Numerical Comparison of Constant and Variable Fluid Properties for MHD Flow Over a Nonlinearly Stretching Sheet

M. Asif Farooq, Razia Sharif, Asif Mushtaq

Abstract—**In this paper, we contemplate a comparison of results between constant and variable fluid properties while considering magnetohydrodynamics (MHD) flow and heat transfer for steady, two dimensional and laminar viscous fluid over a nonlinearly stretching sheet. The governing mathematical model for the underlying problem is drafted in its most general setup, i.e. we consider the the representative partial differential equations(PDEs) for viscous compressible fluid. Similarity transformation is exercised to transform governing nonlinear PDEs into a nonlinear ODEs. The set of two coupled ODEs are solved numerically by shooting technique and bvp4c (built-in MATLAB solver). Besides, we also present and execute a new numerical method for solving coupled nonlinear ODEs, the Simplified Finite Difference Method (SFDM). When comparing with bvp4c and shooting technique, the efficiency of SFDM for the above system has also been shown. Various governing parameters and their effect on temperature and velocity profiles are studied in detail. We have shown that there is a significant difference of results when its comparison is drawn between for constant and varying fluid properties. Skin friction coefficient shows increment in its values while rate of heat transfer shows decrement for variable viscosity when compared with constant viscosity.**

Index Terms—**Magnetohydrodynamics (MHD), Sakiadis flow, variable viscosity, similarity transformations, magnetic field, shooting technique, SFDM, stretching sheet, heat transfer.**

I. Introduction

THE examination of flow over a stretching sheet make us possible to explore novel applications in industries, engineering, metallurgy, manufacturing in metal extrusion, hot rolling, glass fiber production and textiles. It was Crane [1] who first presented a flow where velocity of the stretching sheet depends on the distance from slit. Sakiadis [2], [3], [4] in a series of papers deliberated the values of the flow analysis for axisymmetric, continuous flat surface and continuous cylindrical surface. The preceding work of Sakiadis describe flow and analyze results by using two methods: one is numerical and another is integral method. These studies on continuous flat surfaces open-up possiblities of extensions in different directions. Pantokratoras [5] has shown the effect of viscosity while taking moving continuous flate plate. Andersson and Aarsaeth [6] revisited the problem of Sakiadis flow for variable fluid properties. Almost three decades ago Lai and Kulacki [7] carried out problem for variable viscosity on convective heat transfer, while taking vertical surface in a porous medium. Afzal [8] considered non uniform velocity of stretching surface for heat transfer analysis. The temperature on the surface was non-uniform as well. The work of Andersson and Aarsath was reflected in Bachok et.al. [9]. They regarded the viscous fluid investigate variable viscosity on a fixed or moving flat plate. The MHD heat transfer analysis in the case of non-isothermal sheet has been examined by Chiam [10]. Daniel et al. [11] considered nanofluid with slip effects as well as thermal radiation over a permeable sheet. The MHD flow due to accelerated plate of second grade fluid have been discussed in Salah et al. [12]. Exact solution for MHD heat transfer analysis in generalized Oldroyd-B fluid have been obtained in Liu et al. [13]. Analytical and numerical solutions are acquired in the work of Chiam [10]. Mukhopadhyay et al. [14] has taken a heated surface while the flow is MHD in the presence of the varying viscosity. In a series of papers Pop et al. [15], Ali [16], Prasad et al. [17] and Seddeek [18] examined the effect of variable viscosity over a continuous surfaces. MHD viscoelastic flow past a stretching sheet with transverse magnetic field presented in Andersson [19]. Analytical solution is obtained for nonlinear boundary condition. He has shown that both the magnetic field and viscoelasticity has same effect on flow. In another paper by Andersson et al. [20] power-law fluid has been discussed over a stretching sheet. The effect of magnetic field have been investigated numerically. They have shown that magnetic field make the boundary layer thinner which in turn increase the wall friction. The problem of conducting viscous fluid in a transverse magnetic field over a plane elastic surface is discussed in Pavlov [21] (see also Seddeek[22]). In another work of Chiam [23] they have taken variable hydromagnetic flow with power-law velocity over a stretching surface. Power-law stretching sheet with suction or injection is discussed in Ali [29].

Current work is an extension of the work by Andersson and Aarsaeth [6]. This paper focuses on MHD flow and transfer of heat due to a nonlinear stretching surface with variable viscosity. Comparison has been made between constant and variable viscosity. Three cases i. e. constant viscosity, viscosity dependence on inverse linear temperature and viscosity dependence on exponential temperature have been studied. The current work deals with numerical solutions for various values of governing parameters. The paper is organized as follows. In section 2 we present mathematical model for flow and heat transfer analysis. The special cases for the constant and variable viscosity have been discussed in section 3. The computational procedure is given in section 4. In section 5 we present the graphs and tables followed by discussion of

Manuscript received on April 11, 2019; revised September 25, 2019

M Asif Farooq and Razia Sharif are with the Department of Mathematics, School of Natural Sciences (SNS), National University of Sciences and Technology (NUST), 44000, Islamabad, Pakistan. e-mails: asif.farooq@sns.nust.edu.pk, raziasharif433@yahoo.com

Asif Mushtaq is with the Fakultet for lærerutdanning og kunst- og kulturfag (FLU), Bodø Campus, Nord University, 8026, Norway. email: asif.mushtaq@nord.no

the results.

II. Mathematical Formulation

Here we investigate a steady 2-D, and laminar MHD flow of a Newtonion fluid over a nonlinear stretching sheet. B_0 is the strength of a magnetic field which is applied in normal direction of the sheet. The sheet moves with a non-uniform velocity $U(x)$ in positive x direction. Velocity of the sheet is considered as $U_w(x) = ax^m$, here a is constant and m is an exponent. The temperature of ambient fluid is taken as constant and is denoted by T_o whereas temperature of sheet is of the form $T_w(x) = T_o + cx^n$, where c and n are positive constants. Both the viscous dissipation and induced magnetic field are negligible. The governing equations using above assumptions are given as Andersson and Aarsaeth [6]:

$$\partial_x(\rho u) + \partial_y(\rho v) = 0, \tag{1a}$$

$$\rho(uu_x + vu_y) = \partial_y(\mu u_y) - \sigma B_0^2 u, \tag{1b}$$

$$\rho C_p(uT_x + vT_y) = \partial_y(kT_y), \tag{1c}$$

and the corresponding boundary conditions read as

$$\begin{aligned} &u(x,0) = U_w(x), \;\; v(x,0) = 0, \;\; T(x,0) = T_w(x) \\ &u \to 0, \quad T \to T_0, \quad as\ y \to \infty. \end{aligned} \tag{2}$$

The stretching velocity $U_w(x)$ and temperature $T_w(x)$ are defined as

$$U_w(x) = ax^m$$

$$T_w(x) = T_0 + cx^n$$

where u is the x-component and v is the y-component of velocity. The fluid density is represented by ρ, B_0 shows the strength of the applied magnetic field, dynamic viscosity of the fluid is μ, specific heat is denoted by C_p, temperature of fluid is T and k denotes thermal conductivity. U_w represents the sheet's velocity and wall temperature is denoted by T_w. Introducing the following similarity variables Ali [26].

$$\begin{aligned} \eta &= \sqrt{\frac{(1+m)U(x)}{2\nu_0 x}} \int \frac{\rho}{\rho_0} dy, \\ \psi &= \rho_0 \sqrt{\frac{2\nu_0 x U(x)}{1+m}} f(\eta), \\ \theta(\eta) &= \frac{T - T_0}{T_w - T_0}, \end{aligned} \tag{3}$$

stream function is denoted by ψ and its relation with u and v is given on the same as Andersson and Aarsaeth [6]:

$$\rho u = \frac{\partial \psi}{\partial y}, \qquad \rho v = -\frac{\partial \psi}{\partial x}. \tag{4}$$

Using the above Eq. (4) the x and y components of velocity can be written as

$$\begin{aligned} &u = ax^m f'(\eta), \\ &v = -\rho_0 \sqrt{\frac{2\nu_0 a}{1+m}} x^{\frac{m-1}{2}} \left(\frac{m+1}{2} f(\eta) + \eta \frac{m-1}{2} f'(\eta)\right). \end{aligned} \tag{5}$$

Plug in Eqs. (3), (4) and (5) into (1a), (1b) and (1c) we get the following nonlinear ordinary differential equations (ODEs),

$$\left(\frac{\rho\mu}{\rho_0\mu_0} f''\right)' - Mf' - \beta(f')^2 + ff'' = 0, \tag{6a}$$

$$\left(\frac{\rho k}{\rho_0 k_0}\theta'\right)' + \frac{C_p}{C_{p0}} Pr_0\left(\theta' f - \frac{2n}{1+m}\theta f'\right) = 0, \tag{6b}$$

where Pr_0, β, M shows Prandtl number, velocity ratio parameter and magnetic parameter respectively. These parameters are defined as

$$Pr_0 = \frac{\mu_0 C_{p0}}{k_0}, \; \beta = \frac{2m}{1+m}, \; M = \frac{2\sigma\beta_0^2}{\rho a(1+m)x^{m-1}}. \tag{7}$$

After transformation the boundary conditions (2) take the form

$$\begin{aligned} &f(0) = 0, \quad f'(0) = 1, \quad \theta(0) = 1, \\ &f'(\eta) = 0, \quad \theta(\eta) = 0 \quad as \quad \eta \to \infty \end{aligned} \tag{8}$$

where f' denotes dimensionless velocity and θ denotes dimensionless temperature.

