang, Schifter, & Ng, 2016). Students need to understand that in an equation, the expressions on both sides of the equal sign have the same value and that this equality should always be maintained in the process of solving an equation (e.g., Kieran et al., 2016).

Misconceptions related to the concept of equality in linear equation solving are well documented. These misconceptions are particularly reflected in students' interpretations of the equal sign. Instead of perceiving it as a relational symbol meaning "is the same as," students often have an operational view of the equal sign, that is, they view it as a sign to "do something" or to "calculate the answer" (e.g., Knuth, Stephens, McNeil, & Alibali, 2006). For example, when solving the problem $8 + 4 = \_\_ + 5$, a common mistake is adding the numbers on the left side of the equation and putting a 12 in the blank (Falkner, Levi, & Carpenter, 1999). Such interpretation of the equal sign can begin in the elementary grades and can persist through middle

* Correspondence: m.otten@uu.nl
[1]Freudenthal Group, Faculty of Social and Behavioural Sciences, Utrecht University, PO Box 80140, 3508 TC Utrecht, The Netherlands
Full list of author information is available at the end of the article

school (e.g., Alibali, Knuth, Hattikudur, McNeil, & Stephens, 2007).

One way to help students gain conceptual understanding in equation solving is through the use of models as "ways of thinking about abstract concepts" (Warren & Cooper, 2009, p. 77). Such didactical models can be seen as representations of mathematical problem situations in which the essential mathematical concepts and aspects of the problem situation are reflected, and through which the concrete situation is connected to the more formal mathematics (Van den Heuvel-Panhuizen, 2003). By first being a *model of* a concrete situation where the model is very closely related to a specific problem and later evolving to a *model for* similar problems that are situated in another context, the model can be applied for solving all kinds of new problems (Streefland, 2003).

In mathematics education, several didactical models are used to give students access to particular mathematical concepts, such as the number line or the bar model for teaching fractions. The balance model is another commonly used didactical model. This model is often used to support students' understanding of linear equation solving. Characteristic of the balance model is that its form serves as a model for its function in solving linear equations: the balance can be used to refer to the situation of equality of the expressions on the two sides of an equation. The philosopher and mathematician Gottfried Wilhelm Leibniz (1646–1716) already made this connection when he mentioned the relation between equality in a mathematical situation and a balance with equal things on both sides (Leibniz, 1989).

In the context of a larger research project on algebraic reasoning, we wanted to find indications for setting up a teaching sequence on linear equation solving. We searched for information about the use of the balance model as a possible aid to assist students in developing understanding of solving linear equations. The diverse and scattered picture we got from this initial search prompted us to investigate this more systematically. Therefore, we planned to carry out a systematic review of the literature of how the balance model turns up in the large body of research and professional articles that has been published about teaching linear equation solving. With this review, we aimed at answering the following research question: What role does the balance model play in studies on teaching linear equation solving?

In general, to learn more about a didactical model for teaching students (mathematical) concepts, it is essential to gain insight in various important aspects of a model. The specific way of representing the model is important to take into account, but also information related to possible reasons for choosing this particular model and timing of using the model in a teaching and learning trajectory contribute to getting a complete picture of how the model can be used. Lastly, to be able to evaluate the use of a didactical model for fostering students' conceptual understanding, it is important to incorporate students' learning outcomes as well. To determine the role balance models play in studies on teaching linear equation solving, we looked into what authors reported about *why* such a model is used, *what* types of models are used, *when* they are used, and what learning outcomes are associated with its use. Knowing this can be helpful for teachers, researchers, and developers of instructional material for making informed decisions about choosing the balance model for teaching linear equation solving.

Method

Article search and selection

For selecting articles for the review, we searched in 93 peer-reviewed research journals in the areas of mathematics education, educational sciences, pedagogics, developmental psychology, special education, and technology in education. The search was conducted in Scopus and ERIC for articles in English. The search query consisted of terms such as equation*, equal sign*, equality, equivalence, balanc*, algebra*, mathematic*, unknown*, and solv*, and combinations thereof (see Additional file 1 for the complete search queries). There was no limit with regard to the date of publication.

The search, conducted in March 2017, resulted in 932 hits in Scopus and in 723 hits in ERIC, together resulting in 1655 hits (see Fig. 1). After merging, 333 duplicates were identified and removed, resulting in 1322 articles from 92 journals. Thirty-two articles were removed either because they, despite our search query, turned out to be not in English, or because they did not originate from peer-reviewed journals (e.g., were book chapters), resulting in 1290 articles.

In a six-step procedure, titles and abstracts were screened. Articles that were not in the field of mathematics education, did not touch upon the domain of algebra, did not address equations, were not about linear equations, or did not address teaching or learning linear equations, were excluded. This resulted in 287 articles. In the sixth and final step, the 282 articles of which we could obtain the full text were inspected to make an accurate decision on whether the concept of balance was discussed in relation to linear equation solving. Based on this reading, 29 articles were selected in which the balance model was used to teach linear equation solving. Lastly, snowballing was used to ensure that possible other relevant literature was covered as well, which resulted in five additional articles. Thus, the final collection comprised 34 articles from 22 journals.

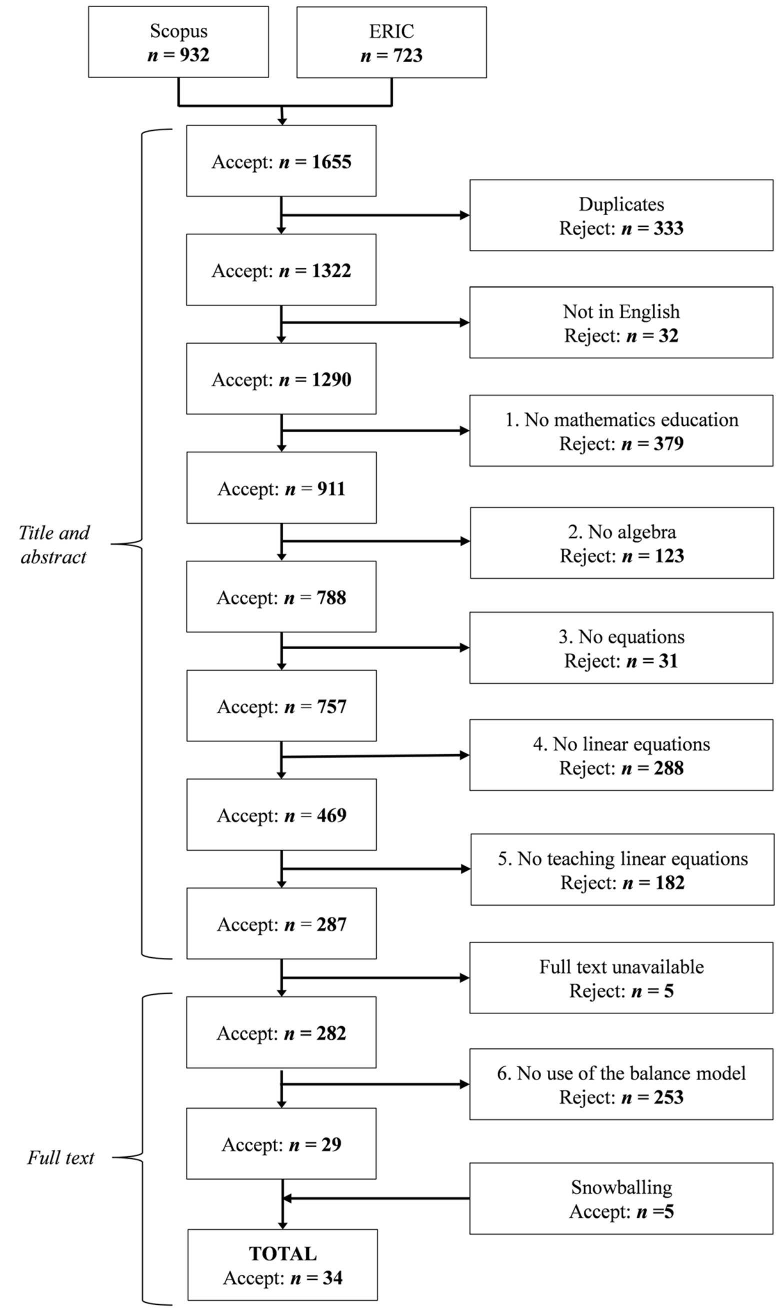

Fig. 1 Flow chart illustrating the systematic search process, resulting in 34 articles

Data extraction

For each of the 34 articles, information was extracted related to the rationales and the limitations of using the balance model, the appearances of the model, the situations in which the model was used, and students' learning outcomes. Information was extracted in case at least one sentence of the article was devoted to either of these four categories. After the inventory of all rationales (in 26 articles) and appearances (in 34 articles), patterns were identified to see whether classes of rationales and types of models could be created. Multiple rationales for using the model and multiple appeareances could be extracted from one article. To describe the situations in which the balance model was used (in 34 articles), information was extracted regarding the grade level of the students, the duration of the intervention, the type of tasks students worked on, and the type of instruction. Students' learning outcomes when using a balance model for teaching linear equation solving were discussed in 19 articles. These different aspects of the reviewed articles are summarized in Table 1.

Results

Why was the balance model used?

