Applications of infinite-dimensional geometry and Lie theory

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Introduction

The present introduction contains a short summary of the articles submitted for habilitation. I will briefly recall the basic material on infinite-dimensional Lie theory and infinite-dimensional geometry needed throughout. Then I will present my contributions to these fields with a view towards applications in numerical and stochastic analysis.

Infinite-dimensional geometry and Lie theory

Infinite-dimensional manifolds and Lie groups arise from problems related to differential geometry, fluid dynamics, and the symmetry of evolution equations. Among the most prominent examples of infinite-dimensional manifolds are manifolds of (differentiable) mappings and the diffeomorphism groups $\mathrm{Diff}(K)$, where K is a smooth and compact manifold. The group $\mathrm{Diff}(K)$ is an infinite-dimensional Lie group [Mic80] which for example arises naturally in fluid dynamics if K is a three-dimensional torus [Arn66, KW09]. The motion of a particle in the fluid corresponds, under periodic boundary conditions, to a curve in $\mathrm{Diff}(K)$. Due to a result by Omori [Omo78] the Lie group $\mathrm{Diff}(K)$ cannot be a Banach manifold (except in trivial cases). Thus many interesting examples force one to leave the realm of Banach spaces and Banach manifolds. Indeed, the reader may wonder what is meant by infinite-dimensional manifold and infinite-dimensional Lie group.

As a working definition, an infinite-dimensional Lie group will be a group which at the same time is an infinite-dimensional manifold that turns the group operations into smooth mappings. An infinite-dimensional manifold will be a topological space which is locally (in charts) homeomorphic to an open subset of an infinite-dimensional space. Moreover, we require the change of charts to be smooth. Beyond the realm of Banach spaces, the usual concept of smoothness is no longer available and we replace it with the requirement that all directional derivatives exist and induce continuous mappings[1]. This approach yields a versatile framework for the study of differential geometry and Lie theory on very general spaces.

Infinite-dimensional Lie groups and their homogeneous spaces will be the objects of our main interest. Founded in its modern form by Milnor [Mil84], infinite-dimensional Lie theory has been developed in the works of Glöckner and Neeb, see [Nee06] and the upcoming [GN]; nowadays it is a well established and active research area. In the infinite-dimensional setting, Lie theory exhibits several novel features and pathologies. For example consider the following well-known statements from Lie theory:

[1]This is the so-called Bastiani calculus [Bas64]. Note that there are various inequivalent ways to generalise calculus beyond Banach spaces, cf. e.g. [KM97] for the *convenient calculus*.

1. Every Lie algebra is the Lie algebra of a Lie group.
2. Every closed subgroup of a Lie group is a Lie subgroup.
3. Every Lie algebra morphism (up to topological obstructions) is the differential of a Lie group morphism.

In infinite-dimensional Lie theory all of the above statements are false in general, [Nee06], and hold at best under some additional assumptions. Hence the Lie theoretic treatment of infinite-dimensional groups requires additional properties. For example, one has to establish the regularity property for Lie groups (cf. [Glö15]) which means that certain ordinary differential equations (ODEs) can be solved on the Lie group.[2] This property turns out to be crucial for the applications we have in mind, because it enables the use of advanced Lie theoretic methods.

In conjunction with Lie theory, we exploit tools from (infinite-dimensional) Riemannian geometry. Recall that a Riemannian metric on a manifold is a choice of inner product for every tangent space which "depends smoothly" on the basepoint [Lan01]. On a finite-dimensional (paracompact) manifold, a standard partition of unity argument allows to construct a Riemannian metric from the Euclidean metric of the ambient space. Hence on every such manifold, powerful tools from Riemannian geometry become available. Generalising Riemannian geometry to infinite-dimensional manifolds, one faces in general the problem that there are no (smooth) partitions of unity (even Banach spaces may not admit smooth partitions of unity, [KM97, Chapter 16]). Further, the inner products will in general not be compatible with the topology of the tangent spaces as they are not Hilbert spaces. Thus the finite-dimensional definition of a Riemannian metric (what we will call a 'strong Riemannian metric', [Lan01, Kli95]) has to be relaxed to admit relevant examples beyond the Hilbert manifold setting. This leads to the notion of a 'weak Riemannian metric', cf. [AMR88, Section 5.2] and [Bru18b], i.e. a smooth choice of inner products on each tangent space which do not necessarily induce the topology of the tangent space. An instructive example is the L^2-inner product, which turns the space $C([0,1],\mathbb{R})$ of continuous functions into a pre-Hilbert space:

$$\langle f, g\rangle_{L^2} := \int_0^1 f(x)g(x)\mathrm{d}x.$$

The L^2-inner product is simple to compute and has the advantage that geodesics are explicit. For two given curves, a geodesic is the family of curves which interpolate pointwise linearly between the curves, cf. [Bru18a, 1.2]. Generalising this to manifold valued mappings (which then form an infinite-dimensional manifold), one obtains weak Riemannian metrics studied for example in shape analysis, fluid dynamics and optimal transport (see [Bru18a, EM70, KW09]). While strong Riemannian metrics exhibit behaviour as expected from the finite-dimensional case, this is no longer true for weak metrics. For example, the geodesic distance between distinct points vanishes for an

[2] Up to now, all known Lie groups on suitably complete spaces are regular, cf. [Nee06, KM97]. Note that beyond Banach spaces there is no general solution theory for ODEs.

equivariant version of the L^2-Riemannian metric, [MM05]. Thus this metric is unsuitable for comparing shapes. This problem has motivated the study of more involved metrics in shape analysis [BBM14]. Instead of general weak Riemannian metrics, we consider only situations in which the metrics either arise or are compatible with certain actions by (infinite-dimensional) Lie groups. This will allow us to establish desirable properties of the Riemannian geometry from the additional structure of the Lie group action. Vice versa, the additional information of the Riemannian geometry will provide tools complementing the structures that arise in Lie theory.

We will now give a brief overview on the works comprising this thesis, categorized under three main topics:

- *Connections between infinite-dimensional Lie groups and higher geometry*,
- *Hopf algebra character groups as Lie groups*, and
- *Applications of the interplay between Lie theory and Riemannian geometry.*

Connections between infinite-dimensional Lie groups and higher geometry

This section is based on the works [SW15, SW16b, SW16a, AS19, Sch19] in which a connection between infinite-dimensional Lie theory and finite-dimensional higher differential geometry is established. By higher differential geometry, we specifically mean Lie groupoids (which form a higher category (in this case a 2-category), hence the term "higher geometry").

Lie groupoids have been used to describe the symmetry of objects with bundle structure. Generalising Lie groups, Lie groupoids allow to describe regimes which lack the symmetry characteristic for groups and their applications. Moreover, large classes of Lie groupoids appear naturally in the study of symplectic or Poisson manifolds. The (Lie) theory of (finite-dimensional) Lie groupoids is a well developed and active field of research (cf. [Mac05, MM03]). In a first approximation, a Lie groupoid $\mathcal{G} = (G \rightrightarrows M)$ is a manifold with a smooth partial multiplication (think of a set of arrows which may only be composed if source and target of the arrows match). Similar to Lie groups, these global objects have an associated infinitesimal object, the so-called Lie algebroid $\mathbf{L}(\mathcal{G})$. In comparison to the situation for finite-dimensional Lie algebras and Lie groups, one may ask whether every Lie algebroid is associated to a Lie groupoid. In general, this is not the case as there is a topological obstruction which was discovered in [CF03]; to a certain extend, this mirrors the situation for infinite-dimensional Lie algebras and Lie groups discussed in the introduction. We claim that this observation is no coincidence and indeed rooted in a deep connection between infinite-dimensional groups and Lie groupoids.

It is well known, that to every Lie groupoid $\mathcal{G} = (G \rightrightarrows M)$ one can associate an (infinite-dimensional) Lie group [SW15, Sch19] of generalised elements, the so-called bisections. A bisection is a map $\sigma\colon M \to G$ which, if composed with the source s

or target t projection maps, yields a diffeomorphism. We can represent the group of bisections as

$$\mathrm{Bis}(\mathcal{G}) := \{\sigma \in C^\infty(M,G) \mid s\circ\sigma = \mathrm{id}_M, t\circ\sigma \in \mathrm{Diff}(M)\}.$$

Recall that if $\mathcal{P}(M) = (M\times M \rightrightarrows M)$ is the pair groupoid of the manifold M, we obtain (up to a trivial identification) $\mathrm{Bis}(\mathcal{P}(M)) = \mathrm{Diff}(M)$. Thus bisection groups generalise diffeomorphism groups. For topological groupoids, the group of (continuous) bisections and its representations are studied in the context of noncommutative geometry and geometric quantisation [Bos11]. Moreover, the Lie algebra of a bisection group is a Lie-Rinehard algebra which is of interest in quantisation and Poisson geometry [Hue04].

In [SW15, Sch19] we developed the Lie theory for bisection groups and certain subgroups in the setting of locally convex Lie groups.[3] Then in [SW16b, SW16a, AS19] we were able to uncover a tight connection between infinite-dimensional Lie groups (namely the bisection groups) and the finite-dimensional Lie groupoids. For example we were able to show that under certain topological assumptions, Lie groupoids are completely determined by their bisections and can be recovered from their bisection groups. Further, these correspondences are even functorial [SW16a]. If one fixes the base manifold M, denote then by $\mathbf{LieGpds}_M$ and $\mathbf{LieAlgbds}_M$ the categories of Lie groupoids or Lie algebroids over M, let $\mathbf{L}$ be the Lie functor which under certain assumptions has an inverse $\mathcal{I}$ called integration [Nee06, Mac05, CF03]. Passing from a groupoid to its bisections corresponds on the inifinitesimal level to the functor $-\Gamma$ assigning an algebroid its Lie algebra of sections (with the negative of the usual Lie bracket) Then our results yield (again under certain assumptions) a reconstruction functor $\mathcal{R}$ which makes the following diagram of functors commute:

$$\begin{tikzcd}
\mathbf{LieGpds}_M \arrow[r, shift left, "\mathbf{L}"] \arrow[d, shift left, "\mathrm{Bis}"] & \mathbf{LieAlgbds}_M \arrow[l, shift left, dashed, "\mathcal{I}"] \arrow[d, shift left, "-\Gamma"] \\
\mathbf{LieGroups} \arrow[r, shift left, "\mathbf{L}"] \arrow[u, shift left, dashed, "\mathcal{R}"] & \mathbf{LieAlgebras} \arrow[l, shift left, dashed, "\mathcal{I}"] \arrow[u, shift left, dashed, "\mathcal{R}"]
\end{tikzcd}$$

This suggests a close connection between the Lie theory of certain infinite-dimensional groups (whose Lie algebra is of Lie-Rinehard type) and finite-dimensional Lie groupoids. We have studied first consequences of this correspondence to the quantisation of (pre-)symplectic manifolds in [SW16b]. Moreover, in [AS19] we established a correspondence between smooth representations of Lie groupoids and smooth representations of the infinite-dimensional bisection groups. This generalises earlier results of [KSM02] and [Bos11] where similar correspondences were considered (albeit in an algebraic/topological setting without differentiability). Finally, we developed the Lie theory for the subgroup of vertical bisections in [Sch19]. These groups are important, because they encode certain information about the underlying Lie groupoid of independent geometric interest (we refer to [CS16] for more information).

[3]That is Lie groups in the setting of Bastiani differentiability. Note that it depends on the infinite-dimensional calculus chosen whether the statement that the bisection group is an infinite-dimensional Lie group is an actual theorem or a triviality. In the Bastiani case it is a theorem.

Hopf algebra character groups as Lie groups

This section is based on the works [BS17, BDS16, BS18, CS18]. Hopf algebras and their character groups appear in numerical analysis [BS17], renormalisation of quantum field theories [CM08, Man08], the theory of rough paths [Lyo98, FH14], and control theory [Foi15, DEG16]. In all of these contexts the Hopf algebras are connected to spaces of (formal) series for which (local) convergence is desirable in applications.

To illustrate and explain the framework used, let us consider the so-called Butcher-Connes-Kreimer-Hopf algebra $\mathcal{H}$ [Man08, 6.3.3] which is constructed as follows:

- As an algebra $\mathcal{H}$ is the (commutative) polynomial algebra generated by the basis $\mathcal{B}$ of unordered rooted trees including the empty tree which we denote by $\mathbf{1}$.
- The coproduct on a tree τ is $\Delta\tau := \sum_\sigma (\tau \setminus \sigma) \otimes \sigma$, where the sum runs over all subtrees σ of τ with the same root as τ and $\tau \setminus \sigma$ is the forest obtained by cutting σ from τ.
- Grading $\mathcal{H}$ by the number of nodes in a tree, $\mathcal{H}$ becomes a graded and connected algebra/coalgebra and therefore a Hopf algebra.