The skin friction coeffcient C_f and Nusselt number Nu_x are defined as follows Mustafa [23]:

$$C_f = \frac{\tau_w}{\rho U_w^2}, \qquad Nu_x = \frac{xq_w}{T_w - T_0}, \tag{9}$$

where τ_w is shear stress and q_w regarded as heat flux, and are defined as :

$$\begin{aligned} \tau_w &= \mu_w x^{\frac{3m-1}{2}} \sqrt{\frac{(1+m)a^3}{2\nu_0}} f''(0), \\ q_w &= \mu_w C_p \Delta T Pr_w^{-1} \sqrt{\frac{a(1+m)}{2\nu_0}} [-\theta'(0)], \end{aligned} \tag{10}$$

using equation (9) and (10) we get

$$\begin{aligned} C_f Re^{1/2} &= \sqrt{\frac{1+m}{2}} f''(0), \\ Nu_x Re^{-1/2} &= k_w \sqrt{\frac{1+m}{2}} [-\theta'(0)], \end{aligned} \tag{11}$$

where Re denotes local Reynolds number.
It should be noted that all the fluid properties considered here are constant except the viscosity which is temperature dependent. Following cases are discussed here as mentioned in Andersson and Aarsaeth[6] .

III. Special Cases

A. Case A: Constant Fluid Properties

For this case we assume all the fluid properties as constant. The dimensionless variables η and stream function ψ take the following form:

$$\eta = \sqrt{\frac{a}{\nu_0}} y, \quad \psi = \rho_0 \sqrt{a\nu_0} x f(\eta). \tag{12}$$

Under above similarity variables, Eqs. (6a) and (6b) take the form:

$$f''' + ff'' - \beta f'^2 - Mf' = 0, \tag{13a}$$

$$\theta'' + Pr_0\left(f\theta' - \frac{2n}{1+m} f'\theta\right) = 0, \tag{13b}$$

the boundary conditions in Eq. (8) remains the same.

B. Case B: Variable Viscosity (Inverse Relation with Temperature)

For this case, we assume only viscosity as a variable that depends linearly on temperature while treating the remaining fluid properties constant which is already explored in these references Andersson and Aarseth [6], Bachok et al [9], Elbashbeshy and Bazid [24].

For this case the momentum boundary layer Eq. (6a) becomes

$$(f''\frac{\mu}{\mu_0})' + ff'' - \beta f'^2 - Mf' = 0. \tag{14}$$

The inverse linear relation between viscosity and temperature is proposed by Lai and Kulacki [7], Pop et al. [12] and Ling and Dybbs [25]. The following is the relation

$$\mu(T) = \frac{\mu_{ref}}{[1 + \gamma(T - T_{ref})]}, \tag{15}$$

where γ is the thermal property of the fluid and its value depends on the reference temperature T_{ref}. If $T_{ref} \approx T_0$, the above formula given in Eq. (15) becomes

$$\mu = \frac{\mu_0}{1 - \frac{T - T_0}{\theta_{ref}(T_w - T_0)}} = \frac{\mu_0}{1 - \frac{\theta(\eta)}{\theta_{ref}}}, \tag{16}$$

here $\theta_{ref} \equiv \frac{-1}{(T_w - T_0)\gamma}$ and $\Delta T = (T_w - T_0)$.
By inserting Eq. (16) into Eq. (14), the resultant equation takes the following form

$$f''' + \frac{\theta'}{\theta_{ref} - \theta}f'' + (\frac{\theta_{ref} - \theta}{\theta_{ref}})(ff'' - Mf' - \beta f'^2) = 0. \tag{17}$$

C. Case C: Variable Viscosity (Exponential Relation with Temperature)

Similar to Case B, viscosity is again taken as variable and its exponential relation with temperature takes the following form Andersson and Aarsaeth [6]:

$$\ln(\frac{\mu}{\mu_{ref}}) = -2.10 - 4.45\frac{T_{ref}}{T} + 6.55(\frac{T_{ref}}{T})^2. \tag{18}$$

Substituting the above formula Eq. (18) in Eq. (14) we get the following equation:

$$\begin{aligned} f''' = {} & -f''\theta'\Delta T(4.45\frac{T_{ref}}{T^2} - 13.1\frac{T_{ref}^2}{T^3}) \\ & + \frac{\mu_0}{\mu}(\beta f'^2 - ff'' + Mf'). \end{aligned} \tag{19}$$

IV. Numerical Procedure

For each Case A, B, and C, we solve numerically the nonlinear ordinary differential equations (ODEs) with the boundary conditions given in Eq. (8) in the following two sections. We apply SFDM for Case A only and compare its outcomes with the shooting technique and bvp4c. We give an explanation about the SFDM in the next section followed by a short depiction of the shooting method and bvp4c.

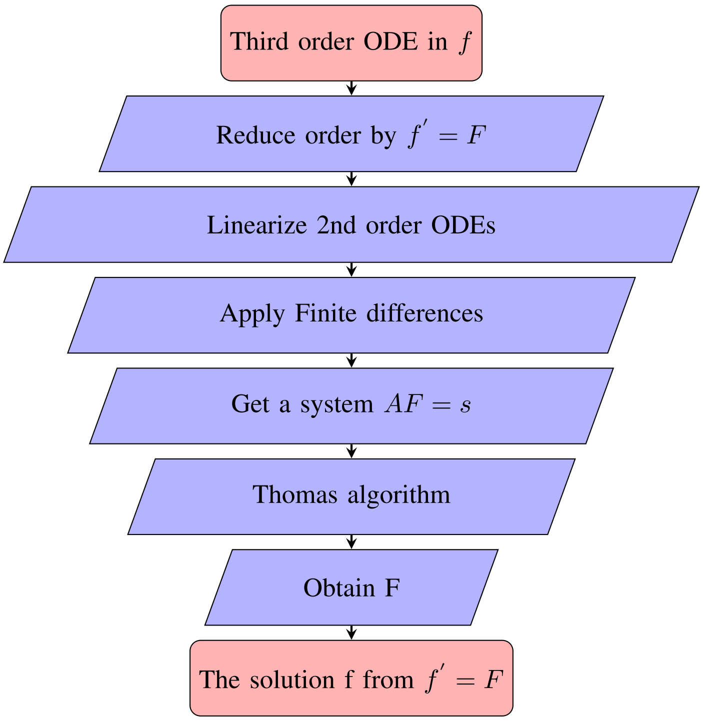

Fig. 1: Flow chart to explain steps in SFDM.

A. The Simplified Finite Difference Method (SFDM)

In this segment, we demonstrate the Simplified Finite Difference (SFDM) newly developed numerical approach. We define $f' = F$ in Eq. (13a) then it is recasted as

$$\frac{d^2F}{d\eta^2} = -f\frac{dF}{d\eta} + \beta F^2 + MF, \tag{20}$$

for the right side of the above equation, we specify the variable ϕ_1 as

$$\phi_1(\eta, F, F') = -f\frac{dF}{d\eta} + \beta F^2 + MF, \tag{21}$$

let us now estimate $\frac{dF}{d\eta}$ in the above equation (21) by approximating the derivative with the forward difference formula

$$\phi_1(\eta, F, F') = -f_i(\frac{F_{i+1} - F_i}{h}) + +\beta F^2 + MF \tag{22}$$

The coefficients of second order ODE read as

$$A_n = -\frac{\partial\phi_1}{\partial F'} = -(-f) = f = f_i \tag{23}$$

$$B_n = -\frac{\partial\phi_1}{\partial F} = -2\beta F - M \tag{24}$$

$$B_n = -2\beta F_i - M \tag{25}$$

$$D_n = f(\eta, F, F') + B_nF_i + A_n\frac{F_{i+1} - F_i}{h} \tag{26}$$

After some manipulation Eq. (20) becomes

$$a_iF_{i-1} + b_iF_i + c_iF_{i+1} = r_i, \qquad i = 1, 2, 3...., N \tag{27}$$

where

$$a_i = 2 - hA_n, b_i = 2h^2B_n - 4, c_i = 2 + hA_n, r_i = 2h^2D_n \tag{28}$$

B. Thomas Algorithm

The obtained tridiagonal algebraic system (27) is solved by the commonly known Tridiagonal Matrix Algorithm (TDMA or Thomas Algorithm). In matrix-vector form it is written as

$$AF = s \tag{29}$$

where

$$A = \begin{bmatrix} b_1 & c_1 & & & & \\ a_2 & b_2 & c_2 & & & \\ & & .... & & & \\ & & & a_{N-2} & b_{N-2} & c_{N-2} \\ & & & & a_{N-1} & b_{N-1} \end{bmatrix} \tag{30}$$

$$F = \begin{bmatrix} F_1 \\ F_2 \\ . \\ . \\ F_{N-1} \end{bmatrix} \qquad s = \begin{bmatrix} s_1 \\ s_2 \\ . \\ . \\ s_{N-1} \end{bmatrix} \tag{31}$$