Rationales for using the balance model were provided in 26 articles. Three main classes of rationales could be identified, which were all related to the specific features of the context of the balance model. Articles constituting the Equality class of rationales all *directly* referred to using the balance to enhance students' understanding of the concept of equality. Direct references to equality are directly focused on the mathematical equality, by emphasizing the analogy between the balance model and equality in an equation. Articles in the remaining two classes of rationales made more *indirect* references to using the balance model for enhancing students' understanding of equality. Indirect references to equality are, for example, offering students physical experiences when manipulating a balance model and thus feel, through the experience of balancing, the concept of equality. Such articles that made a reference to previous or concurrent physical experiences related to the balance model fell in the Physical Experiences class of rationales. Articles that fell into the Models and Representations class of rationales referred to the use of models and representations for enhancing students' conceptual understanding in linear equation solving. Finally, limitations of using the balance model for teaching linear equation solving were also extracted.

Rationales related to the equality concept

A majority of 15 articles (three from the same research project) mentioned rationales for using the balance model related to the concept of equality. It was often stated that understanding the concept of equality can be enhanced by making use of the model of a balance (e.g., Gavin & Sheffield, 2015; Leavy et al., 2013; Mann, 2004; Taylor-Cox, 2003; Warren et al., 2009). Because both sides of a balance model are of equal value and thus exchangeable, the model was described as being very suitable for demonstrating the idea of equality or equilibrium (Figueira-Sampaio et al., 2009) and quantitative sameness (e.g., Warren & Cooper, 2005). In line with this, several authors referred to the use of the balance model to enhance the understanding of the equal sign as a symbol for representing equality (e.g., Vlassis, 2002; Warren & Cooper, 2009). Accordingly, the balance model has often been described as suitable to demonstrate the strategy of doing the same thing on both sides of the equation, in which emphasis on the concept of balance is crucial (e.g., Andrews & Sayers, 2012; Figueira-Sampaio et al., 2009; Marschall & Andrews, 2015), thereby helping students in forming, according to Vlassis (2002), a mental picture of the operations they have to apply. Another mentioned advantage of the balance model is the possibility to keep track "of the entire numerical relationship expressed by the equation while it is being subjected to transformations" (Linchevski & Herscovics, 1996, p. 52), which makes it suitable for demonstration of the cancelation of identical terms on both sides of the equation (see also Filloy & Rojano, 1989).

Rationales related to the physical experiences

The second class of rationales that was identified, mentioned in 11 articles (all from different research projects), was related to learning through physical experiences. In several articles, a reference was made to previous physical experiences related to maintaining balance. Araya et al. (2010) argued that the process of maintaining balance has a primary biological basis and is therefore common physical knowledge for all human beings. Through using the balance model, this biologically primary knowledge can be connected to the abstract idea of maintaining equality in an equation. Others emphasized the similarity between the model and a teeter-totter (or see-saw) and referred to children's (playing) experiences with this object (Alibali, 1999; Kaplan & Alon, 2013).

In other articles, the contribution of concurrent physical experiences with the balance model was pointed out as being beneficial to the learning of linear equations. Warren and Cooper (2009) underlined the importance of movement (for example by acting out a balance) and gestures during the learning trajectory to develop mental models of mathematical ideas. They argued that referring to these experiences in later stages of the learning process can be beneficial. Also, the importance of physical experiences with concrete objects for developing understanding of linear equations was mentioned in several articles. Offering young students experiences with manipulation of balance scales, because through this manipulation, equality can be recognized, defined, created, and maintained, could enhance students' understanding of this concept (Taylor-Cox, 2003). Suh and Moyer (2007) mentioned that using manipulable concrete objects have a sense-making function, through connecting procedural knowledge (manipulations on the objects) and conceptual knowledge of algebraic equations. However, at the same time, these authors pointed out that caution with using such manipulatives for teaching formal equation solving is necessary, because not all students automatically connect their actions on manipulatives with their manipulations on abstract symbols. Also Orlov (1971) commented that the balance model as a physical instrument can help in forming

Table 1 Overview of articles in which the balance model (BM) was used for teaching linear equations

Article	Rationale[b]	Appearance	Use			Students involved	Research design[c]	Intervention in comparison group (CG)?	Learning outcomes (on linear equation solving unless otherwise specified)
			Duration intervention	Instructional setting	Type of equations				
Alibali (1999)	PE	Drawn	40-min session	Individual instruction by teacher	$3 + 4 + 5 = \_\_ + 5$	Grades 3–5; 143 students	Pre-posttest; BM[a]-group and two comparison groups	CG1: received feedback CG2: explanation solution steps	• 36% of the BM-group improved • BM-group outperformed CG1 • CG2 outperformed BM-group
Andrews (2003)		Physical, drawn	1 lesson	Classroom instruction by teacher	$2x + 5 = x + 8$	Grade 7; 4 students			
Andrews and Sayers (2012)	EQ, LI	Drawn	5 lessons	Classroom instruction by teacher	$x + 7 = 9$ $x - 2 = 10$	Grade 8, 3 classes			
Araya et al. (2010)	PE, MR	Drawn	2-h session	Classroom instruction by learning movie	$2x + 1 = 5 + x$	Grade 7; 236 students; no previous algebra instruction	Posttest; BM-group and comparison group	CG: symbolic instruction	• Below average to high achieving students of the BM-group outperformed the CG
Austin and Vollrath (1989)	PE, LI	Physical			$3w + 5 = 11$	"Introductory algebra students"			
Berks and Vlasnik (2014)	MR	Drawn	1 lesson	Classroom instruction by teacher	$4x + 2y = 12$ $y = 2x + 2$	Students with some algebra experience			
Boulton-Lewis et al. (1997)		Drawn	5 lessons	Classroom instruction by teacher	$2x + 5 = 17$	Grade 8; 21 students	Pre-posttest; BM-group		• Few students could model or solve equations with the BM • Most students could solve equations without the BM • Students preferred not to use the BM
Brodie and Shalem (2011)	EQ	Drawn	3–5 lessons	Classroom instruction by teacher	$3 + x = 5$	Grade 8			
Caglayan and Olive (2010)	MR, LI	Drawn	2 lessons	Classroom instruction by teacher	$2c + 1 = 7$ $2x - 1 = 13$	Grade 8; 24 students	Descriptive; BM-group		• BM gives meaning to equations with addition/multiplication • BM does not give meaning to equations with negative values/subtraction

Table 1 Overview of articles in which the balance model (BM) was used for teaching linear equations *(Continued)*

Article	Rationale[b]	Appearance	Use			Students involved	Research design[c]	Intervention in comparison group (CG)?	Learning outcomes (on linear equation solving unless otherwise specified)
			Duration intervention	Instructional setting	Type of equations				
Cooper and Warren (2008)	MR	Physical; drawn	5 years	Classroom instruction by teacher	$? + 11 = 36$ $? - 7 = 6$	Grades 2–6; 220–270 students	Descriptive; BM-group		• Young students can generalize the balance method[d] for simple equations • Older students can generalize the balance method for all operations and use it to solve equations
Figueira-Sampaio, Santos, and Carrijo (2009)	EQ, PE, LI	Group 1: Physical Group 2: Virtual	50-min lesson	Group 1: Classroom instruction by teacher Group 2: Working in pairs with computer	$5x + 50 = 3x + 290$	Grade 6; 46 students	Descriptive; two BM-groups		• Virtual BM-group shows more participation, social interaction, motivation, cooperation, discussion, reflection, and a feeling of responsibility, than the physical BM-group
Filloy and Rojano (1989)	EQ, MR, LI	Drawn	1 session with 5 problems	Individual instruction by teacher	$3 + 2x = 5x$ $10x - 18 = 4x$	Grade 7; three classes	Descriptive; BM-group		• With BM, the step from solving equations with unknowns on one side of the equal sign towards solving equations with unknowns on both sides of the equal sign, is smaller than with the geometrical model • The geometrical model is more appropriate than the BM for modeling equations with subtraction • Assigning values to unknowns can hinder students when using the BM

Table 1 Overview of articles in which the balance model (BM) was used for teaching linear equations *(Continued)*

Article	Rationale[b]	Appearance	Use			Students involved	Research design[c]	Intervention in comparison group (CG)?	Learning outcomes (on linear equation solving unless otherwise specified)
			Duration intervention	Instructional setting	Type of equations				
Fyfe, McNeil, and Borjas (2015)	PE	Physical, drawn	1 lesson	Individual instruction by teacher	$2 + 3 = 2 + \_\_$	Grades 1–3; 389 students			
Gavin and Sheffield (2015)	EQ	Drawn		Classroom instruction by teacher	$12 + 23 = 13 + n$ $51 - n = 50–25$	Grade 6; 305 students			
Jupri, Drijvers, and Van den Heuvel-Panhuizen (2014)		Drawn	1 item on a test		1 kg + 0.5brick = 1brick	Grade 8; 51 students			
Kaplan and Alon (2013)	PE	Virtual	1 session	Individual instruction by teacher and individually working with computer	▲▲ = ●●●	Grades 3–4; 2 students			
Leavy, Hourigan, and McMahon (2013)	EQ	Physical		Classroom instruction by teacher	$8 = \_\_ + 3$	Grade 3			
Linchevski and Herscovics (1996)	EQ, LI	Drawn	1 lesson	Individual instruction by teacher	$8n + 11 = 5n + 50$	Grade 7; 6 students	Descriptive; case studies with BM		• BM is suitable for demonstrating cancelation of identical terms on both sides of the eq. • BM is not suitable for modeling equations with subtraction
Mann (2004)	EQ	Physical, drawn		Classroom instruction by teacher	●●■ = ■ ■ $5 + 6 = \_\_ + 2$	Grade 3; 1 class			
Marschall and Andrews (2015)	EQ, LI	Drawn		Classroom instruction by teacher	$x + 1 = 3$ $4x - 3 = 2x + 5$	Grade 6; 6 classes			
Ngu, Chung, and Yeung (2015)		Drawn	40-min lesson	Individual instruction sheet with BM	$5 + 3n = 10$ $3m - 1 = 5$	Grade 8; 71 students	Pre-posttest; BM-group and comparison group	CG: solving equations with inverse operations	• BM-group improved from pre- to posttest • CG improved more than BM-group • Higher cognitive load for BM-group than CG