Now consider the algebraic dual $\mathcal{H}^*$ and the set of characters

$$\mathcal{G}(\mathcal{H}, \mathbb{R}) := \{\phi \in \mathcal{H}^* | \phi(ab) = \phi(a)\phi(b), \forall a, b \in \mathcal{H} \text{ and } \phi(\mathbf{1}) = 1\}.$$

The coproduct induces the so-called convolution product $\phi \star \psi(\tau) := \phi \otimes \psi(\Delta(\tau))$ which turns $\mathcal{H}^*$ into an algebra and $\mathcal{G}(\mathcal{H}, \mathbb{R})$ into a group. Elements in $\mathcal{G}(\mathcal{H}, \mathbb{R})$ can then be interpreted as infinite Taylor-like series expansions in trees. Characters of the Butcher-Connes-Kreimer algebra appear in several applied contexts, for example:

(1) The Taylor expansions mentioned can be identified with expansions of numerical methods for ordinary differential equations (known as B-series, appearing e.g. in Runge-Kutta schemes [Bro04]). In this context the group is also known as the "Butcher-group" [BS17] and the group product corresponds to composition of numerical methods.

(2) Branched rough paths [Gub10] can be interpreted as paths of a certain regularity with values in a decorated version of $\mathcal{H}$, cf. [BCFP19, HK15, CEMM18]. If one associates to a branched rough path its signature, one obtains a group morphism from the group of rough paths (with concatenation) to the character group of $\mathcal{H}$.

The algebraic and combinatorial properties of character groups of Hopf algebras are well known and the use of these structures in a broad spectrum of applications is a very active area of research. However, the topological and differential structure of groups of characters is often not taken into account. In [BS17, BDS16, BS18] we established the (infinite-dimensional) Lie and structure theory for character groups of graded Hopf algebras. Work in a different direction was carried out in [CS18], where the Hopf algebraic framework was used together with methods from differential geometry to study the convergence behaviour of Lie group methods, cf. [IMNZ00]. These numerical methods are widely used in the numerical solution of ordinary differential equations on Lie groups and homogeneous spaces.

Applications of the interplay between Lie theory and Riemannian geometry

In this section we deal with applications of the interplay between (infinite-dimensional) Lie theory and Riemannian geometry. This common theme connects several applications from different areas of mathematics. Namely, we will considerr applications from

1. shape analysis on spaces with ambient geometry [CES18, CEES17, CES16] and
2. stochastic fluid dynamics [MMS19]

Shape analysis has developed considerably over the last decade and is nowadays used to tackle a variety of problems of pattern and object recognition (cf. [BBM14, Mic16] and the references therein). The use of shapes is natural in applications when one wants to compare curves independently of their parametrisation. To this end, one computes distances between shapes using a Riemannian metric on an infinite-dimensional manifold of shapes (i.e. unparametrised curves arising as a quotient of a manifold of mappings by the action of an infinite-dimensional group). It has been shown in [MM05] that one of the simplest such metrics, an equivariant version of the L^2-metric, has vanishing geodesic distance. Hence it can not be used for shape analysis. To avoid computationally costly metrics, Srivastava et al. introduced in [SKJJ11] the Square Root Velocity Transform (SRVT) on Euclidean spaces. In [CES18, CEES17, CES16] we have constructed generalisations of the SRVT for Lie groups and homogeneous spaces. The resulting Riemannian metrics are computationally advantageous and have non-vanishing geodesic distance. We have been able to use them tackle problems in motion capturing and computer vision, among others.

The second application of the interplay between infinite-dimensional Riemannian geometry and Lie theory is coming from stochastic analysis: Our aim is to treat stochastic versions of the Euler equation for an incompressible fluid using geometric methods. As Arnold pointed out in [Arn66] the Euler equation can be rewritten as an ordinary differential equation on an infinite-dimensional manifold. This approach was used by Ebin and Marsden in their seminal paper [EM70]. Therein, the authors establish local existence and uniqueness of solutions for the Euler equation and the Navier-Stokes equation. Subsequently, many authors used similar techniques to study local existence and uniqueness of partial differential equations (PDEs) which are amenable to these techniques. This class of PDEs, now often called Euler-Arnold PDEs, encompasses important PDEs from e.g. fluid and magnetohydrodynamics as well as the study of imaging problems. We refer to [KW09, II.3] for an introduction and overview to the theory.

Recently, stochastic versions of the Euler equation have been considered in the works of D. Holm and collaborators [Hol15, CFH19].[4] Several authors [GHV14, Bes15] have also considered stochastic versions of Euler equations with different noise terms. In all

[4] Stochastic versions of Euler equations have been used to relate Euler and Navier-Stokes equations, cf. e.g. [Gli11, AC15]. In contrast, the perspective taken in [CFH19] emphasises the stochastic Euler equation as intrinsically interesting for its importance in certain models.

of these works, local existence and uniqueness of solutions for the stochastic versions was established (among other results) using techniques from stochastic analysis and the theory of partial differential equations. These techniques were often tied to the geometry of the domain on which the equation is posed. Our approach, in contrast, employs the geometric techniques of Arnold, Ebin and Marsden [Arn66, EM70] to study a geometric version of the Euler equation in Lagrangian form. To this end, one introduces the Lagrangian variable Φ, with $\dot{\Phi} = u \circ \Phi$. The Lagrangian equation then takes the form

$$\begin{cases} \nabla_{\dot{\Phi}} \dot{\Phi} + \nabla p \circ \Phi = \dot{W} \circ \Phi, \\ \operatorname{div}(\dot{\Phi} \circ \Phi^{-1}) = 0, \qquad \dot{\Phi} \text{ tangent to the boundary.} \end{cases}$$

on a compact manifold K (possibly with boundary). Here p is a fixed pressure function and W a suitable Brownian motion in time (but not in space) on K.[5] The differential operators in the equation are to be understood in terms of an ambient Riemannian metric. We interpret the flow Φ as evolving on the infinite-dimensional manifold of volume preserving diffeomorphisms of Sobolev regularity H^s. Using geometric and stochastic analysis on infinite-dimensional manifolds we prove the following result. The stochastic Euler equation and, more general, stochastic versions of Euler-Arnold equations, can equivalently be formulated as stochastic ordinary differential equations on infinite-dimensional manifolds. We then obtain in a very general local well-posedness result for the solutions of the stochastic Euler equation [MMS19]. In some sense, this result is weaker in that it requires more orders of regularity in the initial data, than comparable results for stochastic variants of the Euler equation, see [GHV14, CFH19]. However, it has the advantage of being agnostic of the underlying manifold K and it is expected that similar methods also yield local well-posedness for stochastic versions of other Euler-Arnold PDEs. Finally, we mention that the idea to use techniques of Ebin and Marsden [EM70] for stochastic differential equations is not new per se: In [Elw82, Gli11] a similar approach was used to study stochastic flows and connections between stochastic versions of Euler and Navier-Stokes equation. Note however, that in both cases no local well-posedness theory for the stochastic Euler equation was developed via the Ebin-Marsden approach. To the best of our knowledge, [MMS19] is the first work to exhibit a complete Ebin-Marsden approach to local well-posedness of SPDEs. We refer to loc.cit. for more information and a complete overview on the relevant stochastic literature.

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[5] As is usual, we interpret the Euler equation as an integral equation in the Stratonovich sense. Note that the noise is an additive noise term which does not depend on the solution itself. This differs from [CFH19].

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Part I.

Connections between infinite-dimensional Lie groups and higher geometry

Research Article

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The Lie group of vertical bisections of a regular Lie groupoid

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Abstract: In this note we construct an infinite-dimensional Lie group structure on the group of vertical bisections of a regular Lie groupoid. We then identify the Lie algebra of this group and discuss regularity properties (in the sense of Milnor) for these Lie groups. If the groupoid is locally trivial, i.e., a gauge groupoid, the vertical bisections coincide with the gauge group of the underlying bundle. Hence, the construction recovers the well-known Lie group structure of the gauge groups. To establish the Lie theoretic properties of the vertical bisections of a Lie groupoid over a non-compact base, we need to generalise the Lie theoretic treatment of Lie groups of bisections for Lie groupoids over non-compact bases.

Keywords: Regular Lie groupoid, Lie algebroid, infinite-dimensional Lie group, regularity of Lie groups, manifold of mappings, local triviality, gauge groupoid, gauge group

MSC 2010: 22E65, 22A22, 58D15, 58H05

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Introduction and statement of results

Lie groupoids have found wide application in differential geometry. In particular, they can be used to formulate the symmetry of objects with bundle structure. They generalise Lie groups, and their Lie theory exhibits features not present in the theory of (finite-dimensional) Lie groups (e.g., the integrability issue of Lie algebroids discussed in [8, 9]). To every (finite-dimensional) Lie groupoid, one can construct an (infinite-dimensional) Lie group, the group of (smooth) bisections of the Lie groupoid [4, 28, 29]. Moreover, one can show that the geometry and representation theory of this group is closely connected to the underlying Lie groupoid. If the Lie groupoid is locally trivial (i.e., represents a principal fibre bundle), one can even recover the Lie groupoid from the infinite-dimensional Lie group [4, 29–31].

In the present note we develop the Lie theory for the group of vertical bisections. A vertical bisection is a smooth map which is simultaneously a section for the source and the target map of the groupoid. We prove that for regular Lie groupoids, the vertical bisections form an infinite-dimensional Lie group which is an initial Lie subgroup of the group of bisections. Before we explain this result, lets motivate the interest in groups of vertical bisections. Firstly, we restrict to the special case of a gauge groupoid $\mathrm{Gauge}(P) := (P \times P/H \rightrightarrows M)$ of a principal H-bundle $P \to M$ over a compact base M.[1] Then the Lie group of bisections $\mathrm{Bis}(\mathrm{Gauge}(P))$ is isomorphic (as an infinite-dimensional Lie group) to the group of (smooth) bundle automorphism $\mathrm{Aut}(P)$, see

1 Gauge groupoids are locally trivial Lie groupoids, and every locally trivial Lie groupoid arises as a gauge groupoid of a principal bundle [19, Section 1.3]. Here M being compact allows us to ignore some technicalities arising in the non-compact case (which is similar to the compact case if one replaces the results in [34] by [1, 32]).

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[29, Example 2.16]. Translating the vertical bisections to the bundle picture, they are identified with the automorphisms of P descending to the identity on the base. Hence, the group of vertical bisections $\mathrm{vBis}(\mathrm{Gau}(P))$ is isomorphic to the gauge group $\mathrm{Gau}(P)$ of the principal H-bundle. Thus, [34] shows that we obtain a Lie group extension

$$\begin{tikzcd} \mathrm{vBis}(\mathrm{Gauge}(P)) \arrow[r, hook] \arrow[d, "\cong"] & \mathrm{Bis}(\mathrm{Gauge}(P)) \arrow[r, two heads] \arrow[d, "\cong"] & \mathrm{Diff}_{[P]}(M) \arrow[d, "\cong"] \\ \mathrm{Gau}(P) \arrow[r, hook] & \mathrm{Aut}(P) \arrow[r, two heads] & \mathrm{Diff}_{[P]}(M), \end{tikzcd} \tag{1}$$

where $\mathrm{Diff}_{[P]}(M)$ is a certain open subgroup of the group $\mathrm{Diff}(M)$ of diffeomorphisms of M. Thus, for locally trivial Lie groupoids, our results on the vertical bisections are not new as they can be derived from the Lie theory of gauge groups [32, 34]. In the present paper we seek to generalise these results to a larger class of Lie groupoids not related to principal bundles.

Secondly, (vertical) bisections are closely connected to the differential geometry of the underlying Lie groupoid. Thinking of a bisection σ as a generalised element of the Lie groupoid, we obtain an inner automorphism [19, Definition 1.4.8] and a surjective morphism onto the, in analogy to the Lie group case so-called, inner automorphisms of the groupoid

$$\pi\colon\ \mathrm{Bis}(\mathcal{G}) \to \mathrm{Inn}(\mathcal{G}) \subseteq \mathrm{Aut}(\mathcal{G}), \quad \sigma \mapsto I_\sigma, \quad I_\sigma(g) := \sigma(\beta(g))g\sigma(\alpha(g))^{-1}.$$

The vertical bisections are mapped precisely to the subgroup of inner automorphisms which preserve source and target fibres. Though the global structure of the groups $\mathrm{Inn}(\mathcal{G})$, $\mathrm{Aut}(\mathcal{G})$ has to our knowledge not yet been studied, these groups are closely connected to the geometry of the Lie groupoid [10, Section 5].[2] Apart from the connection to inner automorphisms, we have shown in [30, 31] that for certain Lie groupoids, the groupoid can be recovered from their groups of bisections. Namely, in [31, Section 4] certain Lie subgroups of the bisections were crucial to this (re-)construction process. So far this process is restricted to locally trivial Lie groupoids and a generalisation would require different ingredients. One candidate which could provide additional structure usable in this (re-)construction could be the group of vertical bisections. Note, however, that this group alone does not carry enough information to deal with the reconstruction of general Lie groupoids.

In the present article, we work in the so-called Bastiani setting of infinite-dimensional analysis (i.e., a mapping is smooth if all iterated directional derivatives exist and are continuous, cf. references in Appendix A). Our main result is the construction of an infinite-dimensional Lie group structure on the group of vertical bisections of a regular Lie groupoid, cf. Theorem 2.8. Moreover, we establish some Lie theoretic properties and clarify the relation of this structure to the Lie group structure on the full group of bisections.