The matrix A is tridiagonal matrix and is written in LU-Factorization by

$$A = LU \tag{32}$$

where

$$L = \begin{bmatrix} \beta_1 & & & & \\ a_2 & \beta_2 & & & \\ & & .... & & \\ & & a_{N-2} & \beta_{N-2} & \\ & & & a_{N-1} & \beta_{N-1} \end{bmatrix} \tag{33}$$

and

$$U = \begin{bmatrix} 1 & \gamma_1 & & & \\ & 1 & \gamma_2 & & \\ & & .... & & \\ & & & 1 & \gamma_{N-2} \\ & & & & 1 \end{bmatrix} \tag{34}$$

where L and U are the lower and upper triangular matrices, respectively. Here the unknowns $(\beta_i, \gamma_i), \quad i = 1, 2, ..., N-1$ are to be related as

$$\beta_1 = -1, \qquad \gamma_1 = 0 \tag{35}$$

$$\beta_i = b_i - a_i\gamma_{i-1}, \qquad i = 2, 3, ..., N-1 \tag{36}$$

$$\beta_i\gamma_i = c_i, \qquad i = 2, 3, ..., N-2 \tag{37}$$

After defining these relations Eq. (29) becomes

$$LUF = s, \qquad UF = z, \qquad \text{and} \qquad Lz = s \tag{38}$$

we have

$$\begin{bmatrix} \beta_1 & & & \\ a_2 & \beta_2 & & \\ & .... & & \\ & & a_{N-2} & \beta_{N-2} \\ & & & a_{N-1} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ . \\ . \\ . \\ z_{N-2} \\ z_{N-1} \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ . \\ . \\ . \\ s_{N-2} \\ s_{N-1} \end{bmatrix} \tag{39}$$

The unknown elements of z can be found by

$$z_1 = s_1/\beta_1, z_i = \frac{s_i - a_i z_{i-1}}{\beta_i}, i = 2, 3, ..., N-1 \tag{40}$$

and

$$\begin{bmatrix} 1 & \gamma_1 & & & \\ & 1 & \gamma_2 & & \\ & & .... & & \\ & & & 1 & \gamma_{N-2} \\ & & & & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ . \\ . \\ . \\ F_{N-2} \\ F_{N-1} \end{bmatrix} = \begin{bmatrix} z_1 \\ z_2 \\ . \\ . \\ . \\ z_{N-2} \\ z_{N-1} \end{bmatrix} \tag{41}$$

We then get

$$F_{i-1} = z_{i-1}, F_i = z_i - \gamma_i F_{i+1}, i = N-2, N-3, ..., 3, 2, 1 \tag{42}$$

which is a solution of Eq. (20). We can easily find f for Eq. (13a) from $f^{'} = F$ which in discretization form is written as

$$\frac{f_{i+1} - f_i}{h} = F_i \tag{43}$$

gives a required solution of Eqs. (13a) with BCs (8).

A similar procedure can also be opted for solutions θ. For the sake of brevity, we only present coefficients for this ODE (13b) and leave the details which follows on the same line as presented above. For example we have energy equation

$$\frac{d^2\theta}{d\eta^2} = Pr_o\left(\frac{2n}{1+m}\theta F - f\frac{d\theta}{d\eta}\right)$$

$$\phi_2(\eta, \theta, \theta^{'}) = Pr_0\left(\frac{2n}{1+m}\theta F - f\frac{d\theta}{d\eta}\right)$$

$$A_{nn} = \frac{\partial \phi_2}{\partial \theta'} = Pr_0 f \tag{44}$$

$$A_{nn} = -\frac{\partial \phi_2}{\partial \theta'} = Pr_0 f_i \tag{45}$$

$$B_{nn} = -\frac{\partial \phi_2}{\partial \theta} = -\frac{2n}{1+m}Pr_0 F \tag{46}$$

$$B_{nn} = -\frac{\partial \phi_2}{\partial \theta} = -\frac{2n}{1+m}Pr_0 F_i \tag{47}$$

Once these coefficients have been obtained, Thomas algorithm can be used again to achieve solution for θ. To save space we skip all the details.

C. Shooting Technique and bvp4c

To solve the Eqs. (6a) and (6b), bvp4c and shooting techniques are also used in Cases A, B and C. The fundamental objective behind the shooting method is to convert the BVP (boundary value problem) into an IVP (initial value problem). A fifth order Runge-Kutta method and root finding algorithm Newton-Raphson method are used to obtain solution of the transformed problem. We verify the results obtained from shooting technique with *bvp4c* [30], which is a built-in solver in MATLAB. For both numerical techniques, we define the variables

$$y_1 = f \tag{48a}$$

$$y_2 = f' \tag{48b}$$

$$y_3 = f'' \tag{48c}$$

$$y_4 = \theta \tag{48d}$$

$$y_5 = \theta' \tag{48e}$$

(a) Case A: The system of first order momentum and energy equations for this case becomes

$$y_1' = y_2, \qquad y_2' = y_3$$

$$y_3' = f''' = -y_1 y_3 + \beta y_2^2 + M y_2 \tag{49}$$

$$y_4' = y_5$$

$$y_5' = \theta'' = Pr_0 \left(\frac{2n}{m+1} y_2 y_4 - y_1 y_5 \right). \tag{50}$$

(b) Case B: For this Case the y_3' takes the form,

$$y_3' = \frac{y_3 y_5}{0.25 + y_4} + \frac{0.25 + y_4}{0.25} (\beta y_2^2 + M y_2 - y_1 y_3), \tag{51}$$

(c) Case C: For this Case the y_3' takes the form ,

$$y_3' = -y_3 y_5 \Delta T (4.45 \frac{T_{ref}}{T^2} - 13.1 \frac{T_{ref}^2}{T_3}) + \frac{\mu_0}{\mu} (\beta y_2^2 - y_1 y_3 + M y_2), \tag{52}$$

$$\frac{\mu}{\mu_0} = \frac{\mu_{ref}}{\mu_0} exp(-2.10 - 4.45 (\frac{T_{ref}}{T}) + 6.65 (\frac{T_{ref}}{T})^2). \tag{53}$$

We use these values in our calculations i.e. $\mu_{ref} = 0.001792 kg/ms$, $\mu_0 = 0.001520 kg/ms$, $T_{ref} = 273K$ and $T_0 = 278K$ Andersson and Aarseth [6]. The energy equations for Cases B and C unaltered as Eq. (50).

V. Results and Discussion

Numerical results for profiles of velocity and temperature are discussed in this part. Results are displayed in tabular and graphical form. Numerical solutions for skin friction $-f''(0)$ and temperature gradient $-\theta'(0)$ for different physical parameters which includes velocity exponent m, magnetic parameter M, temperature index parameter n, Prandtl number Pr and stretching parameter β are presented in different Tables. In Tables II and III, Nusselt number is calculated and compared with previously obtained results by Mustafa [26] and Ali [29]. Table I and V contain results for different values of Prandtl number Pr_0. Table I reveals the calculated SFDM outcomes. The excellent precision of this and other numerical methods can be observed. SFDM efficiency was measured using CPU time. Although SFDM was less effective than bvp4c, we must bear in mind that bvp4c is an integrated solver, but SFDM is not. SFDM yields relatively excellent performance. From Tables I-VI, one can observe that skin friction enhances whereas there is reduction in wall temperature as we raise magnetic parameter. The effect of Prandtl number and temperature index parameter is to enhance wall temperature while skin friction changes slightly. Wall temperature reduces while skin friction enhances with increase in stretching parameter. In Table VI, numerical results are computed for skin friction and Nusselt number for all cases by increasing the Prandtl number. Coefficient of skin friction increases for Case B while it changes slightly for both Case A and Case C but wall temperature enhances for all cases. The effect of viscosity for all the three cases have been studied. Temperature of ambient fluid is $T_0 = 278K$ while temperature of surface is taken as $T_w = 358K$. In Figs 2 and 3 profiles for velocity and temperature are presented for all Cases A, B and C. In comparison with Case A and C velocity profile for Case B have been reduced adjacent to moving surface as shown in Fig 2. The viscosity of fluid adjacent to the surface reduces because of heat transfer. Comparing with the Case B temperature profile for both Cases A and C decreases close to moving surface as shown in Fig 3. Impact of magnetic parameter M, on profiles of temperature and velocity has been shown in Figs (4-9). Temperature profile increases as we increase M and there is decreasing effect on momentum boundary layer for all three Cases A, B and C.