Table 1 Overview of articles in which the balance model (BM) was used for teaching linear equations *(Continued)*

Article	Rationale[b]	Appearance	Use			Students involved	Research design[c]	Intervention in comparison group (CG)?	Learning outcomes (on linear equation solving unless otherwise specified)
			Duration intervention	Instructional setting	Type of equations				
Ngu and Phan (2016)		Drawn	45-min lesson	Individual instruction sheet with BM	$n / 2 = 7$ $x - 9 = 4$	Grade 7; 63 students	Pre-posttest; BM-group and comparison group	CG: solving equations with inverse operations	• BM-group improved from pre- to posttest • CG improved more than BM-group • Positive relation between performance on procedural knowledge and performance on conceptual knowledge for CG but not for BM-group
Ngu, Phan, Yeung, and Chung (2018)		Drawn	Two 40-min lessons	Individual instruction sheet with BM	$3x + 1 = 2x + 8$ $6 - q = 10$	Grades 8–9; 29 students	Pre-posttest; BM-group and comparison group	CG: solving equations with inverse operations	• BM-group improved from pre- to posttest • CG improved more than BM-group • Higher cognitive load for BM-group than CG
Orlov (1971)	PE	Physical	2 years	Classroom instruction by teacher	$5x - x + 2 = 2x + 6$	Grade 8; 200 students	Repeated measures; BM-group and comparison group	CG: experimental program without BM	• BM-group, especially average and above-average students, outperformed CG
Perry, Berch, and Singleton (1995)	PE	Physical	1 lesson	Individual instruction by teacher	$3 + 4 + 5 = \_\_ + 5$	Grades 4–5; 56 students	Pre-posttest; BM-group and comparison group	CG: only verbal instruction	• BM-group outperformed CG
Raymond and Leinenbach (2000)		Drawn	26 lessons	Classroom instruction by teacher	$x + 4 = 2x + 3$	Grade 8; 120 students	Descriptive; BM-group		• BM instruction leads to better performance than textbook instruction • Large performance decrease when returning to textbook after BM • Better than expected performances on standardized algebra test after BM

Table 1 Overview of articles in which the balance model (BM) was used for teaching linear equations *(Continued)*

Article	Rationale[b]	Appearance	Use			Students involved	Research design[c]	Intervention in comparison group (CG)?	Learning outcomes (on linear equation solving unless otherwise specified)
			Duration intervention	Instructional setting	Type of equations				
Rystedt, Helenius, and Kilhamn (2016)	EQ	Drawn	1 lesson	Classroom instruction by teacher	$4x + 4 = 2x + 8$	Grades 6–7; five classes			
Smith (1985)		Physical		In pairs with BM	$8w = 120$	Grades 4–6	Descriptive; BM-group		• BM assisted in exploring/learning basic algebraic principles and enhanced motivation
Suh and Moyer (2007)	PE, MR	Group 1: Virtual Group 2: Drawn	5 lessons	Classroom instruction by teacher; students individually with BM	$2x + 2 = 10$	Grade 3; 36 students	Pre-posttest; two BM-groups		• Both BM-groups improved • Each of the BMs showed unique features to support learning
Taylor-Cox (2003)	EQ, PE	Physical	1 lesson	Classroom instruction by teacher	$A + C + B = C + A + B$	Grade 1			
Vlassis (2002)	EQ, LI	Drawn	16 lessons	Classroom instruction by teacher	$7x + 38 = 3x + 74$ $13x - 24 = 8x + 76$	Grade 8; 40 students	Descriptive; BM-group		• Balance method was used by all students • After BM instruction, students made many mistakes related to negative numbers and unknowns
Warren and Cooper (2005)	EQ, MR	Physical; drawn	4 lessons	Classroom instruction by teacher	$? + 7 = 11$ $? - 4 = 13$	Grade 3; 20 students	Descriptive; BM-group		• Most students could represent equations with the BM and translate the model into symbolic eqs. • BM assisted students in understanding the equal sign and solving for unknowns • Ten students used the balance method for solving a subtraction problem; for others further teaching was necessary

Table 1 Overview of articles in which the balance model (BM) was used for teaching linear equations *(Continued)*

Article	Rationale[b]	Appearance	Use			Students involved	Research design[c]	Intervention in comparison group (CG)?	Learning outcomes (on linear equation solving unless otherwise specified)
			Duration intervention	Instructional setting	Type of equations				
Warren and Cooper (2009)	EQ, PE, MR	Physical; drawn	5 years	Classroom instruction by teacher	$? + 2 = 5$ $? - 3 = 6$	Grades 2–6; 220–270 students;	Descriptive; BM-group		• BM enhanced understanding of language and symbols • Students could generalize balance method for simple equations • Older students could generalize the balance method for all operations
Warren, Mollinson, and Oestrich (2009)	EQ, MR	Physical, drawn		Classroom instruction by teacher	$5 + 1 = 2 + 4$	Kindergarten			

Empty cells indicate that this information was not provided in that article
[a]*BM* Balance model
[b]EQ = rationales related to the equality aspect, PE = rationales related to the physical experiences, MR = rationales related to learning through models and representations, LI = limitation of using the balance model
[c]Information about the research design was only included for articles in which the effect of the balance model on students' learning outcomes was evaluated
[d]"Balance method" refers to the method of solving an equation by performing the same operations on both sides of the equation

abstract mathematical thought, because it represents an intermediate degree between immediate sensory data and mathematical abstraction. In this same line, Fyfe et al. (2015) recommended a sequence based on fading concreteness, where instruction begins with concrete material and fades into abstract mathematical symbols. The real-time feedback some models provide about being in balance, which allows students to verify the results of their manipulations and their reasoning processes and as such to construct knowledge, was also deemed important (Austin & Vollrath, 1989). When combined with social experiences, physical experiences were also said to contribute to the construction of knowledge (Figueira-Sampaio et al., 2009), for example, because it creates shared meaning between teacher and students (Perry et al., 1995).

Rationales related to learning through models and representations

The third class of rationales, mentioned in eight articles (four from the same research project), included a more general argumentation and referred to learning through the use of models and representations. According to Filloy and Rojano (1989), models such as the balance model can provide an opportunity to semantically and syntactically set a foundation for linear equation solving. Here, the meaning of equality and algebraic operations can first be derived from the context, while after students have gone through a process of abstraction, meaning at syntactic level is linked to this meaning of the context. Researchers involved in the Australian Early Algebraic Thinking Project (Cooper & Warren, 2008; Warren & Cooper, 2009) argued that, through models, mathematical ideas are presented externally as concrete material, by iconic representations, language, or symbols, while comprehension of these ideas occurs internally, in mental models or internal cognitive representations of mathematical ideas underlying the external representations. From this point of view, mathematical understanding is determined by the number and strength of the connections in the student's internal network of representations. Also the use of multiple representations in teaching abstract mathematical concepts or strategies was advocated (e.g., Berks & Vlasnik, 2014), because experiencing different modes of representation and making connections between and within these different modes of representation could enhance deep mathematical understanding (Suh & Moyer, 2007). The sense-making function of representations was elaborated on by Caglayan and Olive (2010), who reasoned that students can make sense of abstract symbolic equations through connecting this symbolic equation with the equation as expressed by its representation.

Also other reasons for using representations of the balance model were suggested. For example that it can create a shared language base which students can use when explaining their solutions (Berks & Vlasnik, 2014; Warren et al., 2009; Warren & Cooper, 2005) or that it is supposed to lower students' cognitive load during equation solving processes (see Araya et al., 2010). The latter contrasts with Boulton-Lewis et al. (1997) who hypothesized an increased processing load caused by the use of concrete representations. This might depend on the students' experience and the type of equation problems they have to solve: if students do not really need the help of a concrete representation of the balance

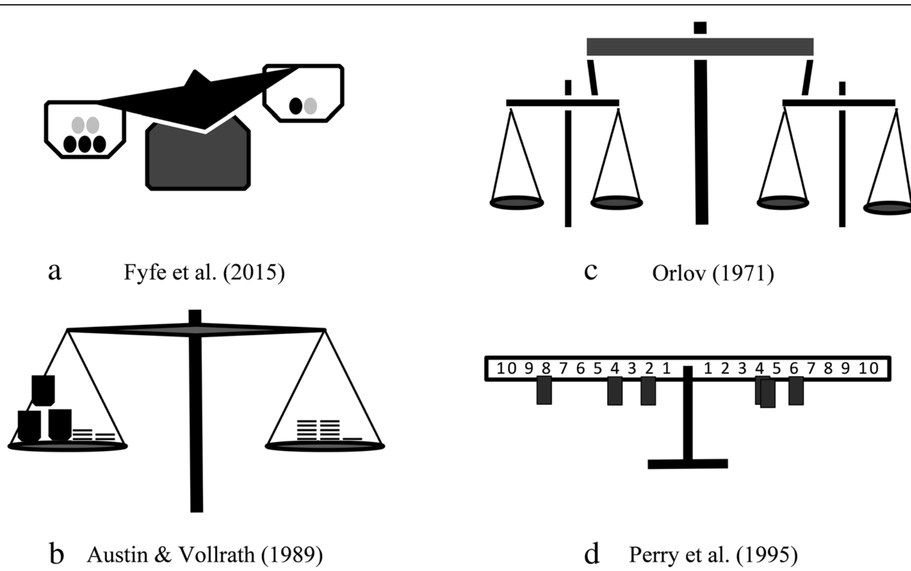

Fig. 2 Physical balance models, examples from four articles (**a**-**d**)