While we concentrate in the present paper on finite-dimensional Lie groupoids, one should be able to extend these results to Lie groupoids with infinite-dimensional space of arrows. If the base of the groupoid is compact, the theory can be adapted using results from [29] which deal with the general case. For non-compact base manifolds, it is conjectured in [16] that similar results can be achieved.

1 Preliminaries and the Lie group of bisections

We shall write $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Hausdorff locally convex real topological vector spaces will be referred to as locally convex spaces. All manifolds will be assumed to be Hausdorff spaces and if a manifold is finite-dimensional, we require that it is σ-compact (for infinite-dimensional manifolds no such requirements are made). For manifolds M, N, we let $C^\infty(M, N)$ denote the set of all (Bastiani) smooth mappings from M to N. Furthermore, we denote by $\mathcal{D}(M, TN)$ the smooth mappings $s\colon M \to TN$ such that $s = 0$ off some compact set $K \subseteq M$ (i.e., the “space of all smooth mappings with compact support”).

2 Another example along these lines can be found in [27, Appendix], where geometric objects such as torsion free connections are constructed using the vertical bisections of the jet groupoid.

1.1. In the following $\mathcal{G} = (G \rightrightarrows M)$ will be a (finite-dimensional) Lie groupoid with source map α and target map β. We denote by $\iota\colon G \to G$ the inversion and by $\mathbf{1}\colon M \to G$ the unit map.

1.2. To the Lie groupoid $\mathcal{G}$, we associate a group of smooth mappings, the so-called *bisection group*. To this end, let

$$\operatorname{Bis}(\mathcal{G}) := \{\sigma \in C^\infty(M, G) \mid \alpha \circ \sigma = \mathrm{id}_M \text{ and} \beta \circ \sigma \in \operatorname{Diff}(M)\},$$

be the set of bisections of $\mathcal{G}$. The set $\operatorname{Bis}(\mathcal{G})$ is a group with respect to the operations

$$\sigma \star \tau(x) := \sigma(\beta \circ \tau(x))\tau(x), \quad \sigma^{-1}(x) = \iota \circ \sigma, \quad x \in M.$$

Proposition 1.3 (cf. [29, Theorem 3.8] and [4, Proposition 1.3]). *Let $\mathcal{G}$ be a finite-dimensional Lie groupoid, then* $\operatorname{Bis}(\mathcal{G})$ *is a submanifold of $C^\infty(M, G)$*[3] *and this structure turns the bisections into an infinite-dimensional Lie group.*

Remark 1.4. Note that [28] establishes the Lie group structure of $\operatorname{Bis}(\mathcal{G})$ in the inequivalent convenient setting of global analysis (cf. [17]). In general our results will imply the results from loc. cit. as they entail continuity of the underlying mappings, which is not automatic in the convenient setting.

In the rest of this section we will prove results and discuss the necessary changes to identify the Lie algebra of bisection groups for Lie groupoids over a non-compact base.

Lemma 1.5 ([4, Corollary A.6]). *Let $\mathcal{G} = (G \rightrightarrows M)$ be a finite-dimensional Lie groupoid. Then the evaluation mapping* $\mathrm{ev}\colon \operatorname{Bis}(\mathcal{G}) \times M \to G$, $(\sigma, m) \mapsto \sigma(m)$ *is a smooth submersion.*

Lemma 1.6. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a finite-dimensional Lie groupoid. Then the canonical action of the bisection group $\gamma\colon \operatorname{Bis}(\mathcal{G}) \times G \to G$, $(\sigma, g) \mapsto \sigma(\beta(g)).g$ is smooth.*

Proof. Note that we can write the action as a composition

$$\gamma(\sigma, g) = m(\mathrm{ev}(\sigma, \beta(g)), g), \quad \sigma \in \operatorname{Bis}(\mathcal{G}),\ g \in G,$$

where $m\colon G \times_M G \to G$ denotes the multiplication map of the Lie groupoid. Since ev is smooth, by Lemma 1.5, we deduce that γ is smooth. □

Remark 1.7. Having established smoothness, similar arguments as in [31, Proposition 2.4] show that the restricted action $\gamma_g\colon \operatorname{Bis}(\mathcal{G}) \to \alpha^{-1}(\alpha(g))$, $\sigma \mapsto \gamma(\sigma, g)$, is a submersion. However, we do not need this result.

We adapt now the approach in [29, Section 3] using smoothness γ to identify the Lie algebra of the bisection group $\operatorname{Bis}(\mathcal{G})$.

Proposition 1.8. *The Lie algebra of* $\operatorname{Bis}(\mathcal{G})$ *is isomorphic to the Lie algebra of smooth compactly supported sections* $\Gamma_c(\mathbf{L}(\mathcal{G}))$ *with the negative of the usual bracket.*

For M compact, Proposition 1.8 was established as [29, Theorem 4.4]. If M is non-compact, the function space topologies are more involved. Though the algebraic calculations carry over verbatim, the proof has to adapt smoothness arguments.

Proof of Proposition 1.8. We assume that M is not necessarily compact and $\operatorname{Bis}(\mathcal{G})$ is endowed with the Lie group structure from [4, Proposition 1.3]. According to loc. cit., the bisections are a submanifold as the preimage of the submersion α_* (pushforward) via $\operatorname{Bis}(\mathcal{G}) = (\alpha_*|_{\beta_*^{-1}(\operatorname{Diff}(M))})^{-1}(\mathrm{id}_M)$. Thus, we identify the Lie algebra $\mathbf{L}(\operatorname{Bis}(\mathcal{G}) \cong T_{\mathbf{1}} \operatorname{Bis}(\mathcal{G}) = \ker T_{\mathbf{1}}\alpha_*$. Following [21, Theorem 10.13], the map

$$\Phi_{M,G}\colon TC^\infty(M, G) \to \mathcal{D}(M, TG), \quad [t \mapsto \eta(t)] \mapsto (m \mapsto (t \mapsto [\eta(t)(m)])),$$

3 If M is non-compact, the topology on $C^\infty(M, G)$ is the so-called fine very strong topology, cf. [16] and see [21] for the construction of the manifold structure. If M is compact, the fine very strong topology coincides with the familiar compact-open C^∞-topology.

is an isomorphism of vector bundles, where tangent vectors are identified with equivalence classes $[\eta]$ of smooth curves $\eta\colon]-\varepsilon, \varepsilon[\to C^\infty(M, G)$ for some $\varepsilon > 0$. Up to the identification $T(\alpha_*) = (T\alpha)_*$, whence the kernel of $T\alpha_*$ is

$$\Gamma_c(\mathbf{L}(\mathcal{G})) := \{\gamma \in \mathcal{D}(M, TG) \mid \text{for all } x \in M, \gamma(x) \in T_{\mathbf{1}_x}\alpha^{-1}(x)\},$$

the space of compactly supported sections of the Lie algebroid $\mathbf{L}(\mathcal{G})$. Recall that the Lie bracket is induced by the bracket of right invariant vector fields on the bisections. We indicate now how to supplement the calculations in [29, Theorem 4.4] when the arguments involve smoothness. The following list compiles the tools and changes:

(i) $\Phi_{M,G}$ restricts to an isomorphism $\varphi_\mathcal{G}\colon \mathbf{L}(\operatorname{Bis}(\mathcal{G})) = T_\mathbf{1}\operatorname{Bis}(\mathcal{G}) \to \Gamma_c(\mathbf{L}(\mathcal{G}))$.
(ii) To $X \in T_\mathbf{1}\operatorname{Bis}(\mathcal{G})$ associate the vector field $\overrightarrow{\varphi_\mathcal{G}(X)}$ on G, defined via $\overrightarrow{\varphi_\mathcal{G}(X)}(g) := T(R_g)(\varphi_\mathcal{G}(X)(\beta(g)))$ (where R_g is the right-translation in the Lie groupoid).
(iii) Note that the natural action $\gamma\colon \operatorname{Bis}(\mathcal{G}) \times G \to G$ is smooth by replacing [29, Proposition 3.11] with Lemma 1.6. Thus, in the proof of [29, Proposition 4.2], we replace [29, Theorem 7.8 (d)] with [21, Lemma 10.15] to prove that.
(iv) For a right invariant vector field X^ρ associated to $X \in T_\mathbf{1}\operatorname{Bis}(\mathcal{G})$, the vector field $X^\rho \times 0$ is related to $\overrightarrow{\varphi_\mathcal{G}(X)}$ via γ.

Now, as in [29, Theorem 4.4], one calculates the Lie bracket and shows that $\varphi_\mathcal{G}$ is an anti-isomorphism of Lie algebras. □

2 The vertical bisections of a regular Lie groupoid

In this section we will discuss the Lie group structure of the group of vertical bisections of a Lie groupoid $\mathcal{G} = (G \rightrightarrows M)$.

Definition 2.1. A *vertical bisection* of $\mathcal{G}$ is a bisection $\sigma \in \operatorname{Bis}(\mathcal{G})$ such that $\beta \circ \sigma = \mathrm{id}_M$. We denote the subgroup of $\operatorname{Bis}(\mathcal{G})$ of all vertical bisections by

$$\operatorname{vBis}(\mathcal{G}) = \{\sigma \in \operatorname{Bis}(\mathcal{G}) \mid \beta \circ \sigma = \mathrm{id}_M\}$$

Example 2.2. Let $\mathcal{G}$ be a Lie groupoid.

(i) If $\mathcal{G}$ is totally intransitive, i.e., source and target mapping coincide and $\mathcal{G}$ is a Lie group bundle, the vertical bisections coincide with the group of bisections.
(ii) If $\mathcal{G}$ is a transitive Lie groupoid, i.e., a gauge groupoid of a principal H-bundle $P \to M$, the vertical bisections coincide with the gauge group of the principal bundle, cf. [29, Example 2.16].

It is not hard to see that the subgroup $\operatorname{vBis}(\mathcal{G})$ is a normal subgroup of $\operatorname{Bis}(\mathcal{G})$, see [10, Proposition 1.1.2]. As the pushforward $\beta_*\colon C^\infty(M, G) \to C^\infty(M, M)$, $f \mapsto \beta \circ f$, is smooth (whence, in particular, continuous), $\operatorname{vBis}(\mathcal{G})$ is a closed subgroup of $\operatorname{Bis}(\mathcal{G})$. Unfortunately, this does not entail that $\operatorname{vBis}(\mathcal{G})$ is a Lie subgroup of $\operatorname{Bis}(\mathcal{G})$, as $\operatorname{Bis}(\mathcal{G})$ is an infinite-dimensional Lie group.

The vertical bisections are exactly the bisections which take their values in the isotropy subgroupoid $\mathcal{I}G$ (cf. Appendix A) of $\mathcal{G}$, i.e.,

$$\operatorname{vBis}(\mathcal{G}) = \{\sigma \in \operatorname{Bis}(\mathcal{G}) \mid \sigma(M) \subseteq \mathcal{I}G\}.$$

If $\mathcal{I}G$ is a Lie subgroupoid (which in general is not, cf. Example A.10), we could identify the vertical bisections as the group of (smooth) bisections of the isotropy subgroupoid. However, there is a large class of Lie groupoids for which at least the connected identity subgroupoid $\mathcal{I}G^\circ$ of the isotropy groupoid $\mathcal{I}G$ (cf. Appendix A) is an embedded Lie subgroupoid.

Definition 2.3. A Lie groupoid $\mathcal{G} = (G \rightrightarrows M)$ is called *regular Lie groupoid* if the Lie groupoid anchor

$$(\alpha, \beta)\colon G \to M \times M, \quad g \mapsto (\alpha(g), \beta(g)),$$

is a mapping of constant rank.

Remark 2.4. Many important classes of Lie groupoids, such as foliation groupoids of regular foliations, transitive groupoids and locally trivial groupoids are regular groupoids, cf. [23, 33] for more information. The regularity condition on the anchor $(\alpha, \beta)\colon G \to M \times M$ is equivalent to requiring that the anchor $\rho\colon \mathbf{L}(\mathcal{G}) \to TM$ of the associated Lie algebroid is of constant rank, cf. [33].