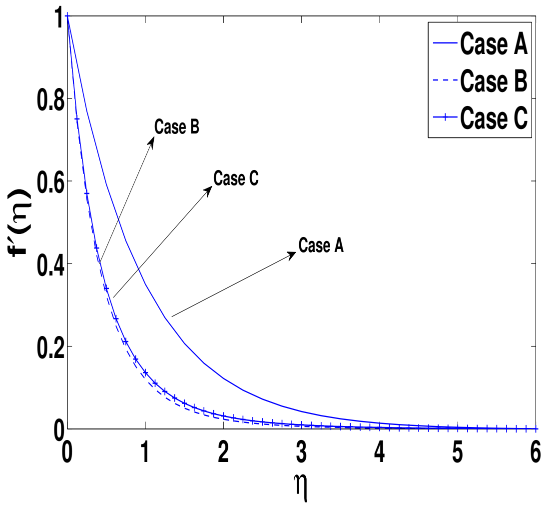

Fig. 2: Variation in dimensionless velocity profiles $f'(\eta)$ for each Case A, B and C with n=1 and M=0.1.

From Figs (10-15), the influence of β (stretching parameter) on velocity profile have been depicted. It can be seen that increment in β parameter causes momentum boundary layer to reduce, while there is an increment in thermal boundary layer for all cases. Physically, $\beta > 0$ shows that the surface is accelerating. The effect of temperature index parameter have been shown in Figs (16-21). For both Cases B and C, the momentum boundary layer thickens while for Case A it is devoid of any effect. The thermal boundary layer shows a decreasing behaviour for all cases. Figs (22-27) shows the effect of Prandtl number on momentum and thermal boundary layer. For Case B and Case C, rise in Prandtl number causes increment in the momentum boundary layer whereas thermal boundary layer reduces for all cases by increasing Prandtl number but in Case A the velocity profile is not affected by Prandtl number.

TABLE I: Results for $-f''(0)$ and $-\theta'(0)$ for various values of parameters (Case A).

					bvp4c		shooting method		SFDM		CPU Time (sec)	
Pr_0	M	β	m	n	$-f''(0)$	$-\theta'(0)$	$-f''(0)$	$-\theta'(0)$	$-f''(0)$	$-\theta'(0)$	bvp4c	SFDM
0.7	0.5	1	1	1	1.2247449	0.73595707	1.2247449	0.73683412	1.223096	0.7344862	2.023469	2.599066
1	-	-	-	-	1.2247449	0.94089967	1.2247449	0.94099339	1.2239	0.9429653	1.355994	2.596358
3	-	-	-	-	1.2247449	1.8655031	1.2247449	1.865517	1.228887	1.857851	0.902247	2.768858
7	-	-	-	-	1.2247449	3.0156599	1.2247449	3.0156921	1.228887	3.001792	0.910613	2.867179
10	-	-	-	-	1.2247449	3.6645662	1.2247449	3.6646523	1.228887	3.64622	0.927072	2.894318
0.7	0.1	-	-	-	1.0488089	0.78093708	1.0488089	0.78096049	1.049641	0.788154	0.670467	4.505192
-	0.2	-	-	-	1.0954451	0.76886566	1.0954451	0.76886391	1.095056	0.7683521	0.692361	4.557601
-	0.3	-	-	-	1.1401754	0.75737841	1.1401754	0.75737874	1.13904	0.7526887	0.681108	3.855351
-	0.4	-	-	-	1.183216	0.74642739	1.183216	0.7464299	1.181602	0.7418342	0.690873	4.059979
10	0.5	0	0	-	0.9294730	4.8059057	0.92947343	4.8060571	0.9279735	4.75504	0.803780	2.652847
-	-	1	1	-	1.2247449	3.6645669	1.2247449	3.6646523	1.228887	3.64622	0.743336	2.989321
-	-	1.33	2	-	1.3090637	3.2282285	1.3090635	3.2282143	1.308382	3.207939	0.748961	2.686516
-	-	1.6	4	-	1.3745053	2.8494207	1.3745033	2.8493841	1.373946	2.835114	0.763796	2.676637
-	-	1.75	7	-	1.4096676	2.6226409	1.4096386	2.6223442	1.409147	2.611411	0.747469	2.646439

TABLE II: Comparison of $C_f {Re_x}^{1/2}$ and ${Re_x}^{-1/2} Nu_x$ for $Pr_0 = 1$ and M=0.

m	Mustafa [26]		Present results		Absolute Error of	
	$Re_x^{1/2} C_f$	$Re_x^{-1/2} Nu_x$	$Re_x^{1/2} C_f$	$Re_x^{-1/2} Nu_x$	$Re_x^{1/2} C_f$	$Re_x^{-1/2} Nu_x$
0	-0.44375	0.44375	-0.443749	0.443749	1×10^{-6}	1×10^{-6}
1	-1.00000	1.00000	-1.00000	1.00000	0	0
2	-1.34845	1.34845	-1.34727	1.34866	0.12×10^{-2}	2.1×10^{-4}

TABLE III: Comparison of $Re_x^{-1/2} Nu_x$ when n=0, m=0, M=0 but for different values of Prandtl number.

Pr_0	Jacobi [27]	Tsou et al. [28]	Ali [29]	Present results	Absolute Error		
0.7	0.3492	0.3492	0.3476	0.3492	0	0	0.1×10^{-2}
1	0.4438	0.44378	0.4416	0.4437	1×10^{-4}	8×10^{-5}	2.1×10^{-3}
10	1.6790	1.6804	1.6713	1.6803	1.3×10^{-3}	1×10^{-4}	9×10^{-3}

TABLE IV: Results for skin friction $-f''(0)$ and temperature gradient $-\theta'(0)$ with different values of M (Case B).

					bvp4c		shooting method		CPU Time (bvp4c)
M	Pr_0	β	n	m	$-f''(0)$	$-\theta'(0)$	$-f''(0)$	$-\theta'(0)$	
0.1	1	1	1	1	2.4530235	0.72122229	2.4530188	0.72122789	0.749931 sec
0.2	-	-	-	-	2.5560358	0.69757654	2.5560372	0.69761235	0.702601 sec
0.3	-	-	-	-	2.6543977	0.67648866	2.6545432	0.67718638	0.716121 sec
0.4	-	-	-	-	2.7488177	0.6574893	2.7492952	0.65959234	0.757713 sec
0.5	-	-	-	-	2.8397797	0.63998507	2.8406015	0.64360739	0.718994 sec

TABLE V: Results for skin friction $-f''(0)$ and temperature gradient $-\theta'(0)$ (Case C).

					bvp4c		shooting method		CPU Time(bvp4c)
Pr_0	M	β	m	n	$-f''(0)$	$-\theta'(0)$	$-f''(0)$	$-\theta'(0)$	
0.7	0.5	1	1	1	2.7086855	0.48648367	2.71058	0.493869	0.707701 sec
1	-	-	-	-	2.7457538	0.66567042	2.74649	0.668392	0.693101 sec
3	-	-	-	-	2.9504369	1.5479812	2.95042	1.54797	0.703286 sec
7	-	-	-	-	3.2162104	2.6638754	3.21623	2.66385	0.705536 sec
10	-	-	-	-	3.3541575	3.294594	3.35427	3.29451	0.689818 sec
0.7	0	-	-	-	2.2358724	0.57909873	2.23586	0.579088	0.713269 sec
-	0.2	-	-	-	2.4376711	0.53485144	2.43782	0.535475	0.683791 sec
-	0.4	-	-	-	2.6217411	0.500965	2.62341	0.507445	0.729590 sec
-	0.5	-	-	-	2.7086796	0.48648475	2.71058	0.493869	0.908213 sec
-	1	-	-	-	3.1053418	0.42979585	3.1211	0.490544	0.729468 sec

TABLE VI: Results for skin friction $-f''(0)$ and temperature gradient $-\theta'(0)$ with different values of Pr_0 for n=1 and M=0.1 (Cases A, B and C).

			bvp4c		shooting method	
Cases	M	Pr_0	$-f^{''}(0)$	$-\theta^{'}(0)$	$-f^{''}(0)$	$-\theta^{'}(0)$
	0.1	0.7				
CaseA			1.0488089	0.78093708	1.0488089	0.78093637
CaseB			2.4220867	0.53201823	2.4220857	0.532034
CaseC			2.3394334	0.55518512	2.33943	0..555191
	0.1	1				
CaseA			1.0488089	0.98710811	1.0488089	0.98710798
CaseB			2.4530225	0.72122065	2.4530162	0.72121671
CaseC			2.3782471	0.7485637	2.37824	0.748559
	0.1	10				
CaseA			1.0488088	3.7084043	1.0488088	3.7085551
CaseB			2.9649588	3.3619367	2.9649931	3.3619032
CaseC			2.9378748	3.3920765	2.93794	3.39204

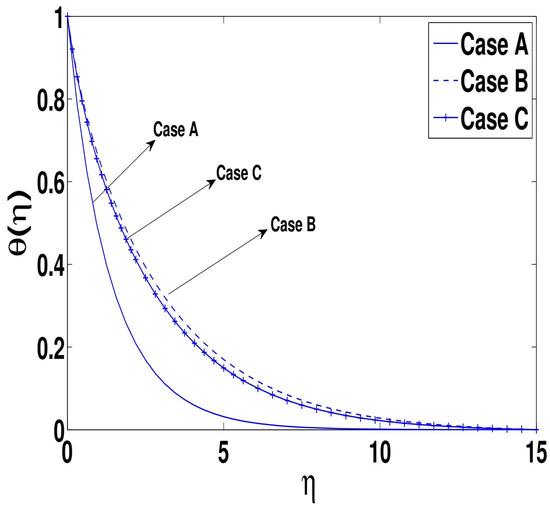

Fig. 3: Variation in dimensionless temperature profiles $\theta(\eta)$ for each Case A, B and C with n=1 and M=0.1.