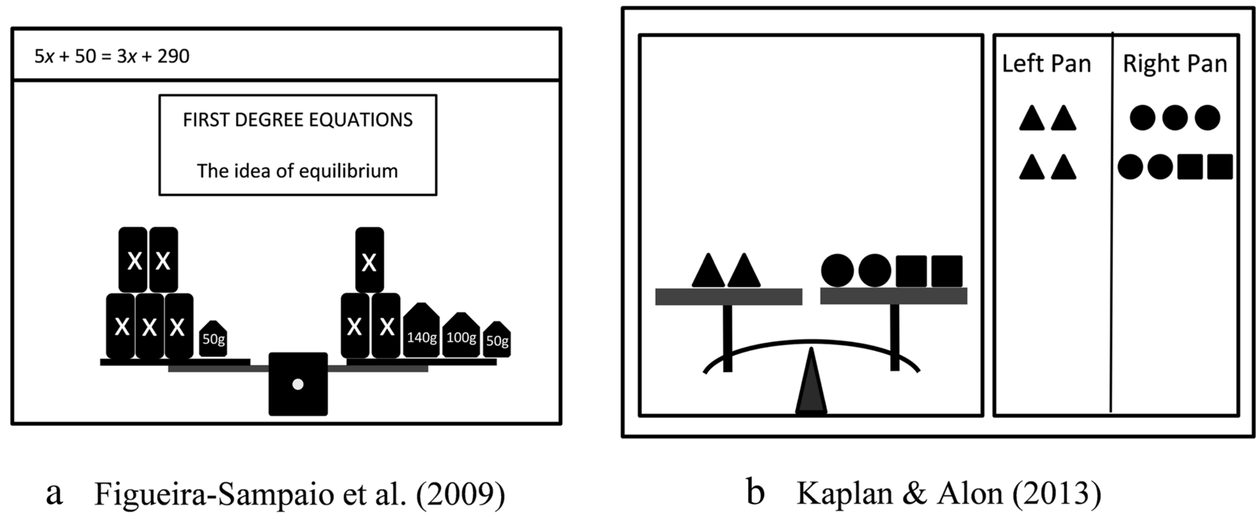

Fig. 3 Virtual balance models, examples from two articles (**a**-**b**)

model anymore and they still have to use it, this could indeed increase processing load.

Limitations of the balance model

Limitations of the balance model were described in eight articles (all from different research projects). In her well-known article, Vlassis (2002) described how eight-grade students were taught formal linear equation solving by making use of the balance model and concluded that although the balance model was able to provide students an "operative mental image" (p. 355) of the to-be-applied equation solving strategies, this model also had some shortcomings. For example, the model was not helpful for equations containing negative numbers or for other equations that are "detached from the model" (p. 354) and that no longer refer to a concrete model. Also, several other articles referred to the restricted possibilities the model has to represent equations with negative quantities or subtractions (e.g., Filloy & Rojano, 1989; Linchevski & Herscovics, 1996). As soon as negative values are involved, such as in the case of the equation $x + 5 = 3$, or equations with subtraction, such as $2x - 3 = 5$, the solution is difficult to express in terms of physical weight which makes it difficult to construct meaning for these equations (Caglayan & Olive, 2010).

Discussion of the findings regarding why the model was used

Although the three classes of rationales all have unique characteristics based on which they can be differentiated, they are also interrelated. The most often mentioned rationale was related to equality; understanding equality is regarded as one of the main conceptual demands associated with linear equation solving (e.g., Kieran et al., 2016). Inherent properties of the balance were connected to the concept of equality and the strategies that can be applied while maintaining the balance. The two remaining rationales were less often mentioned. These rationales contained indirect references to using the balance model for enhancing students' understanding of equality in an equation, through referring to learning through physical experiences or to learning through models and representations.

Articles in the class of rationales related to physical experiences referred either to the biological basis of maintaining balance or to other physical experiences with balance (such as with a teeter-totter), which, through using the balance model, could be connected to the idea of maintaining the balance in an equation. These previous physical experiences with balance could foster students' understanding of equality in an equation. This can be explained from the perspective of embodied cognition theory, stating that the connection of perceptual and physical experiences that we have when we interact with the world is fundamental for developing conceptual knowledge and cognitive learning processes (e.g., Barsalou, 2008; Wilson, 2002). Perceptuo-motor experiences are considered essential for developing mathematical concepts (e.g., Alibali & Nathan, 2012; Núñez, Edwards, & Matos, 1999), and mathematical reasoning is viewed as intricately linked with embodied actions (Abrahamson, 2017; Alibali & Nathan, 2012). When applying embodied cognition theory to teaching and learning solving linear equations, it is assumed that perceptuo-motor knowledge about the action of balancing is a necessary foundation for developing understanding of the mathematical concept of equality (e.g., Antle, Corness, & Bevans, 2013). This perceptuo-motor

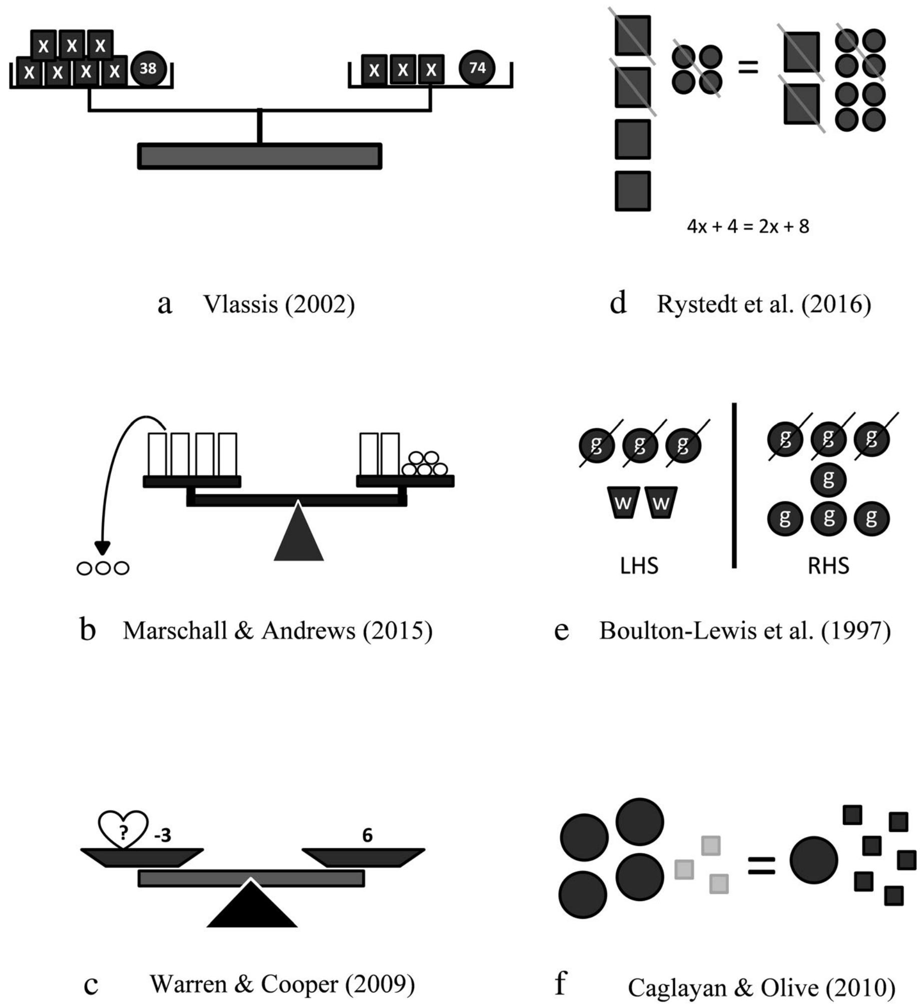

Fig. 4 Drawn balance models, examples from six articles (**a**-**f**)

knowledge is built up from the very pervasive physical experiences we have with balancing from a young age on (Gibbs Jr, 2006), through walking without falling, standing up and sitting down, or holding objects of different weights (Alessandroni, 2018). Furthermore, the other articles in this class of rationales stressed the contribution of concurrent physical experiences with the balance model to the learning of linear equations. Through manipulation of the model, students explore how to maintain its balance; these strategies for maintaining the balance of the model could later be connected to strategies for maintaining equality in an equation. This is also in line with embodied cognition theory: offering students opportunities to revitalize the basic perceptuo-motor knowledge through making use of a model of a balance through which they can imagine (or experience anew) the situation of balancing, could be beneficial for supporting students' understanding of equality in an equation and therefore for teaching linear equation solving.