Lemma 2.5 ([23, Proposition 2.5]). *Let $\mathcal{G}$ be a regular Lie groupoid. Then the connected identity subgroupoid $\mathcal{I}G^\circ$ is an embedded normal Lie subgroupoid of $\mathcal{G}$. Its associated Lie algebroid $\mathbf{L}(\mathcal{I}G^\circ)$ is the isotropy subalgebroid $\mathcal{I}\mathbf{L}(\mathcal{G})$ of $\mathbf{L}(\mathcal{G})$.*[4]

Let us remark here that even for regular Lie groupoids, the subgroupoid $\mathcal{I}G$ is in general not an embedded Lie subgroupoid, as the following example shows:

Example 2.6. Let $\mathbb{T} := \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional torus. Consider the action of $(\mathbb{R}, +)$ on $\mathbb{T} \times]-1, 1[$ via

$$\lambda([x, y], \varepsilon) := ([x + \lambda, y + \lambda\varepsilon], \varepsilon), \quad \lambda \in \mathbb{R}, [x, y] \in \mathbb{T} \text{ and } \varepsilon \in]-1, 1[.$$

The associated action groupoid A is regular as all orbits are diffeomorphic either to circles or lines. Furthermore, the isotropy at a given point $([x, y], \varepsilon) \in A$ is either a copy of $\mathbb{Z}$ in $\mathbb{R} \times \{([x, y], \varepsilon)\}$ if ε is rational or a singleton for ε irrational. As all the points in the same orbit have isotropy of the same type, this implies that the isotropy subgroupoid would have to be at least a one-dimensional submanifold if it were an embedded submanifold. However, this implies that IA cannot be an embedded Lie groupoid as, for example, there is no neighborhood of the point $(1, [x, y], 0) \in IA$ which is diffeomorphic to a non-trivial euclidean space (due to the trivial isotropy groups of the points $([x, y], \varepsilon)$ for $\varepsilon \in \mathbb{R} \setminus \mathbb{Q}$.

Thus, we cannot leverage in the following constructions a smooth structure on the isotropy groupoid. Note that, by restricting ourselves to the smaller class of locally trivial Lie groupoids, such a structure would be available, as then $\mathcal{I}G$ is indeed an embedded submanifold [18, Proposition 1.17]. Instead we will now describe a construction of a Lie group structure on $\mathrm{vBis}(\mathcal{G})$ which works for every regular Lie groupoid. To this end, we leverage that $\mathcal{I}G^\circ$ is an embedded submanifold and consider an auxiliary group

$$\mathrm{vBis}^\circ(\mathcal{G}) := \{\sigma \in \mathrm{Bis}(\mathcal{G}) \mid \sigma(M) \subseteq \mathcal{I}G^\circ\}.$$

Since $\mathcal{I}G^\circ$ is an embedded Lie subgroupoid, we can test smoothness of a bisection taking its values in $\mathcal{I}G^\circ$ with respect to the submanifold structure A.3. Hence, we obtain the following.

Proposition 2.7. *Let $\mathcal{G}$ be a regular Lie groupoid. Then $\mathrm{vBis}^\circ(\mathcal{G}) \subseteq C^\infty(M, G)$ is a submanifold and this structure turns $\mathrm{vBis}^\circ(\mathcal{G})$ into an infinite-dimensional Lie group which is isomorphic to $\mathrm{Bis}(\mathcal{I}G^\circ)$. Moreover, this Lie group satisfies the following:*
(i) *The Lie algebra $\mathbf{L}(\mathrm{vBis}^\circ(\mathcal{G}))$ is isomorphic to the Lie algebra $\Gamma(\mathcal{I}\mathbf{L}(\mathcal{G}))$ of smooth sections of the isotropy algebroid with the negative of the usual bracket.*
(ii) *The inclusion $\iota_{\mathrm{Bis}}\colon \mathrm{vBis}^\circ(\mathcal{G}) \to \mathrm{Bis}(\mathcal{G})$ turns $\mathrm{vBis}(\mathcal{G})$ into an initial Lie subgroup of $\mathrm{Bis}(\mathcal{G})$.*

Proof. Due to Lemma 2.5, we can consider the embedded subgroupoid $\mathcal{I}G^\circ \subseteq \mathcal{G}$. Denoting by $I\colon \mathcal{I}G^\circ \to \mathcal{G}$ the associated embedding, the mapping $I_*\colon C^\infty(M, \mathcal{I}G^\circ) \to C^\infty(M, G), f \mapsto I \circ f$ is a smooth embedding, realising $C^\infty(M, \mathcal{I}G^\circ)$ as a split submanifold of $C^\infty(M, G)$, cf. [21, Proposition 10.8]. We recall from 1.2 that $\mathrm{Bis}(\mathcal{I}G^\circ)$ is a submanifold of $C^\infty(M, \mathcal{I}G^\circ)$ and this structure turns it into a Lie group. Now the canonical identification $I_*(\mathrm{Bis}(\mathcal{I}G^\circ)) = \mathrm{vBis}^\circ(\mathcal{G})$ shows that the Lie group $\mathrm{vBis}^\circ(\mathcal{G}) \cong \mathrm{Bis}(\mathcal{I}G^\circ)$ can be identified as a submanifold of $C^\infty(M, G)$ (cf. [13, Lemma 1.4]). We establish now the properties claimed in the proposition:
(i) The isomorphism $\mathrm{Bis}(\mathcal{I}G^\circ) \cong \mathrm{vBis}^\circ(\mathcal{G})$ identifies the Lie algebra of $\mathrm{vBis}^\circ(\mathcal{G})$ with $\mathbf{L}(\mathrm{Bis}(\mathcal{I}G^\circ)$. Since the isotropy algebroid is the Lie algebroid of $\mathcal{I}G^\circ$, Proposition 1.8 shows that we obtain the Lie algebra claimed in the statement of the proposition.

4 Recall that the isotropy subalgebroid $\mathcal{I}\mathbf{L}(\mathcal{G})$ is given fibre-wise as the kernel $\ker(\rho_x)$, or, equivalently, as the Lie algebra $\mathbf{L}(\alpha^{-1}(x) \cap \beta^{-1}(x))$ of the isotropy subgroup at x, cf. [9, 2.2].

(ii) To see that $\mathrm{vBis}^\circ(\mathcal{G})$ is an initial subgroup of $\mathrm{Bis}(\mathcal{G})$, note first that $\mathrm{vBis}^\circ(\mathcal{G})$ and $\mathrm{Bis}(\mathcal{G})$ are both submanifolds of $C^\infty(M,G)$. Thus, a mapping $f\colon M \to C^\infty(M,G)$ from a C^k-manifold which takes its image in $H \in \{\mathrm{vBis}^\circ(\mathcal{G}), \mathrm{Bis}(\mathcal{G})\}$ is of class C^k if and only if it is a C^k-mapping into H. Now the inclusion ι_{Bis} is an injective group morphism. Composing ι_{Bis} with the inclusion $\mathrm{Bis}(\mathcal{G}) \subseteq C^\infty(M,G)$, we obtain the (smooth) inclusion $\mathrm{vBis}^\circ(\mathcal{G}) \subseteq C^\infty(M,G)$, whence ι_{Bis} is an injective Lie group morphism. It is easy to see that $\mathbf{L}(\iota_{\mathrm{Bis}})\colon \mathbf{L}(\mathrm{vBis}^\circ(\mathcal{G})) \to \mathbf{L}(\mathrm{Bis}(\mathcal{G}))$ is injective, as it is, up to an identification, just the inclusion of subspaces. Hence, we consider a C^k-map $f\colon N \to \mathrm{Bis}(\mathcal{G})$ taking its image in $\mathrm{vBis}^\circ(\mathcal{G})$. Then $\iota_{\mathrm{Bis}}^{-1} \circ f$ is a C^k-map into $\mathrm{vBis}^\circ(\mathcal{G})$, as we can identify it with the C^k-map $f\colon M \to \mathrm{Bis}(\mathcal{G}) \subseteq C^\infty(M,G)$. □

Note that the above proof gives no information about $\mathrm{vBis}^\circ(\mathcal{G})$ being a Lie subgroup of $\mathrm{Bis}(\mathcal{G})$ in the traditional sense (i.e., being an embedded submanifold). However, we will now leverage the structure on $\mathrm{vBis}^\circ(\mathcal{G})$ to construct a Lie group structure on $\mathrm{vBis}(\mathcal{G})$ via the construction principle Proposition A.6.

Theorem 2.8. *Let $\mathcal{G}$ be a regular Lie groupoid. Then the group of vertical bisections* $\mathrm{vBis}(\mathcal{G})$ *is an infinite-dimensional Lie group with* $\mathbf{L}(\mathrm{vBis}(\mathcal{G})) = \mathbf{L}(\mathrm{vBis}^\circ(\mathcal{G}))$. *With respect to this structure, the vertical bisections form an initial Lie subgroup of* $\mathrm{Bis}(\mathcal{G})$. *Moreover,* $\mathrm{vBis}^\circ(\mathcal{G})$ *becomes an open subgroup of* $\mathrm{vBis}(\mathcal{G})$.

Proof. Apply the construction principle Proposition A.6: Set $U = V = \mathrm{vBis}^\circ(\mathcal{G})$ and observe that part (a) just yields the Lie group structure on $\mathrm{vBis}^\circ(\mathcal{G})$ from Proposition 2.7. We wish now to apply part (b) of Proposition A.6 to obtain a Lie group structure on $\mathrm{vBis}(\mathcal{G})$.

To this end, we observe that $\mathcal{I}G^\circ$ forms a normal Lie subgroupoid of $\mathcal{G}$, see 2.5. Let now $\sigma, \tau \in \mathrm{vBis}(\mathcal{G})$. Then one directly verifies from the formula for multiplication and inversion in the bisection group that

$$c_\tau(\sigma)(x) := \tau \star \sigma \star \tau^{-1}(x) = \tau(x)\sigma(x)(\tau(x))^{-1},$$

where $(\tau(x))^{-1}$ denotes the inverse of $\tau(x)$ in the Lie group $\alpha^{-1}(x) \cap \beta^{-1}(x)$. If $\sigma \in \mathrm{vBis}^\circ(\mathcal{G})$, we deduce that $c_\tau(\sigma)(x)$ stays in the normal subgroup $\alpha^{-1}(x) \cap \beta^{-1}(x) \cap \mathcal{I}G^\circ$, whence $\mathrm{vBis}^\circ(\mathcal{G})$ is a normal subgroup of $\mathrm{vBis}(\mathcal{G})$. Hence, we set $W := \mathrm{vBis}^\circ(\mathcal{G})$ and have to prove that $c_\tau\colon W \to W$ is smooth for every $\tau \in \mathrm{vBis}(\mathcal{G})$. As $\mathrm{Bis}(\mathcal{G})$ is a Lie group, $C_\tau\colon \mathrm{Bis}(\mathcal{G}) \to \mathrm{Bis}(\mathcal{G})$, $\delta \mapsto \tau \star \delta \star \tau^{-1}$, is smooth. In Proposition 2.7 we have seen that the inclusion $\iota_{\mathrm{Bis}}\colon \mathrm{vBis}^\circ(\mathcal{G}) \to \mathrm{Bis}(\mathcal{G})$ turns $\mathrm{vBis}^\circ(\mathcal{G})$ into an initial Lie subgroup of $\mathrm{Bis}(\mathcal{G})$. Combining these observations, we conclude from Proposition A.6 (b) that $\mathrm{vBis}(\mathcal{G})$ is a Lie group, as $c_\tau = \iota_{\mathrm{Bis}}^{-1} \circ C_\tau \circ \iota_{\mathrm{Bis}}$ is smooth for every $\tau \in \mathrm{vBis}(\mathcal{G})$.

Since $\mathrm{vBis}^\circ(\mathcal{G})$ is an open Lie subgroup of $\mathrm{vBis}(\mathcal{G})$, it is clear that the Lie algebras of both groups coincide. To see that $\mathrm{vBis}(\mathcal{G})$ is an initial Lie subgroup of $\mathrm{Bis}(\mathcal{G})$, we observe that the inclusion $I_{\mathrm{Bis}}\colon \mathrm{vBis}(\mathcal{G}) \to \mathrm{Bis}(\mathcal{G})$ is an injective morphism of Lie groups with $\mathbf{L}(I_{\mathrm{Bis}})$ injective, since ι_{Bis} is such a morphism. Let now $f\colon N \to \mathrm{Bis}(\mathcal{G})$ be a C^k-map with image in $\mathrm{vBis}(\mathcal{G})$. It suffices to check the C^k-property on every (open) connected component of $\mathrm{vBis}(\mathcal{G})$. Without loss of generality, we may thus assume that f takes its image in a component $C \subseteq \mathrm{vBis}(\mathcal{G})$ such that for some $g \in \mathrm{vBis}(\mathcal{G})$, we have $g^{-1} \star C \subseteq \mathrm{vBis}^\circ(\mathcal{G})$. Denoting left translation by an element ℓ of a Lie group L by λ_ℓ, we see that

$$I_{\mathrm{Bis}}^{-1} \circ f = \lambda_g^{\mathrm{vBis}(\mathcal{G})} \circ \iota_{\mathrm{Bis}}^{-1} \circ \lambda_{g^{-1}}^{\mathrm{Bis}} f,$$

whence $I_{\mathrm{Bis}}^{-1} \circ f$ is a C^k-map. □

We are now in a position to establish regularity (in the sense of Milnor) for the group of vertical bisections. Recall that a Lie group H is C^r-regular if for every C^r-curve $\gamma\colon [0,1] \to \mathbf{L}(H)$), the initial value problem

$$\begin{cases} \eta'(t) = T_{\mathbf{1}}\rho_{\eta(t)}(\gamma(t)), & \rho_g(h) := hg, \\ \eta(0) = \mathbf{1}, \end{cases}$$

has a unique C^{r+1}-solution $\mathrm{Evol}(\gamma) := \eta\colon [0,1] \to H$ and the evolution map $\mathrm{evol}\colon C^r([0,1], \mathbf{L}(H))) \to H$, $\gamma \mapsto \mathrm{Evol}(\gamma)(1)$, is smooth. To employ advanced techniques in infinite-dimensional Lie theory, one needs to require regularity of the Lie groups involved, cf. [12].