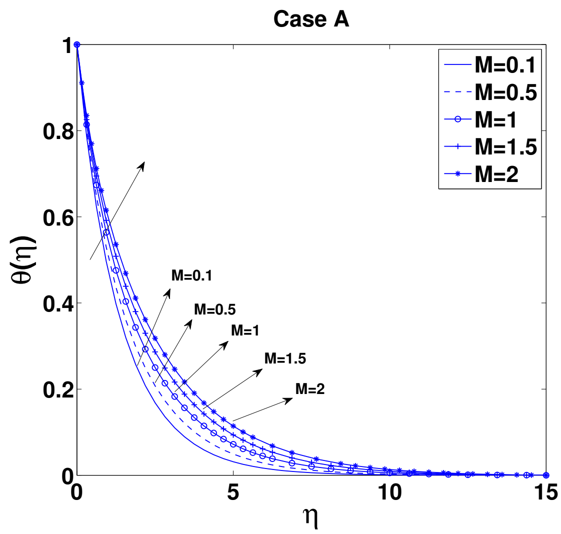

Fig. 5: Variation in M and its impact on the dimensionless temperature profiles $\theta(\eta)$ at n=1, Pr=0.7 and $\beta = 1$.

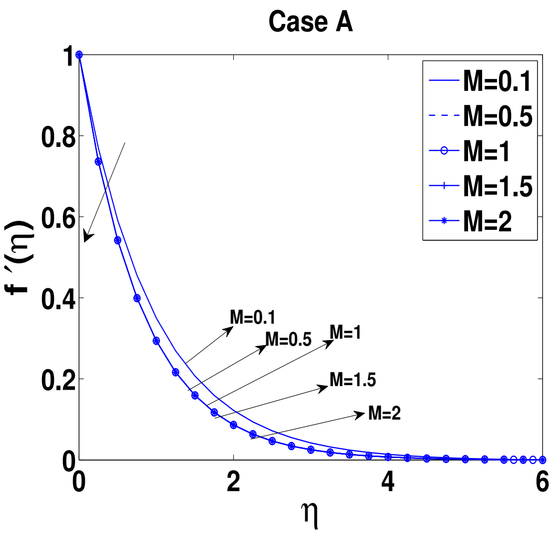

Fig. 4: Variation in M and its impact on the dimensionless velocity profiles $f'(\eta)$ at $\beta = 1$, Pr=0.7 and n=1.

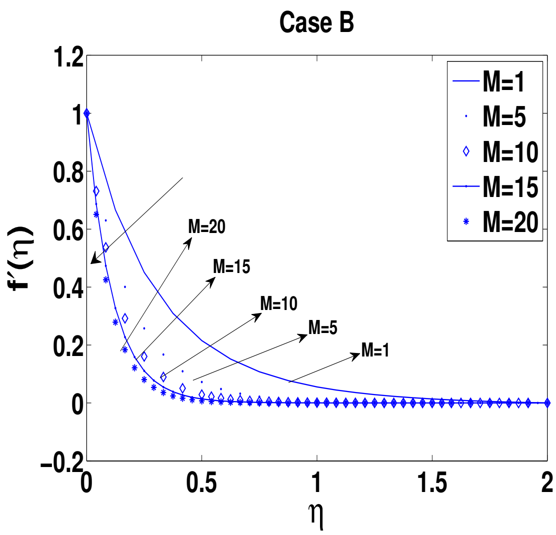

Fig. 6: Variation in M and its impact on the dimensionless velocity profiles $f'(\eta)$ at n=1 and β=1.

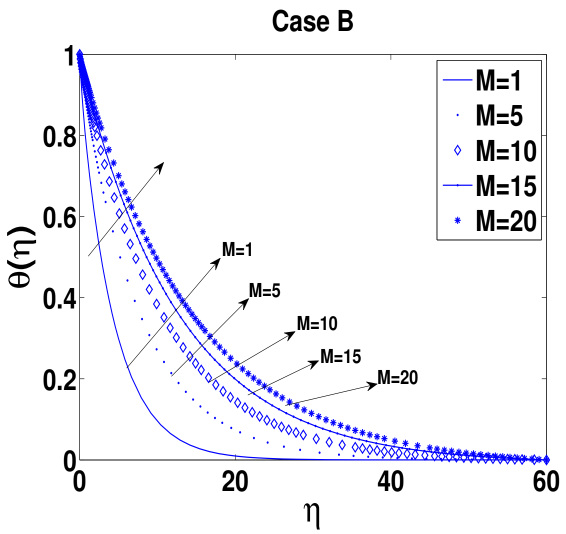

Fig. 7: Variation in M and its impact on the dimensionless temperature profiles $\theta(\eta)$ at n=1 and β=1.

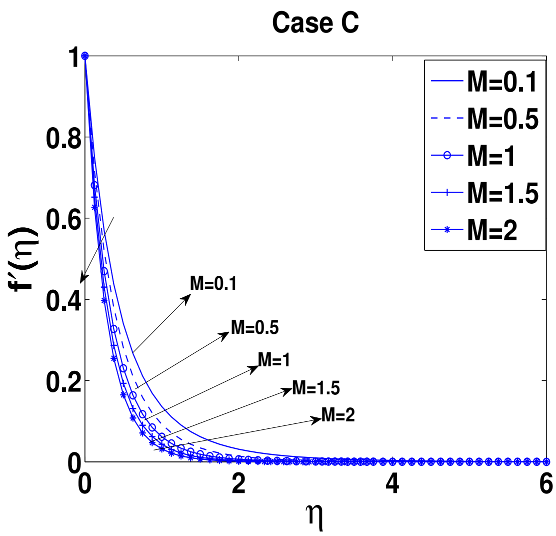

Fig. 8: Variation in M and its impact on the dimensionless velocity profiles $f'(\eta)$ at n=1 and β=1.

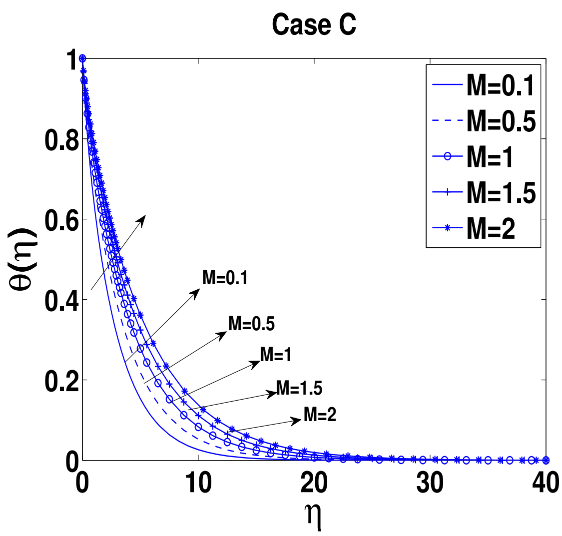

Fig. 9: Variation in M and its impact on the dimensionless temperature profiles $\theta(\eta)$ at n=1 and β=1.

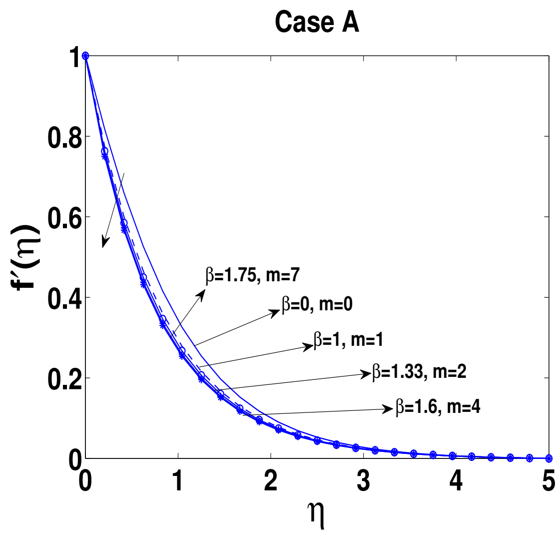

Fig. 10: Variation in β and its impact on the dimensionless velocity profiles $f'(\eta)$ at Pr=0.7.

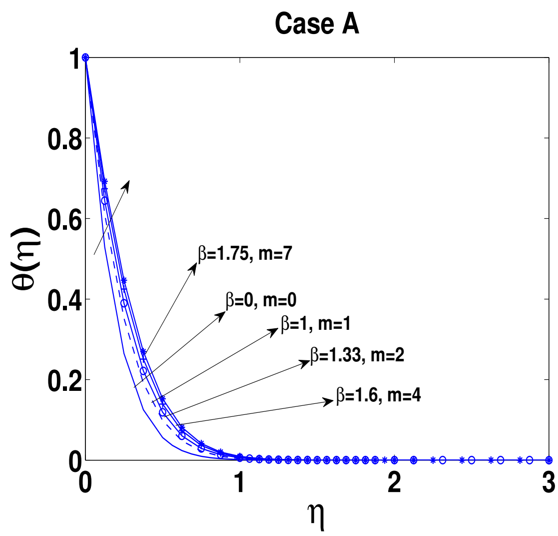

Fig. 11: Variation in β and its impact on the dimensionless temperature profiles $\theta(\eta)$ at Pr=10.