Articles in the class of rationales related to learning through models and representations included more general arguments for enhancing students' understanding of equality in an equation. However, these rationales have some overlap with the rationales related to physical experiences. Both classes are related to perceptuo-motor experiences with balance. In the case of the Models and Representations class, this experience is more related to what the balance looks like. The balance as a device with two arms and a fulcrum in the middle can be used to represent an equation with on two sides of the equal sign an expression of equal value. Learning through models and representations can be connected to ideas of Realistic Mathematics Education (RME). One of the main instructional principles of RME is the use of didactical models with the purpose to bridge the gap between informal, context-related solution methods and the more formal ones, and in this way to stimulate students to come to higher levels of understanding (e.g., Van den Heuvel-Panhuizen, 2003).

What types of balance models were used?

Three types of appearances of the balance models came forward in the reviewed articles: physical, virtual, and drawn balance models. Physical balance models are concrete balance scales. On these scales, students can represent equations by placing real objects, standing for knowns and unknowns, on both sides of the model. Characteristic for these models is that they are dynamic, which means the students can operate on them and get real-time feedback on their actions. In virtual balance models, the balance is implemented in a digital environment. These models are mostly dynamic, in that sense that the balance tilts in response to students' (digital) manipulations and in this way gives real-time feedback. In drawn balance models, a schematic version of a balance is presented on paper or on the blackboard. The representations of these balance models are static: students cannot manipulate them and cannot receive real-time feedback. Whereas in most articles only one type of appearance of the balance model was used, in other articles different types appeared (e.g., Figueira-Sampaio et al., 2009) or a sequence of different appearances was presented, starting with the use of a physical model followed by a drawn balance model (e.g., Fyfe et al., 2015).

Physical balance models

Physical balance models appeared in 14 articles (three from the same research project). We drew schematic versions of several of these physical balance models. These drawings are shown in Fig. 2. The balance displayed in Fig. 2a was used by Fyfe et al. (2015) to represent, for example, $3 + 2 = 1 + 1 + \_\_$. Here, students could put three red and two yellow bears on the left side and one red and one yellow on the right, and then add the missing number to get the scales to balance (for similar models, see also, e.g., Warren et al., 2009). In Austin and Vollrath's balance model (1989; Fig. 2b), the equation $3x + 5 = 11$ is portrayed by, on the left side, three containers with unknown content and five washers and 11 washers on the right side (for similar models, see also, e.g., Andrews, 2003). A more complex example of a balance scale was utilized by Orlov (1971; Fig. 2c). His model contains four scales, two on each side. For example, by putting a weight on the left tray of the left part of the scale, the left arm of the balance scale goes up. In this way, negative numbers and unknowns can also be handled by this model. The last type of described physical balance model is a balance model in which the distance of the objects to the fulcrum can be adapted to represent linear equations such as $8 + 4 + 2 = 4 + 4 + \_\_$ (Perry et al., 1995; Fig. 2d; for a similar model, see also Smith, 1985). Here, all objects have the same weight, but by putting them at a particular position on the beam they represent a particular value.

Virtual balance models

Virtual balance models appeared in three articles (from different research projects). Drawings of the used virtual balance models are shown in Fig. 3. Most of these models display a balance scale quite similar to the physical balance models. However, the digital environment enables more possibilities in representations and functions of the model.

In the digital model used by Figueira-Sampaio and colleagues (2009; Fig. 3a), the equation $5x + 50 = 3x + 290$ is represented by cans with the letter X depicting the unknowns, and small labeled weights (e.g., 50 g, 100 g) depicting the numbers (for a similar model, see also Suh & Moyer, 2007). Here, while students manipulate the virtual balance scale, the corresponding equation is shown in formal algebraic symbols, which makes the link between these manipulations and the changes in the corresponding symbolic equation explicit. A further type of virtual balance model was found in the article of Kaplan and Alon (2013; Fig. 3b). In this model, students can explore relationships between different shapes of unknowns and find new equations based on given ones. For example, on the basis of the equations ▲▲ = ●●● and ▲▲ = ●●■■, a third equation can be created.

Drawn balance models

Drawn balance models appeared in 26 articles (four and three from the same research projects). Drawings of the used drawn balance models are shown in Fig. 4. Here, it is noticeable that some drawn balance models are depicted more realistically (Fig. 4a–c) and others more schematically (Fig. 4d–f), with pictures of objects or symbolic expressions to represent the knowns and unknowns.

While drawn balance models were present in many articles (e.g., Brodie & Shalem, 2011; Mann, 2004; Vlassis, 2002), the way in which the equations are represented in these models varied widely. In the drawn balance model found in the article of Vlassis (2002; Fig. 4a), the equation $7x + 38 = 3x + 74$ is represented by squares for each x and circles in which the numbers are indicated. The unknowns in this model are depicted in an expanded way (i.e., $7x$ and $3x$ are represented as seven separate x's and three separate x's). While in most models all unknowns are depicted separately, in the model of Linchevski and Herscovics (1996), the unknowns and knowns in the equation $8n + 11 = 5n + 50$ are partially shown in an expanded, respectively decomposed way, leading to the equation $5n + 3n + 11 = 5n + 11 + 39$. In this way, students can see that the terms $5n$ and 11 appear on both sides of the equation, which can cancel each other out. In the balances of Marschall and Andrews (2015; Fig. 4b) and Warren and Cooper (2009; Fig. 4c), equations with negative values and subtractions can also be represented. In Fig. 4b, the subtraction in the equation $4x - 3 = 2x + 5$ is represented by an arrow going down from one of the scales, so that the action of "taking away" is made visible. Alternatively, in Fig. 4c, a minus sign is included.

Another way in which drawn balance models appeared in the articles is as an abstract drawing. Here, the balance functions as a metaphor to point students' attention to the concept of equality. In Rystedt and colleagues (2016, Fig. 4d), the equation $4x + 4 = 2x + 8$ is represented with boxes for unknowns and dots for numbers. In articles in which such a metaphorical use of the balance model was present (e.g., Caglayan & Olive, 2010), this use was often accompanied by the instruction that the *balance* in the equation had to be *maintained* when solving the equation (Boulton-Lewis et al., 1997), or by gestures representing a balance scale (Rystedt et al., 2016). The use of a drawn balance model, especially for models with an abstract drawing, often went together with the use of manipulatives. For example, in the model of Boulton-Lewis and colleagues (1997; Fig. 4e), the schematically notated equation $2x + 3 = 7$ is represented by two white cups and three green counters on the left-hand side (indicated by LHS) and seven green counters on the right-hand side (indicated by RHS), while other colored cups and counters are used to represent subtractions or negative unknowns and numbers (for a similar approach, see, e.g., Suh & Moyer, 2007). Another example is the drawn balance model used by Caglayan and Olive (2010; Fig. 4f), where in the equation $4x - 3 = x + 6$ the "-3" is represented by gray tiles instead of black ones. Moreover, the equal sign is directly represented in this model.

Discussion of the findings regarding the types of used balance models

Drawn models appeared the most and virtual models the least, while the use of a physical model was often followed by the use of a drawn model. When looking into the relationship between the rationales and the appearances of the models, it seems that the use of a physical balance model most often goes together with rationales related to learning through physical experiences and the equality aspect. For the virtual models, all rationales appear more or less equally often, and the drawn balance models go most often together with the equality aspect rationale and rationales related to learning through models and representations. Except for rationales related to learning through physical experiences, the remaining two classes of rationales most often go together with the use of a drawn balance model. The drawn model appears to be the most flexible model, which means that it was used with all classes of rationales.

Although all three appearances of the model have the balance as a basic concept, they differ in their nature. Whereas the physical balance model and partly the virtual balance have a dynamic nature and as such can provide real-time feedback to the students about their actions, the drawn balance model is static. Drawn models, either presented on paper or on the blackboard, can nonetheless be extended with dynamic aspects by using manipulatives. For all three types of appearances of the model it applies that most models consist at least of a fulcrum, a horizontal balancing beam, and on both sides a scale. In addition to this configuration of the balance model, in other models, extra features are added. Through the addition of these features, the reach of the balance model is extended to represent a wider range of problems. For example, the additional scales in the physical model of Orlov (1971; Fig. 2c), the arrow going down from the scales of the drawn balance model in the article of Marschall and Andrews (2015; Fig. 4b), and the different colored manipulatives added to the drawn model of Boulton-Lewis and colleagues (1997; Fig. 4e) are all examples of variations of the balance model allowing the representation of negative numbers and unknowns. Such additional features provide a solution for the restricted possibilities that this model has (e.g., Vlassis, 2002), for example by allowing for the representation of equations with negative quantities or subtractions. In fact, this flexibility of the balance model is exactly how models should work. When used as didactical models (Van den Heuvel-Panhuizen, 2003), models should be flexible and not only suitable for solving one type of equation. One way of ensuring this flexibility is by allowing for adaptations without losing its primary function. However, bearing in mind the concept of *model of … - model for …* (Streefland, 2003), didactical models are not meant as a tool that everlasting has to be used for problem solving at a concrete, context-connected level. Instead, the idea is that in a later phase of the learning process, when a basis is laid for solving linear equations and the students have to solve more difficult equations, the student's thinking can still be supported by, and related to, the model without concretely representing the equation in a physical model.

When was the balance model used?