Proposition 2.9. *The Lie group* $\mathrm{vBis}(\mathcal{G})$ *is C^r-regular for every $r \in \mathbb{N}_0 \cup \{\infty\}$, whenever M is compact or $\mathcal{G}$ is a transitive Lie groupoid.*

Proof. We distinguish two cases:

Case 1: M is compact. Since $\mathrm{vBis}^\circ(\mathcal{G}) \cong \mathrm{Bis}(\mathcal{I}G^\circ)$ is an open subgroup of $\mathrm{vBis}(\mathcal{G})$, we see that $\mathrm{vBis}(\mathcal{G})$ is C^r-regular if and only if $\mathrm{Bis}(\mathcal{I}G^\circ)$ is C^r-regular. However, the C^r-regularity of $\mathrm{Bis}(\mathcal{I}G^\circ)$ was established in [29, Theorem 5.5].

Case 2: $\mathcal{G}$ is transitive. In this case $\mathcal{G}$ can be identified as a gauge groupoid of a principal H-bundle $P \to M$. As explained in the introduction, see (1), we can identify the compactly supported vertical bisections $\mathrm{vBis}(\mathcal{G})$ with the group of compactly supported gauge transformations $\mathrm{Gau}_c(P)$. However, $\mathrm{Gau}_c(P)$ (and thus also $\mathrm{vBis}(\mathcal{G})$) is C^r-regular for every $r \in \mathbb{N}_0 \cup \{\infty\}$ by a combination of [11, Theorem A and Corollary 8.3]. □

We expect Proposition 2.9 to hold for all vertical bisection groups, as all bisection groups of finite-dimensional Lie groupoids are expected to be regular. For groupoids over a non-compact base, these results require mild generalisations of the results obtained in [29, 31]. The main issue is that the function space topologies are much more involved in this case, see [16, 21]. Working around this would require extensive localisation arguments (on a cover of compact sets), which poses no conceptual problem, but would lead quite far away from the main line of reasoning. Thus, we have not established the result in full generality.

Note that from the construction, it is not clear whether $\mathrm{vBis}(\mathcal{G})$ is a Lie subgroup of $\mathrm{Bis}(\mathcal{G})$ and not even if $\mathrm{vBis}(\mathcal{G})$ is a submanifold of $C^\infty(M, G)$. Up to this point we can only obtain the following:

Lemma 2.10. *The Lie group topology of* $\mathrm{vBis}(\mathcal{G})$ *from Theorem 2.8 is the subspace topology induced by* $C^\infty(M, G)$. *Moreover, a mapping* $f\colon M \to C^\infty(M, G)$ *whose image is contained in* $\mathrm{vBis}(\mathcal{G})$ *is of class* C^k *if and only if it is* C^k *as a mapping into* $\mathrm{vBis}(\mathcal{G})$.

Proof. The statement about $f\colon N \to C^\infty(M, G)$ follows from $\mathrm{vBis}(\mathcal{G})$ being an initial Lie subgroup of $\mathrm{Bis}(\mathcal{G})$, Theorem 2.8, and the fact that $\mathrm{Bis}(\mathcal{G})$ is a submanifold of $C^\infty(M, G)$.

To see that the topology on $\mathrm{vBis}(\mathcal{G})$ coincides with the subspace topology induced by the inclusion $\mathrm{vBis}(\mathcal{G}) \subseteq C^\infty(M, G)$, recall that the Lie group topology on $\mathrm{Bis}(\mathcal{G})$ is the subspace topology induced by $C^\infty(M, G)$ (cf. [4, Proposition 1.3]). Further $\mathrm{vBis}^\circ(\mathcal{G})$ carries the subspace topology of $C^\infty(M, G)$, Proposition 2.7, whence the subspace topology is induced by $\mathrm{Bis}(\mathcal{G})$. Now for $\tau \in \mathrm{vBis}(\mathcal{G})$, the left translations $\lambda_\tau\colon \mathrm{Bis}(\mathcal{G}) \to \mathrm{Bis}(\mathcal{G})$, $\sigma \mapsto \tau \star \sigma$, is a homeomorphism mapping $\mathrm{vBis}(\mathcal{G})$ to $\mathrm{vBis}(\mathcal{G})$. We conclude that every component of $\mathrm{vBis}(\mathcal{G})$ carries the subspace topology induced by the inclusion $\mathrm{vBis}(\mathcal{G}) \subseteq \mathrm{Bis}(\mathcal{G}) \subseteq C^\infty(M, G)$. The proof is complete. □

The vertical bisections encode isotropy information of the underlying Lie groupoid. If the Lie groupoid contains only 'small' isotropy groups, the subgroup of vertical bisections is a very small subgroup as the next example shows.

Example 2.11. Consider a proper étale Lie groupoid $\mathcal{G}$,[5] i.e., a Lie groupoid with proper anchor map such that α, β are local diffeomorphisms. Then the isotropy subgroup $\mathcal{G}_x := \alpha^{-1}(x) \cap \beta^{-1}(x)$ is discrete. Hence, $\mathcal{I}G^\circ = \mathbf{1}(M) \subseteq G$ and we have $\mathrm{vBis}^\circ(\mathcal{G}) = \{\mathbf{1}\}$. From the construction of the Lie group $\mathrm{vBis}(\mathcal{G})$ via Proposition A.6 in Theorem 2.8, it is clear that $\mathrm{vBis}(\mathcal{G})$ is a discrete Lie group.

Remark 2.12. Albeit the smooth structure on $\mathrm{vBis}(\mathcal{G})$ is constructed by translating the smooth structure of $\mathrm{vBis}^\circ(\mathcal{G})$ along the diffeomorphisms λ_τ, it is still not clear whether $\mathrm{vBis}(\mathcal{G})$ is a submanifold of $C^\infty(M, G)$, as it is not clear that the connected components of $\mathrm{vBis}^\circ(\mathcal{G})$ can be separated from each other in the topology of $C^\infty(M, G)$.

However, if the isotropy subgroupoid is an embedded Lie subgroupoid, we can indeed obtain $\mathrm{vBis}(\mathcal{G})$ as a submanifold of $C^\infty(M, G)$.

5 Proper étale Lie groupoid are also known as "orbifold groupoids" as they represent orbifolds. See [24] for more information.

Proposition 2.13. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a regular Lie groupoid such that the isotropy subgroupoid $\mathcal{I}G$ is an embedded Lie subgroupoid, e.g., if $\mathcal{G}$ is a locally transitive Lie groupoid. Then the Lie group* $\mathrm{vBis}(\mathcal{G})$ *from Theorem 2.8 is a submanifold of $C^\infty(M, G)$.*

Proof. Since $\mathcal{I}G$ is an embedded submanifold of G, we can argue as in the proof of Proposition 2.7 to see that $\mathrm{vBis}(\mathcal{G}) \cong \mathrm{Bis}(\mathcal{I}G) \subseteq C^\infty(M, G)$ is a submanifold. Moreover, the manifold structure turns $\mathrm{vBis}(\mathcal{G})$ into a Lie group. To distinguish the new Lie group structure from the one inherited from Theorem 2.8, we write $\widetilde{\mathrm{vBis}}(\mathcal{G})$ for this Lie group. As $\widetilde{\mathrm{vBis}}(\mathcal{G})$ is an embedded submanifold of $C^\infty(M, G)$, we can argue as in the proof of (in particular part (b)) of Proposition 2.7 to see that $\widetilde{\mathrm{vBis}}(\mathcal{G})$ is an initial Lie subgroup of $\mathrm{Bis}(\mathcal{G})$. As any subgroup of a given Lie subgroup carries at most one structure as an initial Lie subgroup [26, Lemma II.6.2], we have $\mathrm{vBis}(\mathcal{G}) = \widetilde{\mathrm{vBis}}(\mathcal{G})$ as infinite-dimensional Lie groups.

Finally, assume that $\mathcal{G}$ is locally transitive, i.e., its anchor $(\alpha, \beta)\colon G \to M \times M$ is a submersion. Then $\mathcal{I}G = (\alpha, \beta)^{-1}(\Delta M)$ is an embedded submanifold of G, where ΔM is the diagonal embedded in $M \times M$. □

To provide a different geometric interpretation of the vertical bisections, we deviate from our usual convention and consider an infinite-dimensional Lie groupoid. In [31, Definition 2.1] we have constructed an (infinite-dimensional) action groupoid from the natural action of $\mathrm{Bis}(\mathcal{G})$ on M:

2.14. The group $\mathrm{Bis}(\mathcal{G})$ acts on M, via the natural action of $\mathrm{Diff}(M)$ on M composed with the morphism $\beta_*\colon \mathrm{Bis}(\mathcal{G}) \to \mathrm{Diff}(M)$, $\sigma \mapsto \beta \circ \sigma$. We can thus define an action Lie groupoid $\mathcal{B}(\mathcal{G}) := \mathrm{Bis}(\mathcal{G}) \ltimes M$, with source and target projections defined by $\alpha_{\mathcal{B}}(\sigma, m) = m$ and $\beta_{\mathcal{B}}(\sigma, m) = \beta(\sigma(m))$, respectively. The multiplication on $\mathcal{B}(\mathcal{G})$ is defined by

$$(\sigma, \beta_{\mathcal{G}}(\tau(m))) \cdot (\tau, m) := (\sigma \star \tau, m).$$

The Lie groupoid $\mathcal{B}(\mathcal{G})$ plays a crucial rôle in the reconstruction of the Lie groupoid $\mathcal{G}$ from its group of bisections (see [29, Section 2] for more information on this process). From the definition of the action Lie groupoid, one immediately obtains that the vertical bisections determine the isotropy subgroupoid, i.e., $\mathcal{I}\mathcal{B}(\mathcal{G}) = \mathrm{vBis}(\mathcal{G}) \ltimes M$.

2.15. As a final remark, the group $\mathrm{vBis}(\mathcal{G})$ can be generalised (as observed by H. Amiri) by considering $\{f \in C^\infty(G, G) \mid \alpha \circ f = \alpha = \beta \circ f,\ x \mapsto xf(x) \in \mathrm{Diff}(G)\}$. This set turns out to be a subgroup of the group $S_{\mathcal{G}}(\alpha)$ from [3] (which generalises $\mathrm{Bis}(\mathcal{G})$). Similar techniques to the ones in the present paper can be used to turn the generalised group into a Lie group.

A Infinite-dimensional calculus, Lie groups and Lie groupoids

In this appendix we recall some basic facts on the infinite-dimensional analysis used throughout the text. For more information, we refer the reader to [5], and the generalizations thereof (see [14] and [2], also [15, 21, 25], and [22]). As already remarked, we are working in the Bastiani calculus, where f is C^k-map if all iterated directional derivatives up to order k exist and are continuous.

Definition A.1. For a smooth map $f\colon M \to N$ between manifolds, we say (see [13, 15]) that f is
(i) a *submersion* if for each $x \in M$, we can choose a chart ψ of M around x and a chart ϕ of N around $f(x)$ such that $\phi \circ f \circ \psi^{-1}$ is the restriction of a continuous linear map with continuous linear right inverse,
(ii) an *immersion* if for every $x \in M$, there are charts such that we can always achieve that $\phi \circ f \circ \psi^{-1}$ is the restriction of a continuous linear map admitting a continuous linear left inverse,
(iii) an *embedding* if f is an immersion and a topological embedding.

Note that the above definitions (of submersions etc.) are adapted to the infinite-dimensional setting we are working in. In general, they are not equivalent to the usual characterisations known from the finite-dimensional setting. See, e.g., [13].

A.2 (Submanifolds). Let M be a C^k-manifold. A subset $N \subseteq M$ is called a *submanifold* if, for each $x \in N$, there exists a chart $\phi\colon U_\phi \to V_\phi \subseteq E_\phi$ of M with $x \in U_\phi$, and a closed vector subspace $F \subseteq E_\phi$ such that

$\phi(U_\phi \cap N) = V_\phi \cap F$. Then N is a C^k-manifold in the induced topology, using the charts $\phi|_{U_\phi \cap N} : U_\phi \cap N \to V_\phi \cap F$.

A.3. If N is a submanifold of a smooth manifold M and $f: L \to M$ a map on a smooth manifold L such that $f(L) \subseteq N$, then f is smooth if and only if its corestriction $f|^N : L \to N$ is smooth for the smooth manifold structure induced on N.

Lie groups and Lie groupoids

We follow here [14, 22, 26] for the basic theory concerning infinite-dimensional Lie groups (modeled on locally convex spaces) and [19, 20] for (finite-dimensional) Lie groupoids and Lie algebroids.[6]

Remark A.4. For a Lie group G, we write $\mathbf{1}$ for the unit element. As in the finite-dimensional setting, one can associate to G a Lie algebra $\mathbf{L}(G) \cong T_{\mathbf{1}} G$ whose Lie bracket is constructed from the Lie bracket of left invariant vector fields on G.