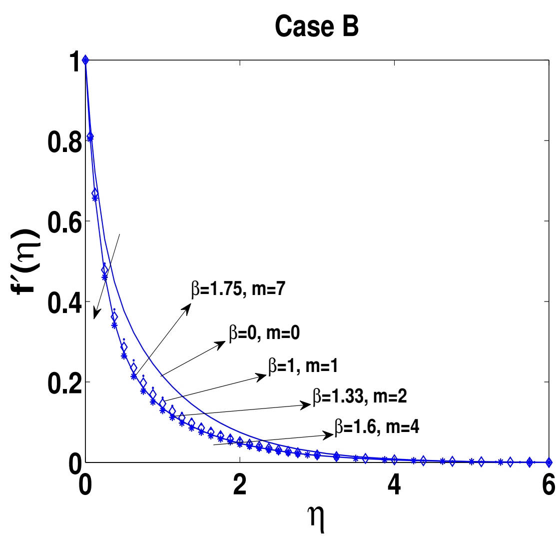

Fig. 12: Variation in β and its impact on the dimensionless velocity profiles $f'(\eta)$ at Pr=10, n=1 and M=0.5.

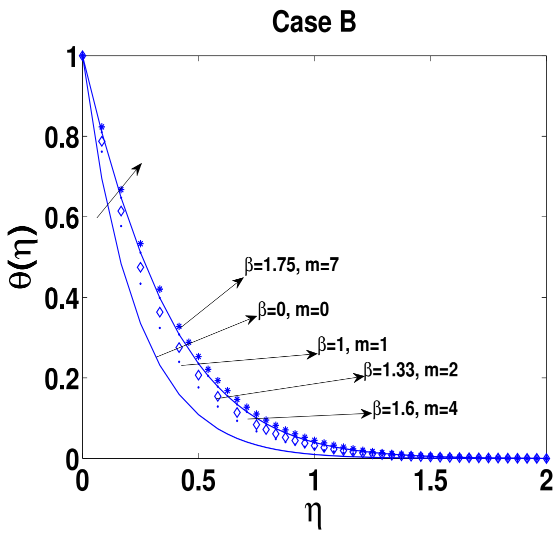

Fig. 13: Variation in β and its impact on the dimensionless temperature profiles $\theta(\eta)$ at Pr=10, n=1 and M=0.5.

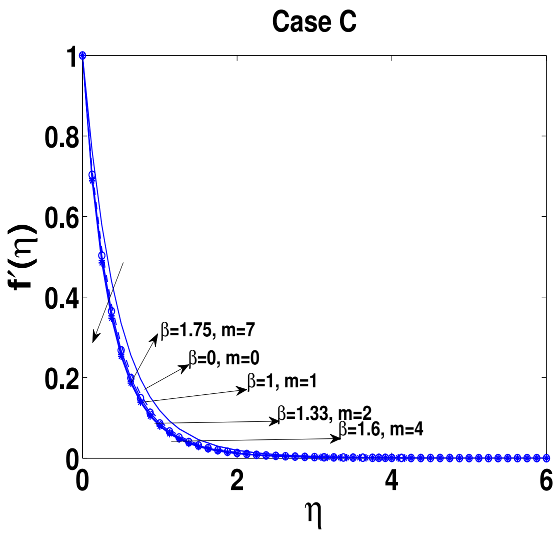

Fig. 14: Variation in β and its impact on the dimensionless velocity profiles $f'(\eta)$ at Pr=0.7, n=1 and M=0.5.

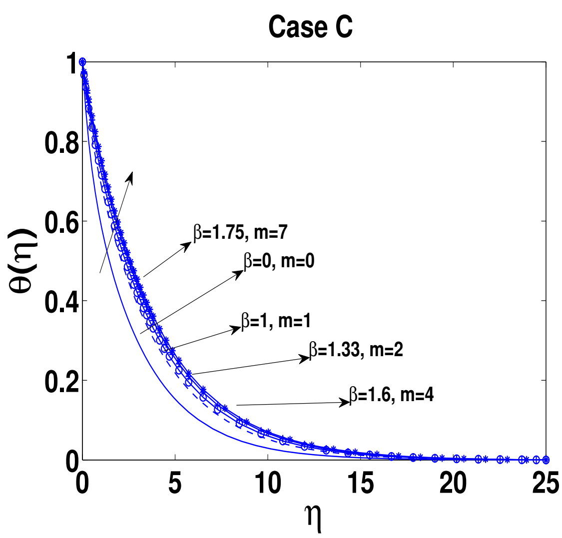

Fig. 15: Variation in β and its impact on the dimensionless temperature profiles $\theta(\eta)$ at Pr=0.7, n=1 and M=0.5.

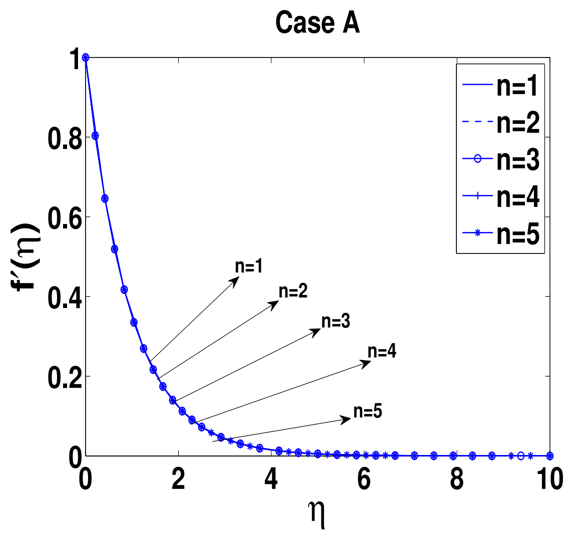

Fig. 16: Variation in n and its impact on the dimensionless velocity profiles $f'(\eta)$ at m=1 and Pr=1.

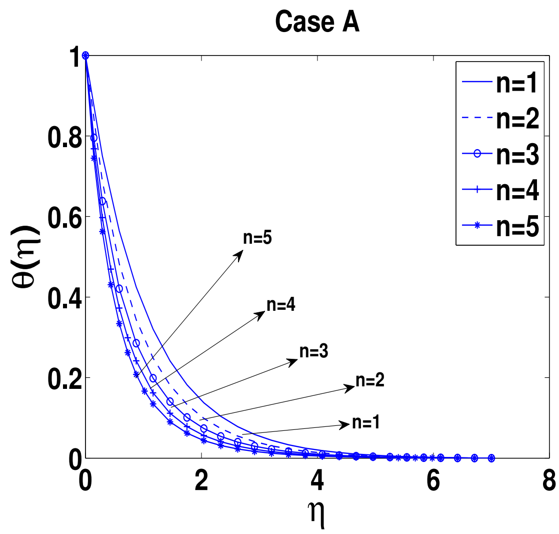

Fig. 17: Variation in n and its impact on the dimensionless temperature profiles $\theta(\eta)$ at m=1 and Pr=1.

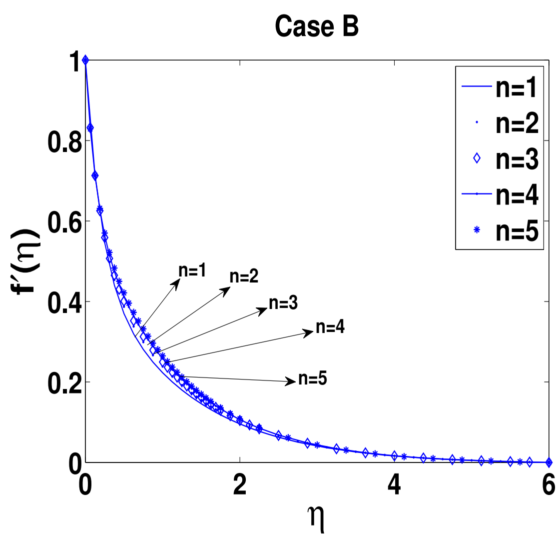

Fig. 18: Variation in n and its impact on the dimensionless velocity profiles $f'(\eta)$ at m=1 and Pr=10.