The situations in which the balance model was used in the articles when describing the teaching of linear equation solving, varied considerably with respect to the grade level of the students involved, the duration of the intervention with the model, the type of equation problems that students worked on, and the type of instruction that was provided to the students.

Grade levels and intervention duration

The balance model was used to teach linear equation solving to students from Kindergarten to Grade 9. Students up to Grade 6, who do not have previous experience with algebra, had their first encounter with linear equations through the balance model, which came forward in different studies (e.g., Warren & Cooper, 2005). In studies with students from Grades 7–9, who already have some basic experience with linear equation solving (with the exception of the seventh-grade students in the study by Araya et al., 2010), the balance model was introduced as a tool for solving equations (Vlassis, 2002) or used to illustrate the balance method (i.e., perform the same operations on both sides of the equation; Ngu & Phan, 2016). The duration of the interventions in which the balance model was used was also very diverse. The shortest interventions comprised one activity or one lesson (e.g., Figueira-Sampaio et al., 2009; Rystedt et al., 2016), while in other studies the balance model was integrated in a multiple-year teaching trajectory (e.g., Orlov, 1971; Warren & Cooper, 2009).

Type of equation problems

With very young students (e.g., Kindergarten, Grades 1–2), the balance model was mostly used for exploration of the first ideas of equality and the equal sign (e.g., Taylor-Cox, 2003; Warren et al., 2009). Students' task was for example to weigh different objects to find out which were the same and which were different. For somewhat older students (e.g., Grades 3–6), the balance model was for example used to assist them in solving simple addition problems such as $8 = \_\_ + 3$ (e.g., Leavy et al., 2013). Here, eight objects were put on the left side of the balance and three objects on the right side, and the students' task was to figure out what they could do to make both sides equal. The model was also used to introduce algebraic symbols to students without prior algebra experience, so that they could link the model to the abstract symbols. Then students' task was for example to manipulate the objects on the scales in such a way that they could determine the weight of the unknown object, while in the meantime in the digital environment the corresponding symbolic equation was shown (e.g., Figueira-Sampaio et al., 2009, see Fig. 3a; Suh & Moyer, 2007). In research with students with some algebra experience (i.e., from Grade 7 on), students' task was for example to represent symbolic equations by making use of the balance model and to use this representation to transform and solve the equations (Caglayan & Olive, 2010; see Fig. 4f). Or students' task was to solve an equation by making use of a physical balance model, while subsequently to represent the equation and the solution steps symbolically (Andrews, 2003). There were also articles in which two balance models with different unknowns were presented simultaneously to create a system of equations and to evoke the algebraic strategy of substitution (e.g., Austin & Vollrath, 1989; Berks & Vlasnik, 2014). Here, students' task was to combine the information of the equations to find the values of the unknowns.

In most studies, students' task was to determine the value of the unknown(s). However, there were also articles in which the main purpose was to discover different possibilities to maintain the balance of the model, without focusing on finding values of unknowns. For example, in the study by Kaplan and Alon (2013), the goal was to create multiple balanced scales and to analyze the relationships between unknowns (see Fig. 3b). Also in other articles, the balance model was used to discover different possibilities to maintain equality (Orlov, 1971) or to discover which "legal moves" (Raymond & Leinenbach, p. 288) could be made without disturbing the balance.

Lastly, there were large varieties between studies concerning maintaining the balance model when teaching equations. For example, in Warren and Cooper (2005), first a physical balance model and later a drawn balance model were used to model equations containing positive values and additive operations (e.g.,$? + 7 = 11$). After some lessons, these students also solved equations containing subtraction (e.g.,$? - 4 = 13$), but these equations were not represented with the balance model. In other studies, the use of the balance model was maintained longer during the learning process. For example, one of the teachers in the study by Marschall and Andrews (2015) did not only use the model for teaching equations containing positive values and addition, but extended the use of the model to represent equations such as $4x - 3 = 2x + 5$ (see Fig. 4b; for using the model for other type of equations, see also, e.g., Boulton-Lewis et al., 1997, see Fig. 4e; Orlov, 1971, see Fig. 2c).

Type of instruction

When working with the balance model, students either received classroom instruction by a teacher (e.g., Warren & Cooper, 2009) or via a learning movie (Araya et al., 2010), or they received individual instruction by a teacher (e.g., Perry et al., 1995), through instruction sheets (Ngu, Chung, & Yeung, 2015), or through working individually or in pairs with the balance (e.g., students working with the virtual balance in Figueira-Sampaio et al., 2009). Classroom instruction often concerned the teacher manipulating a balance model in front of the classroom (e.g., students working with the physical balance model in Figueira-Sampaio et al., 2009), while during individual instruction, students more often got opportunities to actively work with the balance model themselves (e.g., Suh & Moyer, 2007).

Discussion of the findings regarding when the balance model was used

In what situations the balance model was used was very diverse in the different studies. For what equation problems the balance model was used appeared to be related to students' experience with solving linear equations. For students up to Grade 6, without previous experience with algebra, most tasks concentrated around exploring the basic ideas of balance and solving simple equations (e.g., $8 = \_\_ + 3$), which went hand in hand with the rationale that such activities can be beneficial for developing understanding of equality and a relational understanding of the equal sign. Physical and virtual balance models were relatively often used to teach linear equation solving to students without prior algebra experience. In most of these studies, equations only contained positive values and additive operations. The studies conducted with students without prior experience in general underpinned the use of the balance model for teaching linear equation solving more thoroughly than studies with students with some algebra experience. The rationale that was relatively often mentioned in relation to teaching students without prior algebra experiences is the rationale related to the physical experiences, which fits the using of the physical balance model to teach these students. This also aligns with the common trend of using concrete materials for teaching young students rather than for teaching older students and with research showing that the use of concrete materials in mathematics education is in particular beneficial for children aged 7–11, in the mathematical domains of fractions and algebra (Carbonneau, Marley, & Selig, 2013).

With regard to studies conducted with students *with* prior algebra experience (in general students from Grades 7 and higher), students' tasks when working with the balance model were most often to model, to transform, and to solve equations by means of the balance model. Also in these studies, the rationale related to the equality aspect was most prominent. On the contrary, most of the studies in which no rationale for using the model was provided were also conducted with students with prior algebra experience. Most studies in which a limitation of using the balance model was mentioned involved these students. Drawn balance models were mostly used to teach students with prior algebra experience and in more than half of these studies, students were also taught equations containing negative values and subtraction.

Learning outcomes

Nineteen articles evaluated students' learning outcomes related to the use of the balance model. The research design of these studies and the most important learning outcomes are summarized in Table 1. Most studies were descriptive in nature and less than one-third of the studies used a pre-posttest design combined with a comparison group. As described in "When was the balance model used?" section, the studies showed large variations as regards the age and algebra experience of the students in their sample, the duration of the intervention, the tasks students worked on, and the type of instruction students received. Similar variations were detected upon examining the learning outcomes of the different studies. For example, Araya et al. (2010) found very positive results of using a learning movie with a drawn balance model in Grade 7 with students without prior algebra experience. These students outperformed students in the comparison group who received symbolic linear equation solving instruction. Also, Suh and Moyer (2007) reported positive effects of using balance models to teach third-grade students linear equation solving. Contrastingly, Boulton-Lewis et al. (1997) found that eighth-grade students had difficulties with modeling and solving linear equations when making use of the balance model. These students preferred not to use the model. The studies by Ngu and colleagues (2015), 2016, 2018) consistently showed similar or lower performances for Grade 7–9 students who used the method of performing the same operations on both sides of the equation—which was taught by making use of an instruction sheet with the balance model—compared to students who used the inverse method—which was taught as by referring to the change side, change sign-rule—for solving equations. In this latter approach, in which for example $x - 4 = 6$ becomes $x = 6 + 4$, students can conceptualize the inverse operation of -4 becoming $+4$ as a means to preserve the equality of equations. Therefore, the understanding of this inverse principle at a structural level is considered to be very relevant for students' learning of algebraic thinking (see, e.g., Ding, 2016). Interesting to notice here is that, although viewed superficially, the balance method differs from the inverse method, this latter method bears much resemblance to "doing the same on both sides." When taking the example of $x - 4 = 6$, then this rule means that on both sides 4 has to be added. This makes $x - 4 + 4 = 6 + 4$, which after simplifying results into $x = 6 + 4$. In other words, the main difference between "doing the same things on both sides" and "change sides, change sign" involves that one comes directly to the result by skipping the intermediate step of adding 4 on both sides. However, despite the close relationship between these two approaches and the related underlying principles, in only a few articles of our review study when authors refer to the use of the balance model, they also refer to the inverse method. This indicates that there has not been much research in which both approaches have been put in relation or contrasted.

The large variation between studies in which the balance model was used and the lack of studies with an experimental research design make it very difficult to draw unequivocal conclusions about the effects of using the balance model on students' learning outcomes. Nevertheless, some trends can be identified. Overall, most mixed and negative results are found for studies with somewhat older students (Grades 7–9) who already had some (basic) experience in solving linear equations (e.g., Ngu, Phan, Yeung, & Chung, 2018; Vlassis, 2002). The main reasons for this finding could be that the balance models in these studies, which were all drawn, were used for teaching a broad range of equations, including more difficult equations such as equations containing negative numbers and unknowns (e.g., Boulton-Lewis et al., 1997; Caglayan & Olive, 2010; Vlassis, 2002). In general, more positive results were found for studies conducted with younger students (e.g., Suh & Moyer, 2007; Warren & Cooper, 2005) or with students without prior knowledge on equation solving (e.g., Araya et al., 2010). In these studies, more often a physical model (e.g., Perry et al., 1995; see Fig. 2d) or a virtual model (e.g., Figueira-Sampaio et al., 2009; see Fig. 3a) was used, which in some cases in later stages was followed by a drawn model (e.g., Warren & Cooper, 2005). In most of these studies, the balance model was used to teach linear equations containing only positive values and addition. However, there were some exceptions. For example, Orlov (1971) found positive results for teaching different types of linear equations (including negative values and subtraction) to eighth-grade students by making use of a physical balance model (see Fig. 2c).