Definition A.5. Let G, H be (infinite-dimensional) Lie groups and let $\varphi: H \to G$ be an injective morphism of Lie groups. We call H an *initial Lie subgroup* if the induced Lie algebra morphism $\mathbf{L}(\varphi): \mathbf{L}(H) \to \mathbf{L}(G)$ is injective, and for each C^k-map $f: N \to G$ ($k \in \mathbb{N} \cup \{\infty\}$) from a C^k-manifold N to G whose image $\mathrm{im}(f)$ is contained in H, the corresponding map $\varphi^{-1} \circ f: N \to H$ is C^k.

We furthermore need the following construction principle for Lie groups, whose proof for manifolds modeled on Banach spaces (which generalises verbatim to our more general setting) can be found in [7, Chapter III, Section 1.9, Proposition 18].

Proposition A.6. *Let G be a group and let U, V be subsets of G such that $\mathbf{1} \in V = V^{-1}$ and $VV \subseteq U$. Suppose that U is equipped with a smooth manifold structure such that V is open in U, which turns the inversion $\iota: V \to V \subseteq U$ and the multiplication $\mu: V \times V \to U$ – induced by the group – into smooth maps. Then the following hold:*

(i) *There is a unique smooth manifold structure on the subgroup $G_0 := \langle V \rangle$ of G generated by V such that G_0 becomes a Lie group, V is open in G_0, and such that U and G_0 induce the same smooth manifold structure on V.*

(ii) *Assume that for each g in a generating set of G, there is an open identity neighborhood $W \subseteq U$ such that $gWg^{-1} \subseteq U$ and $c_g: W \to U, h \mapsto ghg^{-1}$ is smooth. Then there is a unique smooth manifold structure on G turning G into a Lie group such that V is open in G and both G and U induce the same smooth manifold structure on the open subset V.*

A.7. A groupoid $\mathcal{G} = (G \rightrightarrows M)$, with source map $\alpha: G \to M$ and target map $\beta: G \to M$, is a *Lie groupoid* if the following holds: G and M are smooth manifolds, α and β are C^∞-submersions and the multiplication map $G^{(2)} \to G$, the inversion map $G \to G$ and the identity-assigning map $M \to G$, $x \mapsto \mathbf{1}_x$, are smooth. Recall that one can associate to every Lie groupoid $\mathcal{G}$ a Lie algebroid, which we denote by $\mathbf{L}(\mathcal{G})$.

Definition A.8. Let $F: \mathcal{H} \to \mathcal{G}$ be a morphism of Lie groupoids. We call $\mathcal{H}$

- *immersed subgroupoid* of $\mathcal{G}$ if F and the induced map on the base are injective immersions,
- *embedded subgroupoid* of $\mathcal{G}$ if F and the induced map on the base are embeddings.

Definition A.9. For a Lie groupoid $\mathcal{G} = (G \rightrightarrows M)$ we define the following:

- Denote, for $m \in M$, by C_m the connected component of $\mathbf{1}_m$ in $\alpha^{-1}(m)$. Then

$$C(\mathcal{G}) := \bigcup_{n \in M} C_m.$$

6 The concept of infinite-dimensional Lie groupoid is clear, cf. [29, 31] and [6], though not needed for most of the text.

By [19, Proposition 1.5.1], we obtain a wide Lie subgroupoid $C(\mathcal{G}) \rightrightarrows M$ of $\mathcal{G}$, called the *identity-component subgroupoid* of $\mathcal{G}$.

- The *isotropy subgroupoid* $\mathcal{I}G := \{g \in G \mid \alpha(g) = \beta(g)\}$, and the identity component subgroupoid of the isotropy groupoid $\mathcal{I}G^\circ := C(\mathcal{I}G)$.

Endowed with the subspace topology, $\mathcal{I}G$ (and also $\mathcal{I}G^\circ$) are topological bundles of Lie groups. In general, the isotropy subgroupoid is not a Lie subgroupoid:

Example A.10. Let $\mathcal{A} = (\mathbb{S}^1 \times \mathbb{R}^2 \rightrightarrows \mathbb{R}^2)$ be the action groupoid associated to the canonical action of the circle group $\mathbb{S}^1$ on $\mathbb{R}^2$ via rotation. Then

$$\mathcal{I}\mathcal{A} = \mathbb{S}^1 \times \{0\} \cup \bigsqcup_{x \in \mathbb{R}^2 \setminus \{0\}} \{1\} \times \{x\} \subseteq \mathbb{S}^1 \times \mathbb{R}^2$$

is not a submanifold of $\mathbb{S}^1 \times \mathbb{R}^2$, and thus $\mathcal{I}\mathcal{A}$ cannot be a Lie groupoid of $\mathcal{A}$.

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7 cf. https://mathoverflow.net/questions/329939/isotropy-subgroupoid-of-a-regular-lie-groupoid

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(Re)constructing Lie groupoids from their bisections and applications to prequantisation

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A B S T R A C T

This paper is about the relation of the geometry of Lie groupoids over a fixed compact manifold M and the geometry of their (infinite-dimensional) bisection Lie groups. In the first part of the paper we investigate the relation of the bisections to a given Lie groupoid, while the second part is about the construction of Lie groupoids from candidates for their bisection Lie groups. The procedure of this second part becomes feasible due to some recent progress in the infinite-dimensional Frobenius theorem, which we heavily exploit. The main application to the prequantisation of (pre)symplectic manifolds comes from an integrability constraint of closed Lie subalgebras to closed Lie subgroups. We characterise this constraint in terms of a modified discreteness conditions on the periods of that manifold.

0. Introduction

The Lie group structure on bisection Lie groups was established in [28,31], along with a smooth action of the bisections on the arrow manifold of a Lie groupoid. In this paper, we develop a tight relation between Lie groupoids and their bisection Lie groups by making use of this action. As a first step, we show how this action can be used to reconstruct a Lie groupoid from its bisections. In general, this will not be possible, since there may not be enough bisections for a reconstruction to work. However, under mild assumptions on the Lie groupoid, e.g., the groupoid being source connected, the reconstruction works quite well. It is

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worthwhile to note that in Chapter 2 the analytical tools we need are of moderate complexity, all results follow from a thorough usage of the concept of a submersion.

In the intermediate Chapter 3, we analyse in some more detail the structure that bisections have in addition to merely being Lie groups. For instance, they act on M and the stabilisers are closely related to the vertex groups of the Lie groupoid. To this end, we use the results of Glöckner on submersion properties in infinite-dimensions [6,7].

The next step is then to take the insights from Chapter 2 and Chapter 3 in order to formulate structures and conditions on a Lie group that turn it into a Lie group of bisections of a Lie groupoid. This becomes analytically more challenging, and the procedure is only possible due to a heavy usage of the recent results from [7], building on a generalised Frobenius Theorem of integrable co-Banach distributions [3,10]. The result, developed in Chapter 4, is the concept of a transitive pair, which describes in an efficient way the necessary structure that is needed on a Lie group in order to relate it in a natural way to the bisection group of a Lie groupoid. However, we restrict here to transitive (respectively locally trivial) Lie groupoids, the general theory will be part of future research.

Finally, in Chapter 5 we then apply the integration theory for abelian extensions from [23] in order to derive a transitive pair from an integration of an extension of $\mathcal{V}(M)$, given by a closed 2-form $\omega \in \Omega^2(M)$ to an extension of $\mathrm{Diff}(M)_0$. The crucial point here is the integration of a certain Lie subalgebra to a closed Lie subgroup of the integrated extension, for which we derive a new discreteness condition in terms of the associated period groups.

We now go into some more detail and explain the main results. Suppose $\mathcal{G} = (G \rightrightarrows M)$ is a Lie groupoid. This means that G, M are smooth manifolds, equipped with submersions $\alpha, \beta\colon G \to M$ and an associative and smooth multiplication $G \times_{\alpha,\beta} G \to G$ that admits a smooth identity map $1\colon M \to G$ and a smooth inversion $\iota\colon G \to G$. Then the bisections $\mathrm{Bis}(\mathcal{G})$ of $\mathcal{G}$ are the sections $\sigma\colon M \to G$ of α such that $\beta \circ \sigma$ is a diffeomorphism of M. This becomes a group with respect to

$$(\sigma \star \tau)(x) := \sigma((\beta \circ \tau)(x))\tau(x) \text{ for } x \in M.$$

This group is an (infinite-dimensional) Lie group (cf. [31]) if M is compact, G is modelled on a metrisable space and the groupoid $\mathcal{G}$ admits a local addition adapted to the source projection α, i.e. it restricts to a local addition on each fibre $\alpha^{-1}(x)$ for $x \in M$ (cf. [17,31]).

By construction of the Lie group structure, we obtain a natural action $\mathrm{Bis}(\mathcal{G}) \times M \to M$, $(\sigma, m) \mapsto \beta(\sigma(m))$ of $\mathrm{Bis}(\mathcal{G})$ on M. This action gives rise to the bisection action groupoid $\mathcal{B}(\mathcal{G}) := (\mathrm{Bis}(\mathcal{G}) \ltimes M \rightrightarrows M)$. Observe that the action is constructed from the joint evaluation map

$$\mathrm{ev}\colon\ \mathrm{Bis}(\mathcal{G}) \times M \to G, \quad (\sigma, m) \mapsto \sigma(m)$$

and the target projection of the groupoid $\mathcal{G}$. While the target projection is a feature of the groupoid $\mathcal{G}$, the evaluation map yields a groupoid morphism over M from $\mathcal{B}(\mathcal{G})$ to $\mathcal{G}$. Thus information on the groupoid can be recovered from the group of bisections and the base manifold via the joint evaluation map. The idea is to recover the smooth structure of the arrow manifold from the evaluation map based on the following result:

Theorem A. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid, where M is compact, G is modelled on a metrisable space and $\mathcal{G}$ admits an α-adapted local addition. Then the joint evaluation* ev *is a submersion. If $\mathcal{G}$ is in addition source-connected, i.e. the source fibres are connected manifolds, then* ev *is surjective.* □

Note that as an interesting consequence of Theorem A we obtain also information on the evaluation of smooth maps from a compact manifold M into a manifold N modelled on a metrisable space. In this case, the evaluation map

$$\mathrm{ev}\colon C^\infty(M,N)\times M\to N,\quad (f,m)\mapsto f(m)$$

is a surjective submersion.

A crucial point of our approach will be that the joint evaluation map is a surjective submersion. Unfortunately, this will not be the case in general as there are Lie groupoids without enough bisections (see Remark 2.18 b) for an example). In this case there is no hope to recover the manifold of arrows, whence not all information on the groupoid is contained in its group of bisections. However, it turns out that at least the identity subgroupoid can always be reconstructed. Moreover, Theorem A still gives a sufficient criterion for the surjectivity of ev. It is sufficient that the Lie groupoid is source connected, and in general this condition can not be dispensed with. As a byproduct, we obtain generalisations of some results about the existence of global bisections through each point (cf. [37]) to infinite dimensions. Hence we can reconstruct the groupoid from its bisections and the base manifold for a fairly broad class of Lie groupoids. Namely, we obtain the following reconstruction result.

Theorem B. *Let $\mathcal{G}=(G\rightrightarrows M)$ be a Lie groupoid, where M is compact, G is modelled on a metrisable space and $\mathcal{G}$ admits an α-adapted local addition. If the joint evaluation map* ev *is surjective, e.g. $\mathcal{G}$ is source-connected, then the Lie groupoid morphism*

$$\mathrm{ev}\colon \mathcal{B}(\mathcal{G})=(\mathrm{Bis}(\mathcal{G})\ltimes M\rightrightarrows M)\to\mathcal{G}$$

is the groupoid quotient of $\mathcal{B}(\mathcal{G})$ by a normal Lie subgroupoid. In this case, the Lie group of bisections and the manifold M completely determine the Lie groupoid $\mathcal{G}$. □

Note that Theorem B really is a *re*construction theorem, i.e., we need a Lie groupoid to begin with as a candidate for the quotient. Hence, the original groupoid is needed and Theorem B does not provide a way to construct $\mathcal{G}$ without knowing it beforehand. The problem here is twofold. At first, we need to know the kernel of ev, or some equivalent information, that allows us to determine the groupoid $\mathcal{G}$ as a quotient of $(\mathrm{Bis}(\mathcal{G})\ltimes M\rightrightarrows M)$. The other problem is an analytical problem: quotients of (infinite-dimensional) Lie groupoids and Lie groups usually do not admit a suitable smooth structure. In particular, it is not known if the familiar tools, e.g. Godement's criterion, carry over to the infinite-dimensional setting beyond the Banach setting.