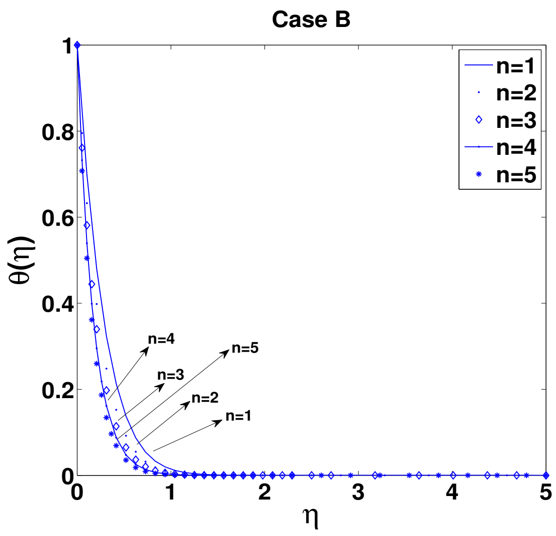

Fig. 19: Variation in n and its impact on the dimensionless temperature profiles $\theta(\eta)$ at m=1 and Pr=10.

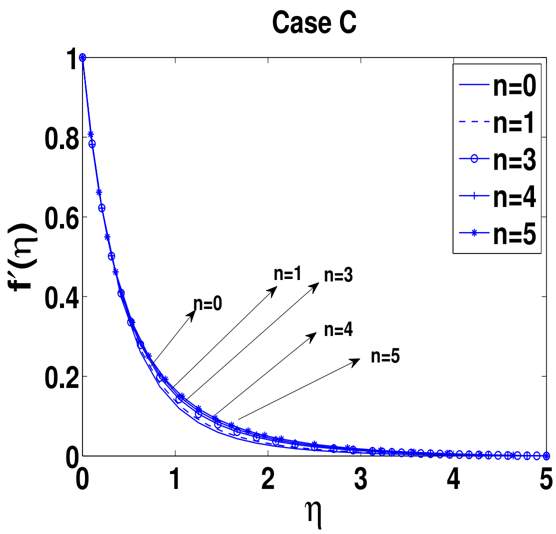

Fig. 20: Variation in n and its impact on the dimensionless velocity profiles $f'(\eta)$ at m=1 and Pr=0.7.

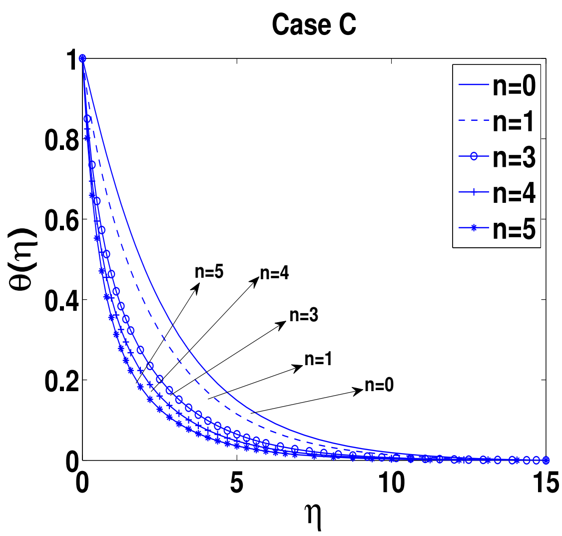

Fig. 21: Variation in n and its impact on the dimensionless temperature profiles $\theta(\eta)$ at m=1 and Pr=0.7.

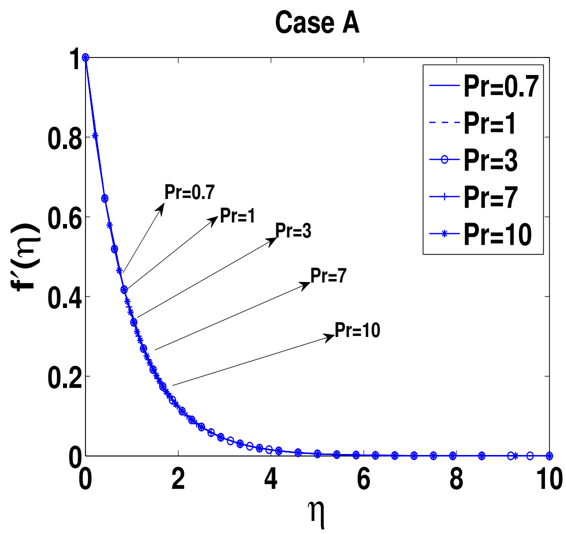

Fig. 22: Variation in Pr and its impact on the dimensionless velocity profiles $f'(\eta)$ at M=0.1.

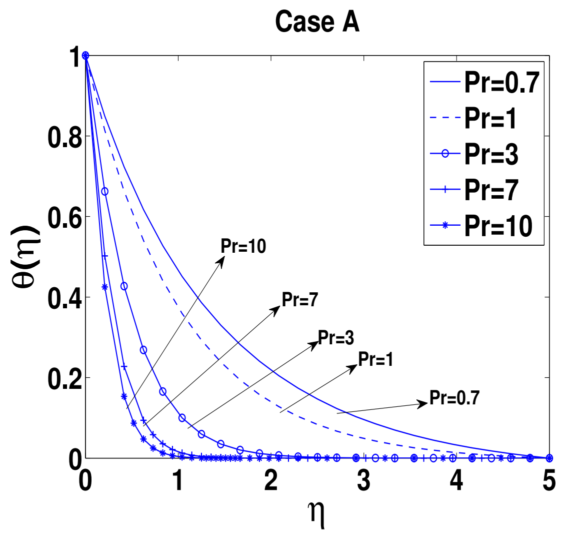

Fig. 23: Variation in Pr and its impact on the dimensionless temperature profiles $\theta(\eta)$ at M=0.1.

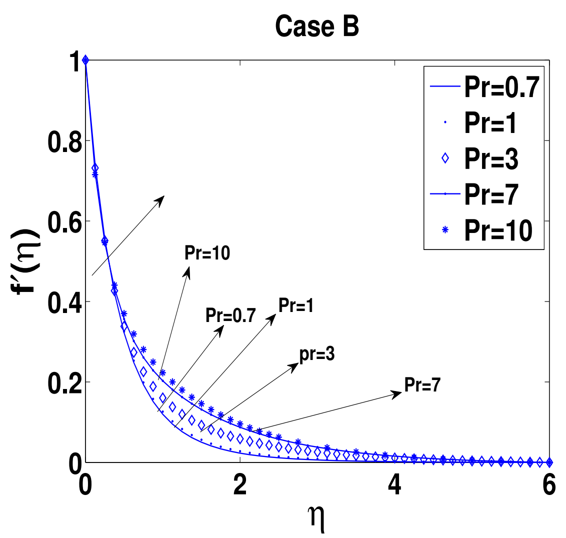

Fig. 24: Variation in Pr and its impact on the dimensionless velocity profiles $f'(\eta)$ at M=0.1.

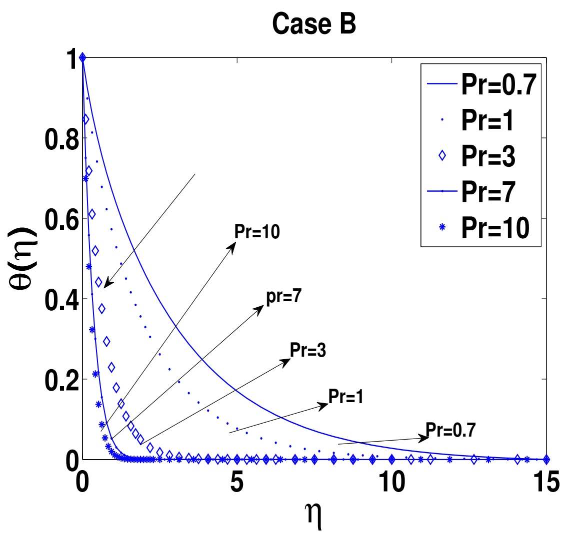

Fig. 25: Variation in Pr and its impact on the dimensionless temperature profiles $\theta(\eta)$ at M=0.5.

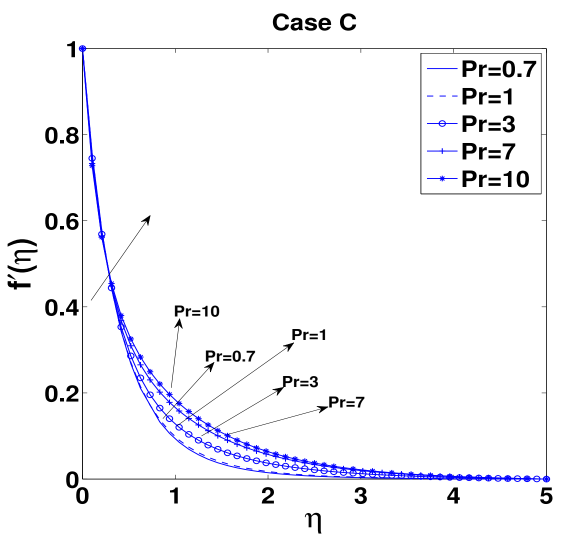

Fig. 26: Variation in Pr and its impact on the dimensionless velocity profiles $f'(\eta)$ at M=0.5.