Discussion of the learning outcomes

Overall, the balance model seems to have more positive effects on learning outcomes related to linear equation solving for (younger) students without prior knowledge on linear equation solving. A possible explanation might be that for younger students, the balance model is used for laying a conceptual basis for linear equation solving, while for older students, who already have such a basis in solving linear equations, the model is more often used to revitalize this basis. Younger students have their first experience with exploring the concept of equality and with linear equation solving by means of the balance model. The tasks of older students when working with the balance model are more often to model, to transform, or to solve equations. In other words, the balance model then is used to revitalize their knowledge on linear equation solving and assist in solving all kinds of new equations. Warren and Cooper (2005) provide an example of using the balance model to support students in solving equations containing subtraction. In their teaching sequence, they first used a physical model to let students develop understanding of the concept of equality as "balance" and the strategy of doing the same things on both sides. Later, students could use this strategy for solving symbolically notated problems on paper, which also contained subtraction.

Conclusions

Our systematic review reveals a rather kaleidoscopic image of the balance model as an aid for teaching linear equations. The findings on *why* a balance model was used, *what* types of models were used, *when* they were used, and what learning outcomes were associated with its use, are diverse. Nevertheless, we could identify some trends within this scattered picture. Physical and virtual balance models were more often mentioned in the articles for teaching students during their first encounter with linear equation solving. Also, authors of these articles were more explicit about their rationales for using the balance model, with most rationales related to the equality aspect and students' physical experiences. The equations taught to these students mostly only contained positive values and addition, and these studies in general reported positive effects of using the balance model on students' learning outcomes of linear equation solving. Drawn balance models were more often used for students who already had some previous algebra experience. Additional features (such as manipulatives) were often added to these models, so that a wider range of problems could be represented, such as equations with negative values and subtraction. Articles in which drawn balance models were used were less explicit about their reasons for using the balance model, and in general reported more mixed and negative effects of using the balance model on teaching linear equation solving. However, it is important to note that within these trends, there were still many differences between studies, for example concerning the duration of the intervention and the type of instruction provided to the students.

These trends should of course be interpreted with caution. First of all, our results are entirely based on what the authors of the articles reported. In some articles, the authors did not explicitly report their rationales for using the balance model, which meant that they could not be identified by our analysis. Secondly, although we searched for articles in which the use of the balance model was discussed in 93 peer-reviewed journals to ensure a good coverage of the research literature, we only had a relatively small final sample of 34 articles that met our inclusion criteria. In addition, within the limited time we had available for this study, we could not consider including textbooks or other curriculum documents. Furthermore, we decided to do the review on articles in which the balance model was used for teaching linear equation solving and leave out other mathematical topics in which the balance model could be used. Lastly, a limitation that also should be mentioned is that our study only focused on the balance model and we did not compare it with other often used methods for helping students to solving linear equations such as the change side change sign-rule. Clearly, more research is necessary in this respect.

Our study was meant to create an overview of the role the balance model plays in teaching linear equation solving that might provide teachers, researchers, and developers of instructional materials with a source for making informed instructional decisions. Our analysis of the 34 peer-reviewed journal articles shows that there exists a considerable diversity in the rationales for using the model, the appearances of the model, the situations in which the model is used, and the found learning outcomes. This offers many possibilities for making use of the balance model. However, at the same time, our study reveals a clear lack of in-depth knowledge about when which type of balance models can be used effectively. For gaining this knowledge, more research is necessary, in particular (quasi-) experimental studies, allowing to investigate the effects of using models of different appearances (e.g., physical, virtual, and drawn models, with or without additional features such as added scales or the use of manipulatives) and the effects of different situations of using the model (e.g., for students with or without prior algebra experience, a short-time use of the model or a more extensive intervention, with one type of instruction or another) on students' learning outcomes. To provide a more theoretical grounding for the use of the balance model as an aid for teaching linear equations, it is important that the type of model that is used and the situations in which it is used are explicitly related to the rationales for using them. In summary, we can conclude that the balance model, which at first sight may seem to be a rather simple model—and maybe therefore is often used to teach students linear equation solving—is actually a rather complex model, of which still a lot has to be discovered to be used optimally in education.

Additional file

Additional file 1: Search queries. (DOCX 20 kb)

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Authors' contributions
The design of the study and the formulation of the research questions was done by all authors. Mara Otten conducted the systematic search procedure, the selection of articles, and the first analysis of the articles of which the findings were continuously discussed with Marja van den Heuvel-Panhuizen and Michiel Veldhuis. All authors participated in writing and revising the manuscript and approved the final version.

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Author details
[1]Freudenthal Group, Faculty of Social and Behavioural Sciences, Utrecht University, PO Box 80140, 3508 TC Utrecht, The Netherlands. [2]Freudenthal Institute, Faculty of Science, Utrecht University, Utrecht, The Netherlands. [3]Nord University, Bodø, Norway. [4]iPabo University of Applied Sciences, Amsterdam, The Netherlands.

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References

References marked with an asterisk indicate studies included in the systematic literature review.

Abrahamson, D. (2017). Embodiment and mathematics learning. In K. Peppler (Ed.), *The SAGE encyclopedia of out-of-school learning* (pp. 247–252). Thousand Oaks: SAGE Publications, Inc..

Alessandroni, N. (2018). Varieties of embodiment in cognitive science. *Theory & Psychology, 28*, 227–248. https://doi.org/10.1177/0959354317745589.

*Alibali, M. W. (1999). How children change their minds: Strategy change can be gradual or abrupt. *Developmental Psychology*, 35, 127–145. doi:https://doi.org/10.1037/0012-1649.35.1.127.

Alibali, M. W., Knuth, E. J., Hattikudur, S., McNeil, N. M., & Stephens, A. C. (2007). A longitudinal look at middle-school students' understanding of the equal sign and equivalent equations. *Mathematical Thinking and Learning, 9*, 221–247. https://doi.org/10.1080/10986060701360902.

Alibali, M. W., & Nathan, M. J. (2012). Embodiment in mathematics teaching and learning: Evidence from learners' and teachers' gestures. *Journal of the Learning Sciences, 21*, 247–286. https://doi.org/10.1080/10508406.2011.611446.

*Andrews, P. (2003). Opportunities to learn in the Budapest mathematics classroom. International Journal of Science and Mathematics Education, 1, 201–225. doi:https://doi.org/10.1023/B:IJMA.0000016874.07706.5a.

*Andrews, P., Sayers, J. (2012). Teaching linear equations: Case studies from Finland, Flanders and Hungary. The Journal of Mathematical Behavior, 31, 476–488. doi:https://doi.org/10.1016/j.jmathb.2012.07.002.

Antle, A. N., Corness, G., & Bevans, A. (2013). Balancing justice: Comparing whole body and controller-based interaction for an abstract domain. *International Journal of Arts and Technology, 6*, 388–409. https://doi.org/10.1504/IJART.2013.058285.

*Araya, R., Calfucura, P., Jiménez, A., Aguirre, C., Palavicino, M. A., Lacourly, N., ..., Dartnell, P. (2010). The effect of analogies on learning to solve algebraic equations. Pedagogies: An International Journal, 5, 216–232. doi:https://doi.org/10.1080/1554480X.2010.486160.

*Austin, J., Vollrath, H. J. (1989). Representing, solving, and using algebraic equations. Mathematics Teacher, 82(8), 608–612. Retrieved from http://www.history.didaktik.mathematik.uni-wuerzburg.de/vollrath/papers/053.pdf

Ballheim, C. (1999). Readers respond to what's basic. *Mathematics Education Dialogues, 3*, 11.

Barsalou, L. W. (2008). Grounded cognition. *Annual Review of Psychology, 59*, 617–645. https://doi.org/10.1146/annurev.psych.59.103006.093639.

*Berks, D. R., Vlasnik, A. N. (2014). Working the system. Mathematics Teacher, 107, 542–546. doi:https://doi.org/10.5951/mathteacher.107.7.0542.