Nevertheless, one can extract some information on the quotient from Theorem B. The groupoid quotient is controlled by certain subgroups of the group of bisections which arise from the kernel of the joint evaluation. Namely, the source fibre over m in the kernel corresponds to the Lie subgroup $\mathrm{Bis}_m(\mathcal{G})=\{\sigma\in\mathrm{Bis}(\mathcal{G})\mid\sigma(m)=1_m\}$ of all bisections which take m to the corresponding unit. Observe that $\mathrm{Bis}_m(\mathcal{G})$ sits inside the Lie subgroup $\mathrm{Loop}_m(\mathcal{G}):=\{\sigma\in\mathrm{Bis}(\mathcal{G})\mid\beta(\sigma(m))=m\}$ of all elements whose image at m is an element in the vertex group. Both subgroups will later turn out to be important in the construction of Lie groupoids from their bisections. In this context Lie theoretic properties, like regularity in the sense of Milnor, of these subgroups are crucial to our approach. Thus we first study them in a separate section. Regularity (in the sense of Milnor) of a Lie group roughly means that a certain class of differential equations can be solved on the Lie group. Many familiar results from finite-dimensional Lie theory carry over only to regular Lie groups (cf. [8] for a survey). Our results then subsume the following theorem.

Theorem C. *Let $\mathcal{G}=(G\rightrightarrows M)$ be a Banach Lie groupoid, then for each $m\in M$ the inclusions*

$$\mathrm{Bis}_m(\mathcal{G})\subseteq\mathrm{Loop}_m(\mathcal{G})\subseteq\mathrm{Bis}(\mathcal{G})$$

turn $\mathrm{Bis}_m(\mathcal{G})$ and $\mathrm{Loop}_m(\mathcal{G})$ into split Lie subgroups which are regular in the sense of Milnor. As submanifolds these subgroups are even co-Banach submanifolds. □

These Lie subgroups are of interest, as the quotient $\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ reconstructs the source fibre of $\mathcal{G}$ over the point m and $\mathrm{Loop}_m(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ reconstructs the vertex group at m. Moreover, Theorem C enables us to construct a natural smooth structure on these quotients which coincides a posteriori with the manifold structure on the fibre and the vertex group, respectively.

We now use the results obtained so far to turn the reconstruction result given by Theorem B into a construction result, at least in the locally trivial case. This means that we start with Lie groups, some extra structure on them and then produce a Lie groupoid such that the groups are related to the Lie group of bisections. Recall that a locally trivial Lie groupoid is completely determined by its source fibre and the vertex group over a given point. Hence, if we fix a point $m \in M$, the problem to construct a Lie groupoid reduces in the locally trivial case to reconstructing a manifold (modelling the source fibre) and a Lie group (modelling the vertex group). This motivates the notion of a *transitive pair* (cf. Definition 4.1). A transitive pair (θ, H) consists of a transitive Lie group action $\theta \colon K \times M \to M$ and a normal subgroup H of the m-stabiliser K_m of θ, such that H is a regular and co-Banach Lie subgroup of K_m. The guiding example is here the transitive pair $(\beta_{\mathcal{G}} \circ \mathrm{ev} \colon \mathrm{Bis}(\mathcal{G}) \times M \to M, \mathrm{Bis}_m(\mathcal{G}))$ induced by the bisections of a locally trivial Lie groupoid $\mathcal{G}$ with connected base M. The notion of transitive pair can be thought of as a generalisation of a Klein geometry for fibre-bundles (see Remark 4.19 for further information). We then obtain the following construction principle.

Theorem D. *Let $(\theta \colon K \times M \to M, H)$ be a transitive pair. Then there is a locally trivial Banach Lie groupoid $\mathcal{R}(\theta, H)$ together with a Lie group morphism $a_{\theta,H} \colon K \to \mathrm{Bis}(\mathcal{R}(\theta, H))$.* If $(\theta, H) = (\beta_{\mathcal{G}} \circ \mathrm{ev}, \mathrm{Bis}_m(\mathcal{G}))$ for some locally trivial Banach Lie groupoid $\mathcal{G}$, then $a_{\theta,H}$ is an isomorphism. □

One should think of the construction principle from Theorem D as an analogue of the reconstruction result in Theorem B for locally trivial Lie groupoids. The crucial difference here is that Theorem D makes no reference to the Lie groupoid, but constructs it purely from the given transitive pair. Note that the Lie group morphism $a_{\theta,H}$ in Theorem D will in general not be an isomorphism. However, the morphisms are interesting in their own right due to the fact that the definition of a transitive pair is quite flexible. It allows us to construct for a wide range of Lie groups with transitive actions on M Lie group morphisms into the Lie group of bisections $\mathrm{Bis}(\mathcal{R}(\theta, H))$. Moreover, these Lie group morphisms carry geometric information and thus connect the group actions of both Lie groups.

In the last section we then invoke the integration theory of abelian extensions of infinite-dimensional Lie algebras from [23] to construct transitive pairs from a closed 2-form ω on a 1-connected and compact manifold M. We formulate the results here for $\mathrm{Diff}(M)_0$, whereas in the text we allow for more general $K \leq \mathrm{Diff}(M)$. If ω is prequantisable, then the prequantisation provides a gauge groupoid and thus an integration of the Lie algebroid extension. By the results from [31] and [23], the associated Lie algebra extension also integrates to an extension of Lie groups. In the other direction, we show that the integration of the extensions of Lie algebras to transitive pairs is in fact a two-step process. The first step is concerned with the integration of the extension

$$C^\infty(M) \to C^\infty(M) \oplus_{\overline{\omega}} \mathcal{V}(M) \to \mathcal{V}(M) \tag{1}$$

of Lie algebras to an extension of Lie groups, where $\overline{\omega}$ is the abelian cocycle $(X, Y) \mapsto \omega(X, Y)$. By the results of [23], the integration of (1) is governed by the discreteness of the *primary* periods, i.e., the periods of the Lie group $\mathrm{Diff}(M)_0$ for the equivariant extension $\overline{\omega}^{\mathrm{eq}}$ of $\overline{\omega}$. The second step is then the integration of the Lie subalgebra of $C^\infty(M) \oplus_{\overline{\omega}} \mathcal{V}(M)$, that corresponds to the vector fields vanishing in the base-point, to a closed Lie subgroup. This is governed by the discreteness of the *secondary* periods, i.e., the periods of (M, ω) modulo the periods of $(\mathrm{Diff}(M)_0, \overline{\omega}^{\mathrm{eq}})$ (see Remark 5.5 and Remark 5.10 for a precise definition).

Theorem E. *Let M be a compact and 1-connected manifold with base-point m and $\omega \in \Omega^2(M)$ be closed. If the extension (1) integrates to an extension of Lie groups, then (M, ω) is prequantisable if and only if the secondary periods are discrete. The latter is equivalent to the integrability of the subalgebra $C^\infty_m(M) \oplus_{\overline{\omega}} \mathcal{V}_m(M)$ to a closed Lie subgroup, where $C^\infty_m(M)$ and $\mathcal{V}_m(M)$ denote the functions (respectively vector fields) that vanish in m.* □

Finally, we would like to remark that the constructions of Lie groupoids given in the present paper yield functors on suitable categories of Lie groups and Lie groupoids. These functors are closely connected to the bisection functor. However, there is no need for these results in the present paper as we are only concerned with the (re-)construction of Lie groupoids. Thus we will largely avoid categorical language and postpone a detailed investigation of these functors to [32].

1. Locally convex Lie groupoids and the Lie group of bisections

In this section we recall basic facts and conventions on Lie groupoids and bisections used in this paper. We refer to [16] for an introduction to (finite-dimensional) Lie groupoids and the associated group of bisections. The notation for Lie groupoids and their structural maps also follows [16]. However, we do not restrict our attention to finite dimensional Lie groupoids. Hence, we have to augment the usual definitions with several comments. Note that we will work all the time over a fixed base manifold M.

Definition 1.1. Let $\mathcal{G} = (G \rightrightarrows M)$ be a groupoid over M with source projection $\alpha\colon G \to M$ and target projection $\beta\colon G \to M$. Then $\mathcal{G}$ is a *(locally convex and locally metrisable) Lie groupoid over M*[1] if

- the objects M and the arrows G are locally convex and locally metrisable manifolds,
- the smooth structure turns α and β into surjective submersions, i.e., they are locally projections[2]
- multiplication $m\colon G \times_{\alpha,\beta} G \to G$, object inclusion $1\colon M \to G$ and inversion $\iota\colon G \to G$ are smooth. □

Definition 1.2. The *group of bisections* $\mathrm{Bis}(\mathcal{G})$ of $\mathcal{G}$ is given as the set of sections $\sigma\colon M \to G$ of α such that $\beta \circ \sigma\colon M \to M$ is a diffeomorphism. This is a group with respect to

$$(\sigma \star \tau)(x) := \sigma((\beta \circ \tau)(x))\tau(x) \text{ for } x \in M. \tag{2}$$

The object inclusion $1\colon M \to G$ is then the neutral element and the inverse element of σ is

$$\sigma^{-1}(x) := \iota(\sigma((\beta \circ \sigma)^{-1}(x))) \text{ for } x \in M. \qquad \square \tag{3}$$

In [31] we have established a Lie group structure on the group of bisections of a (locally convex) Lie groupoid which admits a certain type of local addition. To understand the Lie group structure on $\mathrm{Bis}(\mathcal{G})$ one uses local additions (cf. Definition B.7) which respect the fibres of a submersion. This is an adaptation of the construction of manifold structures on mapping spaces [35,13,17] (see also Appendix B).

Definition 1.3. We say that a Lie groupoid $\mathcal{G} = (G \rightrightarrows M)$ admits an *adapted local addition* if G admits a local addition which is adapted to the source projection α (or, equivalently, to the target projection β). □

Recall from [31, Section 3] the following facts on the Lie group structure of the group of bisections:

[1] See Appendix B for references on differential calculus in locally convex spaces.
[2] This implies that the occurring fibre-products are submanifolds of the direct products, see [35, Appendix C].

Theorem 1.4. *Suppose M is compact and $\mathcal{G} = (G \rightrightarrows M)$ is a locally convex and locally metrisable Lie groupoid over M which admits an adapted local addition. Then* $\mathrm{Bis}(\mathcal{G})$ *is a submanifold of* $C^\infty(M, G)$ *(with the manifold structure from Theorem B.9). Moreover, the induced manifold structure and the group multiplication*

$$(\sigma \star \tau)(x) := \sigma((\beta \circ \tau)(x))\tau(x) \text{ for } x \in M$$

turn $\mathrm{Bis}(\mathcal{G})$ *into a Lie group modelled on* $E_1 := \{\gamma \in C^\infty(M, TG) \mid \forall x \in M,\ \gamma(x) \in T_{1_x}\alpha^{-1}(x)\}$. □

As the Lie group $\mathrm{Bis}(\mathcal{G})$ is a submanifold of $C^\infty(M, G)$ the exponential law for smooth maps B.9 c) applies to maps defined on $\mathrm{Bis}(\mathcal{G})$. In particular, as in [31, Proposition 3.11] one easily concludes that the natural action of the bisections on the arrows and the evaluation of bisections are smooth:

Proposition 1.5. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid such that M is a compact manifold and $\mathcal{G}$ admits an adapted local addition. Then*

a) *the natural action* $\gamma \colon \mathrm{Bis}(\mathcal{G}) \times G \to G, (\psi, g) \mapsto \psi(\beta(g)) \cdot g$ *is smooth and for $g \in G$ the restricted action*

$$\gamma_g \colon \mathrm{Bis}(\mathcal{G}) \to \alpha^{-1}(\alpha(g)), \psi \mapsto \psi(\beta(g)) \cdot g$$

is smooth;

b) *the evaluation map* $\mathrm{ev} \colon \mathrm{Bis}(\mathcal{G}) \times M \to G, (\sigma, m) \mapsto \sigma(m)$ *is smooth and satisfies*

$$\mathrm{ev}(\sigma, m) = \sigma(m) = \gamma(\sigma, 1_m). \tag{4}$$

Furthermore, the map $\mathrm{ev}_m \colon \mathrm{Bis}(\mathcal{G}) \to \alpha^{-1}(m), \sigma \mapsto \sigma(m)$ *is smooth for each $m \in M$.*

2. Reconstruction of the Lie groupoid from its group of bisections

In this section, the close link between Lie groupoids and their Lie groups of bisections is established. More precisely, we will show that each Lie groupoid is the quotient of its Lie group of bisections. In the end, we discuss some examples and applications of this link.

Throughout this section assume that $\mathcal{G} = (G \rightrightarrows M)$ is a locally metrisable Lie groupoid that admits an adapted local addition and that has a compact space of objects M.