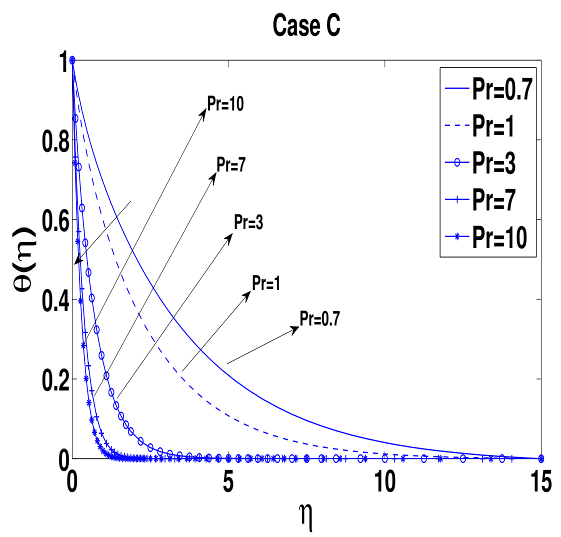

Fig. 27: Variation in Pr and its impact on the dimensionless temperature profiles $\theta(\eta)$ at M=0.5.

VI. Conclusions

The present paper examines numerical investigation for MHD flow and rate of heat transfer for viscous fluid with changeable fluid properties over nonlinear stretching surface. Governing parameters such as velocity exponent m, Prandtl number Pr, temperature index parameter n, stretching parameter β and magnetic parameter M and their effect on MHD flow has been examined. Main focus of our study was to compare viscosity as temperature dependent and treatment of viscosity as a constant. The nonlinear PDEs together with the boundary conditions are converted to nonlinear ODEs by using suitable similarity parameters. Shooting technique and *bvp4c* are used to find numerical solution of resulting ODEs. The results are summarized as:

- Skin friction coefficient and thermal boundary layer both increases with increment in magnetic parameter while velocity profile declines.
- Parameter m reduces momentum boundary layer for all cases whereas thermal boundary layer thickens.
- Prandtl number causes thermal boundary layer to reduce whereas enhances momentum boundary layer. While it causes a slight change in skin friction and enhances wall temperature.

Acknowledgements

We are thankful to Tausif Iqra for her contribution in developing new numerical procedure after discussion with M Asif Farooq. Also, Asif Mushtaq would like to thank Mathematics Teaching and Learning, research group within the Department of Mathematics, Bodø, Nord university for the partial support.

References

[1] L. J. Crane, "Flow past a stretching plane", *Z. Angew. Math. Phys.*, vol. 21, pp. 645-647, 1970.
[2] B. C. Sakiadis, "Boundary-layer behavior on continuous solid surfaces: I. Boundary-layer equations for two-dimensional and axisymmetric flow", *AIChE.*, vol. 7 26-28, 1961.
[3] B. C. Sakiadis, "Boundary-layer behavior on continuous solid surfaces: II. The boundary layer on a continuous flat surface", *AIChE.*, vol. 7, pp. 221-225, 1961.
[4] B. C. Sakiadis, "Boundary-layer behavior on continuous solid surfaces: III. The boundary layer on a continuous cylindrical surface", *AIChE.*, vol. 7, pp. 467-472, 1961.
[5] A. Pantokratoras, "Further results on the variable viscosity on flow and heat transfer to a continuous moving flat plate", *Int. J. Eng. Sci.*, vol. 42, no. 17-18, pp. 1891-1896, 2004.
[6] H. Andersson and J. Aarseth, "Sakiadis flow with variable fuid properties revisited", *Int. J. Eng. Sci.*, vol. 45, no. 2-8, pp. 554-561, 2007.
[7] F. C. Lai and F. A. Kulacki, "The effect of variable viscosity on convective heat transfer along a vertical surface in a saturated porous medium", *Int. J. Heat Mass Transfer.*, vol. 33, no. 5, pp. 1028-1031, 1990.
[8] N. Afzal, "Heat transfer from a stretching surface", *Int. J. Heat Mass Transfer.*, vol. 36, no. 4, pp. 1128-1131, 1993.
[9] N. Bachok, A. Ishak and I. Pop, "Boundary layer flow and heat transfer with variable fluid properties on a moving flat plate in a parallel free stream", *J. Appl. Math.*, vol. 2012, pp. 1-12, 2012.
[10] T. C. Chiam, "Magnetohydrodynamic heat transfer over a non-isothermal stretching sheet", *Acta Mech.*, vol. 122, no. 1-4, pp. 169-179, 1997.
[11] Y. S. Daniel, Z. A. Aziz, Z. Ismail, and F. Salah, "Slip Effects on Electrical Unsteady MHD Natural Convection Flow of Nanofluid over a Permeable Shrinking Sheet with Thermal Radiation", Engineering Letters, vol. 26, no. 1, pp. 107-116, 2018.

[12] F. Salah, Z. A. Aziz and D. L. C. Ching, ”On accelerated MHD flows of second grade fluid in a porous medium and rotating frame”, IAENG International Journal of Applied Mathematics, vol. 43, no. 3, pp. 106-113, 2013.
[13] Y. Liu, L. Zheng, X. Zhang, and F. Zong, ”The oscillating flows and heat transfer of a generalized Oldroyd-B fluid in magnetic field”, IAENG International Journal of Applied Mathematics, vol. 40, no. 4, pp. 276-281, 2010.
[14] S. Mukhopadhyay, G. C. Layek and S. k. A. Samad, ”Study of MHD boundary layer flow over a heated stretching sheet with variable viscosity”, *Int. J. Heat Mass Transfer.*, vol. 48, no. 21-22, pp. 4460-4466, 2005.
[15] I. Pop, R. S. R. Gorla and M. Rashidi, ”The effect of variable viscosity on flow and heat transfer to a continuous moving flat plate”, *Int. J. Eng. Sci.*, vol. 30, no. 1, pp. 1-6, 1992.
[16] M. E. Ali, ”Heat transfer characteristics of a continuous stretching surface”, *Warme Stoffu- bert.*, vol. 29, no. 4, pp. 227-234, 1994.
[17] K. V. Prasad, K. Vajravelu and P. S. Datti, ”The effects of variable fluid properties on the hydro-magnetic flow and heat transfer over a non-linearly stretching sheet”, *Int. J. Thermal Sci.*, vol. 49, no. 3, pp. 603-610, 2010.
[18] M. A. Seddeek, ”The effect of variable viscosity on hydromagnetic flow and heat transfer past a continuously moving porous boundary with radiation”, *ICHMT.*, vol. 27, no. 7, pp. 1037-1046, 2000.
[19] H. I. Andersson, ”MHD flow of a visco-elastic fluid past a stretching surface”, *Acta Mech.*, vol. 95, no. 1-4, pp. 227-230, 1992.
[20] H. I. Andersson, K. H. Bech and B. S. Dandapat, ”Magnetohydrodynamic flow of a power- law fluid over a stretching sheet”, *Int. J. Non-Linear Mechanics.*, vol. 27, no. 6, pp. 929-936, 1992.
[21] K. B. Pavlov, ”Magnetohydrodynamic flow of an incompressible viscous fluid caused by deformation of a plane surface”, *Magnitnaya Gidrodinamika (U.S.S.R.)*, vol. 4, no. 1, pp. 146-147, 1974.
[22] M. A. Seddeek, ”Effects of radiation and variable viscosity on a MHD free convection flow past a semi-infinite flat plate with an aligned magnetic field in the case of unsteady flow”, *Int. J. Heat Mass Transfer.*, vol. 45, no. 4, pp. 931-935, 2002.
[23] T. C. Chiam, ”Hydromagnetic flow over a surface stretching with a power-law velocity”, *Int. J. Eng. Sci.*, vol. 33, no. 3, pp. 429-435, 1995.
[24] E. M. A. Elbashbeshay and M. A. A. Bazid, ”The effect of temperature-dependent viscosity on heat transfer over a continuous moving surface”, *J. Phys. D: Appl. Phys.*, vol. 33, no. 21, pp. 2716-2721, 2000.
[25] J. X Ling and A. Dybbs, ”Forced convection over a flat plate submersed in a porous medium:variable viscosity case”, *Am. Soc. Mech. Eng.*, vol. 114 , pp. 87-123, 1987.
[26] M. Mustafa, ”Viscoelastic flow and heat transfer over a non-linearly stretching sheet: OHAM solution”, *J. Appl. Fluid Mech.*, vol. 9 , no. 3, pp. 1321-1328, 2016.
[27] A. M. Jacobi, ”A scale analysis approach to the correlation of continuous moving sheet (backward boundary layer) forced convective heat transfer”, *J. Heat Transfer.*, vol. 115 , no. 4, pp. 1058-1061, 1993.
[28] F. K. Tsou, E. M. Sparrow and R. J. Goldstein, ”Flow and heat transfer in the boundary layer on a continuous moving surface”, *Int. J. Heat Mass Transfer.,* vol. 10, no. 2, pp. 219-235, 1967.
[29] M. E. Ali, ”On thermal boundary layer on a power law stretching surface with suction or injection”, *Int. J. Heat Mass Flow.,* vol. 16, no. 4, pp. 280-290, 1995.
[30] L. F. Shampine, J. Kierzenka and M. W. Reichelt, ”Solving boundary value problems for ordinary differential equations in MATLAB with bvp4c”, *Tutorial notes.*, pp. 1-27, 2000.