*Boulton-Lewis, G., Cooper, T. J., Atweh, B., Pillay, H., Wilss, L., Mutch, S. (1997). Processing load and the use of concrete representations and strategies for

solving linear equations. The Journal of Mathematical Behavior, 16, 379–397. doi:https://doi.org/10.1016/S0732-3123(97)90014-6.
*Brodie, K., Shalem, Y. (2011). Accountability conversations: Mathematics teachers' learning through challenge and solidarity. Journal of Mathematics Teacher Education, 14, 419–439. doi:https://doi.org/10.1007/s10857-011-9178-8.
Bush, S. B., & Karp, K. S. (2013). Prerequisite algebra skills and associated misconceptions of middle grade students: A review. *The Journal of Mathematical Behavior, 32*, 613–632. https://doi.org/10.1016/j.jmathb.2013.07.002.
*Caglayan, G., Olive, J. (2010). Eighth grade students' representations of linear equations based on a cups and tiles model. Educational Studies in Mathematics, 74, 143–162. doi:https://doi.org/10.1007/s10649-010-9231-z.
Carbonneau, K. J., Marley, S. C., & Selig, J. P. (2013). A meta-analysis of the efficacy of teaching mathematics with concrete manipulatives. *Journal of Educational Psychology, 105*, 380–400. https://doi.org/10.1037/a0031084.
Carraher, D. W., Schliemann, A. D., & Schwartz, J. (2007). Early algebra is not the same as algebra early. In J. J. Kaput, D. W. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 235–272). Mahwah: Lawrence Erlbaum Associates/National Council of Teachers of Mathematics.
*Cooper, T. J., Warren, E. (2008). The effect of different representations on years 3 to 5 students' ability to generalise. *ZDM*, 40, 23–37. doi:https://doi.org/10.1007/s11858-007-0066-8.
Ding, M. (2016). Opportunities to learn: Inverse relations in U.S. and Chinese textbooks. *Mathematical Thinking and Learning, 18*, 45–68. https://doi.org/10.1080/10986065.2016.1107819.
Falkner, K. P., Levi, L., & Carpenter, T. P. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics, 6*(4), 232–236 Retrieved from https://eric.ed.gov/?id=EJ600209.
*Figueira-Sampaio, A. S., Santos, E. E. F., Carrijo, G. A. (2009). A constructivist computational tool to assist in learning primary school mathematical equations. Computers & Education, 53, 484–492. doi:https://doi.org/10.1016/j.compedu.2009.03.012.
*Filloy, E., Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. For the Learning of Mathematics, 9(2), 19–25. Retrieved from http://www.jstor.org/stable/40247950?seq=1#page_scan_tab_contents
*Fyfe, E. R., McNeil, N. M., Borjas, S. (2015). Benefits of "concreteness fading" for children's mathematics understanding. Learning and Instruction, 35, 104–120. doi:https://doi.org/10.1016/j.learninstruc.2014.10.004.
*Gavin, M. K., Sheffield, L. J. (2015). A balancing act: Making sense of algebra. Mathematics Teaching in the Middle School, 20, 460–466. doi:https://doi.org/10.5951/mathteacmiddscho.20.8.0460.
Gibbs, R. W., Jr. (2006). *Embodiment and cognitive science*. New York: Cambridge University Press.
Huntley, M. A., & Terrell, M. S. (2014). One-step and multi-step linear equations: A content analysis of five textbook series. *ZDM, 46*, 751–766. https://doi.org/10.1007/s11858-014-0627-6.
*Jupri, A., Drijvers, P., Van den Heuvel-Panhuizen, M. (2014). Difficulties in initial algebra learning in Indonesia. Mathematics Education Research Journal, 26, 683–710. doi:https://doi.org/10.1007/s13394-013-0097-0.
*Kaplan, R. G., Alon, S. (2013). Using technology to teach equivalence. Teaching Children Mathematics, 19, 382–389. doi:https://doi.org/10.5951/teacchilmath.19.6.0382.
Kieran, C. (1997). Mathematical concepts at the secondary school level: The learning of algebra and functions. In T. Nunes & P. Bryant (Eds.), *Learning and teaching mathematics: An international perspective* (pp. 133–158). East Sussex: Psychology Press.
Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels. Building meaning for symbols and their manipulation. In F. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (Vol. 2, pp. 707–762). Charlotte: Information age publishing.
Kieran, C., Pang, J., Schifter, D., & Ng, S. F. (2016). *Early algebra: Research into its nature, its learning, its teaching*. New York: Springer (open access eBook). https://doi.org/10.1007/978-3-319-32258-2.
Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*, 297–312. https://doi.org/10.2307/30034852.
*Leavy, A., Hourigan, M., McMahon, Á. (2013). Early understanding of equality. Teaching Children Mathematics, 20, 246–252. doi:https://doi.org/10.5951/teacchilmath.20.4.0246.
Leibniz, G. W. (1989). To queen Sophie Charlotte of Prussia, on what is independent of sense and matter (1702). In R. Ariew & D. Garber (Eds.), (Eds., Trans.), *Philosophical essays* (pp. 186–192). Indianapolis: Hackett Publishing Company.
*Linchevski, L., Herscovics, N. (1996). Crossing the cognitive gap between arithmetic and algebra: Operating on the unknown in the context of equations. Educational Studies in Mathematics, 30, 39–65. doi:https://doi.org/10.1007/BF00163752.
*Mann, R. L. (2004). Balancing act: The truth behind the equals sign. Teaching Children Mathematics, 11(2), 65–70. Retrieved from https://www.nctm.org/Publications/teaching-children-mathematics/2004/Vol11/Issue2/Balancing-Act_-The-Truth-behind-the-Equals-Sign/
*Marschall, G., Andrews, P. (2015). Polish teachers' conceptions of and approaches to the teaching of linear equations to grade six students: An exploratory case study. Research in Mathematics Education, 17, 220–238. doi:https://doi.org/10.1080/14794802.2015.1107498.
*Ngu, B. H., Chung, S. F., Yeung, A. S. (2015). Cognitive load in algebra: Element interactivity in solving equations. Educational Psychology, 35, 271–293. doi:https://doi.org/10.1080/01443410.2013.878019.
*Ngu, B. H., Phan, H. P. (2016). Comparing balance and inverse methods on learning conceptual and procedural knowledge in equation solving: A cognitive load perspective. Pedagogies: An International Journal, 11, 63–83. doi:https://doi.org/10.1080/1554480X.2015.1047836.
*Ngu, B. H., Phan, H. P., Yeung, A. S., Chung, S. F. (2018). Managing element interactivity in equation solving. Educational Psychology Review, 30, 255–272. doi:https://doi.org/10.1007/s10648-016-9397-8.
Núñez, R. E., Edwards, L. D., & Matos, J. F. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics, 39*, 45–65. https://doi.org/10.1023/A:1003759711966.
*Orlov, K. (1971). Experimental verification of the use of the mathematical balance in secondary teaching. Educational Studies in Mathematics, 3, 192–205. doi:https://doi.org/10.1007/BF00305448.
*Perry, M., Berch, D., Singleton, J. (1995). Constructing shared understanding: The role of nonverbal input in learning contexts. Journal of Contemporary Legal Issues, 6, 213–235. Retrieved from http://heinonline.org/HOL/Page?handle=hein.journals/contli6&div=14&g_sent=1&casa_token=lB1QPO7tHvMAAAAA:tm46P4IjNPW_1Y7w-ZEQf50oJ8FjP-cJj2aw9Td9ppqiUv-bN74-NS1yWW6Gpsk8w8ZGUe8W&collection=journals
*Raymond, A. M., Leinenbach, M. (2000). Collaborative action research on the learning and teaching of algebra: A story of one mathematics teacher's development. Educational Studies in Mathematics, 41, 283–307. doi:https://doi.org/10.1023/A:1004091631981.
*Rystedt, E., Helenius, O., Kilhamn, C. (2016). Moving in and out of contexts in collaborative reasoning about equations. The Journal of Mathematical Behavior, 44, 50–64. doi:https://doi.org/10.1016/j.jmathb.2016.10.002.
*Smith, L. R. (1985). Mathematics on the balance beam. School Science and Mathematics, 85, 494–498. doi:https://doi.org/10.1111/j.1949-8594.1985.tb09653.x.
Streefland, L. (2003). Learning from history for teaching in the future. *Educational Studies in Mathematics, 54*, 37–62. https://doi.org/10.1023/B:EDUC.0000005236.36499.fc.
*Suh, J., Moyer, P. S. (2007). Developing students' representational fluency using virtual and physical algebra balances. Journal of Computers in Mathematics and Science Teaching, 26(2), 155–173. Retrieved from https://digitalcommons.usu.edu/cgi/viewcontent.cgi?article=1031&context=teal_facpub
*Taylor-Cox, J. (2003). Algebra in the early years. Young Children, 58(1), 14–21. Retrieved from http://oldweb.naeyc.org/journal/btj/200301/Algebra.pdf
Van den Heuvel-Panhuizen, M. (2003). The didactical use of models in realistic mathematics education: An example from a longitudinal trajectory on percentage. *Educational Studies in Mathematics, 54*, 9–35. https://doi.org/10.1023/B:EDUC.0000005212.03219.dc.
*Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. Educational Studies in Mathematics, 49, 341–359. doi:https://doi.org/10.1023/A:1020229023965.
*Warren, E., Cooper, T. J. (2005). Young children's ability to use the balance strategy to solve for unknowns. Mathematics Education Research Journal, 17, 58–72. doi:https://doi.org/10.1007/BF03217409.
*Warren, E., Cooper, T. J. (2009). Developing mathematics understanding and abstraction: The case of equivalence in the elementary years. Mathematics Education Research Journal, 21, 76–95. doi:https://doi.org/10.1007/BF03217546.

*Warren, E., Mollinson, A., Oestrich, K. (2009). Equivalence and equations in early years classrooms. Australian Primary Mathematics Classroom, 14(1), 10–15. Retrieved from https://eric.ed.gov/?id=EJ855964
Wilson, M. (2002). Six views of embodied cognition. *Psychonomic Bulletin & Review*, *9*, 625–636. https://doi.org/10.3758/BF03196322.

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