Definition 2.1. The Lie group $\mathrm{Bis}(\mathcal{G})$ acts on M, via the action induced by $(\beta_\mathcal{G})_* \colon \mathrm{Bis}(\mathcal{G}) \to \mathrm{Diff}(M)$ and the natural action of $\mathrm{Diff}(M)$ on M. This gives rise to the action Lie groupoid $\mathcal{B}(\mathcal{G}) := \mathrm{Bis}(\mathcal{G}) \ltimes M$, with source and target projections defined by $\alpha_\mathcal{B}(\sigma, m) = m$ and $\beta_\mathcal{B}(\sigma, m) = \beta_\mathcal{G}(\sigma(m))$. The multiplication on $\mathcal{B}(\mathcal{G})$ is defined by

$$(\sigma, \beta_\mathcal{G}(\tau(m))) \cdot (\tau, m) := (\sigma \star \tau, m).$$

Clearly, any morphism $f \colon \mathcal{G} \to \mathcal{H}$ of Lie groupoids over M induces a morphism $f_* \times \mathrm{id}_M \colon \mathcal{B}(\mathcal{G}) \to \mathcal{B}(\mathcal{H})$ of Lie groupoids. □

Remark 2.2. The Lie groupoid $\mathcal{B}(\mathcal{G})$ admits an adapted local addition (for $\alpha_\mathcal{B}$ and thus also for $\beta_\mathcal{B}$). In fact, this is the case for the Lie group $\mathrm{Bis}(\mathcal{G})$ and the finite-dimensional manifold M separately (cf. [13, p. 441]), and on $\mathrm{Bis}(\mathcal{G}) \times M$ one can simply take the product of these local additions. In addition, the evaluation map $\mathrm{ev} \colon \mathrm{Bis}(\mathcal{G}) \times M \to G$ is a morphism of Lie groupoids over M

$$\begin{pmatrix} \mathrm{Bis}(\mathcal{G}) \times M \\ \alpha_{\mathcal{B}} \downarrow \downarrow \beta_{\mathcal{B}} \\ M \end{pmatrix} \xrightarrow{\mathrm{ev}} \begin{pmatrix} G \\ \alpha_{\mathcal{G}} \downarrow \downarrow \beta_{\mathcal{G}} \\ M \end{pmatrix}. \qquad \square$$

We will now study the analytic properties of the morphism $\mathrm{ev}\colon \mathrm{Bis}(\mathcal{G}) \times M \to G$. It will turn out that under mild assumptions on $\mathcal{G}$ the groupoid is a groupoid quotient of the action groupoid $\mathcal{B}(\mathcal{G})$ via the evaluation map. The key point to establish this result is to prove that ev actually induces a quotient map, i.e. we need ev to be a surjective submersion. Let us first deal with some preparations:

Remark 2.3. Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid and fix $m \in M$ and $\tau \in \mathrm{Bis}(\mathcal{G})$. Our goal is to split the space of sections $\Gamma(\tau^* T^\alpha G)$ into a product of (closed) subspaces. To this end define

$$\Gamma(\tau^* T^\alpha G)_m := \{X \in \Gamma(\tau^* T^\alpha G \mid X(m) = 0_{\tau(m)}\}.$$

Choose a bundle trivialisation $\lambda\colon M_\lambda \to \tau^*\pi^\alpha(M_\lambda) \times E$ of the bundle $\tau^*\pi^\alpha\colon \tau^* T^\alpha G \to M$ such that $m \in \tau^*\pi^\alpha(M_\lambda)$. Since M is compact, we can choose a smooth cut-off function $\rho\colon M \to [0,1]$ with $\rho_\lambda(m) = 1$ and $\rho|_{M \setminus \pi^\alpha(M_\lambda)} \equiv 0$. Then we obtain a (non-canonical) isomorphism of topological vector spaces

$$I_\lambda\colon \Gamma(\tau^* T^\alpha G) \to \Gamma(\tau^* T^\alpha G)_m \times T^\alpha_{\tau(m)} G, \quad X \mapsto (X - (\rho \circ \tau^*\pi^\alpha) \cdot \lambda^{-1}(\tau^*\pi^\alpha, \mathrm{pr}_2 \circ \lambda(X(m))), X(m)), \quad (5)$$

its inverse is given by

$$I_\lambda^{-1}(X_0, y) := X_0 + (\rho \circ \tau^*\pi^\alpha) \cdot \lambda^{-1}(\tau^*\pi^\alpha, \mathrm{pr}_2 \circ \lambda(y)) \quad (6)$$

This turns $\Gamma(\tau^* T^\alpha G)_m$ into a complemented subspace of $\Gamma(\tau^* T^\alpha G)$. Moreover, if $\mathcal{G}$ is a Banach–Lie groupoid then $T^\alpha_{\tau(m)} G$ is a Banach space and thus $\Gamma(\tau^* T^\alpha G)_m$ turns into a co-Banach subspace of $\Gamma(\tau^* T^\alpha G)$. $\square$

Let us establish the submersion property for the restricted action of the group of bisections on the manifold of arrows.

Proposition 2.4. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid and $g \in G$. Then the restricted action*

$$\gamma_g\colon\ \mathrm{Bis}(\mathcal{G}) \to \alpha^{-1}(\alpha(g)), \sigma \mapsto \gamma(\sigma, g) = \sigma(\beta(g)).g$$

is a submersion.

Proof. From Proposition 1.5 a) we infer that γ_g is smooth. Hence, we only have to prove that γ_g is locally a projection. To see this fix $\tau \in \mathrm{Bis}(\mathcal{G})$ and a chart $\kappa\colon U_\kappa \to V_\kappa \subseteq E$ of the manifold $\alpha^{-1}(\alpha(g))$ such that $\gamma_g(\tau) \in U_\kappa$. Furthermore, choose a bundle trivialisation λ of $\tau^* T^\alpha G$ and construct the vector space isomorphism (5) for λ.

Now consider the canonical chart (O_τ, φ_τ) of $\mathrm{Bis}(\mathcal{G})$. The set $I_\lambda \circ \varphi_\tau(O_\tau)$ is an open zero-neighbourhood in $\Gamma(\tau^* T^\alpha G)_{\beta(g)} \times T^\alpha_{\tau(\beta(g))} G$. Shrinking O_τ, we can assume that $O_\tau \subseteq \gamma_g^{-1}(U_\kappa)$ and that there are open zero-neighbourhoods $U \subseteq \Gamma(\tau^* T^\alpha G)_{\beta(g)}$ and $W \subseteq T^\alpha_{\tau(\beta(g))} G$ such that $U \times W = I_\lambda \circ \varphi_\tau(O_\tau)$. In conclusion, we obtain a commutative diagram

$$\begin{array}{ccccc} \mathrm{Bis}(\mathcal{G}) \supseteq O_\tau & \xrightarrow{\gamma_g} & \alpha^{-1}(\alpha(g)) & \xrightarrow{\kappa} & E \\ {\scriptstyle I_\lambda \circ \varphi_\tau}\downarrow & & & & \Vert \\ U \times W & & \xrightarrow{\tilde{\gamma}_g} & & E \end{array}$$

with $\tilde{\gamma}_g := \kappa \circ \gamma_g \circ (I_\lambda \circ \varphi_\tau)^{-1}|_{U \times W}$. Denote by Σ the adapted local addition of $\mathcal{G}$ and consider the right translation $R_g \colon \alpha^{-1}(\beta(g)) \to \alpha^{-1}(\alpha(g)), h \mapsto hg$. Then (6) and the definition of φ_τ^{-1} yield

$$\begin{aligned}\tilde{\gamma}_g(X, y) &= \kappa(\gamma_g(\Sigma(X + (\rho \circ \pi^\alpha) \cdot \lambda^{-1}(\pi^\alpha, \mathrm{pr}_2\, \lambda(y)))) = \kappa(\Sigma(\underbrace{X(\beta(g))}_{=0} + \underbrace{(\rho(\beta(g)))}_{=1} \cdot y).g) \\ &= \kappa(\Sigma(y).g) = \kappa \circ R_g \circ \Sigma(y).\end{aligned} \tag{7}$$

Note that by (7) the map $\tilde{\gamma}_g$ does neither depend on the choice of the trivialisation λ nor on the cut-off function ρ_λ.

By definition of the adapted local addition, Σ restricts to a diffeomorphism $W \to \Sigma(W) \subseteq \alpha^{-1}(\beta(g))$. Moreover, κ is a chart and the right translation R_g is a diffeomorphism. Hence $\psi := \kappa \circ R_g \circ \Sigma|_W \colon W \to E$ is a diffeomorphism onto its (open) image, mapping $W \subseteq T^\alpha_{\tau(\beta(g))}G$ to an open subset in E. Now (7) yields a commutative diagram

$$\begin{tikzcd}
\mathrm{Bis}(\mathcal{G}) \supseteq O_\tau \arrow[r, "\gamma_g"] \arrow[d, "(\mathrm{id}_U, \psi) \circ I_\lambda \circ \varphi_\tau"'] & \alpha^{-1}(\alpha(g)) \arrow[r, "\kappa"] & E \arrow[d, equal] \\
U \times \psi(W) \arrow[rr, "\mathrm{pr}_2"] & & E
\end{tikzcd}$$

from which we conclude that γ_g is on O_τ a projection. As $\tau \in \mathrm{Bis}(\mathcal{G})$ was chosen arbitrarily, γ_g is a submersion. $\square$

The following corollary is now an immediate consequence of (4) and Proposition 2.4:

Corollary 2.5. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid, then for each $m \in M$ the evaluation map*

$$\mathrm{ev}_m \colon \mathrm{Bis}(\mathcal{G}) \to \alpha^{-1}(m), \quad \sigma \mapsto \sigma(m)$$

is a submersion.

Furthermore, we can establish the existence of bisections through certain arrows which coincide with the object inclusion outside of pre-chosen open sets.

Note that in the following we will denote by the symbol "$U \Subset X$" that some set U is an open subset of the topological space X.

Lemma 2.6. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid and $m \in M$ fixed. For any m-neighbourhood $U \Subset M$ there exists a 1_m-neighbourhood $W \Subset \alpha^{-1}(m) \cap \beta^{-1}(U)$ such that for each $g \in W$ there is a bisection $\sigma_g \in \mathrm{Bis}(\mathcal{G})$ with $\sigma_g(m) = g$ and $\sigma_g(y) = 1_y$ for all $y \in M \setminus U$.*

Proof. Choose a C^∞-function $\lambda \colon M \to [0, 1]$ such that $\lambda(m) = 1$ and $\lambda|_{M \setminus U} \equiv 0$. We denote by

$$\varphi_1 \colon \Gamma(1^* T^\alpha G) \supseteq \Omega \to \mathrm{Bis}(\mathcal{G}), X \mapsto \Sigma \circ X$$

the canonical manifold chart of $\mathrm{Bis}(\mathcal{G})$ (see Theorem 1.4). Computing with local representatives, it is easy to see that the map $m_\lambda \colon \Gamma(1^* T^\alpha G) \to \Gamma(1^* T^\alpha G), X \mapsto (x \mapsto \lambda(x) \cdot X(x))$ is continuous linear. Hence, there is an open zero-neighbourhood $P \Subset \Omega \subseteq \Gamma(1^* T^\alpha G)$ with $m_\lambda(P) \subseteq \Omega$.

Now define $W := \mathrm{ev}_m(\varphi_1(P)) \cap \beta^{-1}(U)$. Observe that W is an open 1_m-neighbourhood in $\alpha^{-1}(m) \cap \beta^{-1}(U) \Subset \alpha^{-1}(m)$ since P is open and ev_m is a submersion by Corollary 2.5. Moreover, for $g \in W$ we have

$g = s_g(m)$ for some $s_g \in \varphi_1(P)$. Define $X_g := m_\lambda \circ \varphi_1^{-1}(s_g)$ for each $g \in \mathcal{W}$. By construction X_g is contained in Ω since m_λ takes the section $\varphi_1^{-1}(s_g) \in P$ to Ω. Hence $\sigma_g := \varphi_1(X_g)$ makes sense and is a bisection of $\mathcal{G}$. From $\lambda(m) = 1$ we derive that $\sigma_g(m) = s_g(m) = g$. Moreover, for $x \in M \setminus U$ we obtain by definition of a local addition

$$\sigma_g(x) = \Sigma(\underbrace{\lambda(x)}_{=0}\varphi_1^{-1}(s_g)(x)) = \Sigma(0_{T_{1_x}^\alpha G}) = 1_x.$$

Hence $\sigma_g(x) = 1_x$ for all $x \in M \setminus U$. □

We can now prove a variant of [37, Theorem 3.2] for infinite-dimensional Lie groupoids over compact base. The proof of [37] carries over verbatim if one uses Lemma 2.6, whence we omit it.

Proposition 2.7. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid and $g \in \alpha^{-1}(m)$ for $m \in M$. Suppose that $W \Subset \alpha^{-1}(m)$ is connected and contains g and 1_m and there is $U \subseteq M$ with $\beta(W) \subseteq U$. Then there exists $\sigma_g \in \operatorname{Bis}(\mathcal{G})$ with $\sigma_g(m) = g$ and $\sigma_g(x) = 1_x$ for all $x \in M \setminus U$.*

Having dealt with the pointwise evaluation, we will now use the results obtained so far to also show that the *joint* evaluation is a submersion.

Proposition 2.8. *Let M be a compact manifold, Q be a locally metrisable manifold and $s\colon Q \to M$ be a submersion such that Q admits a local addition that is adapted to s. Then the joint evaluation map*

$$\operatorname{ev}\colon \Gamma(M \xleftarrow{s} Q) \times M \to Q, \quad (\sigma, m) \mapsto \sigma(m) \tag{8}$$

is a submersion. Here, $\Gamma(M \xleftarrow{s} Q) := \{\sigma \in C^\infty(M, Q) \mid s \circ \sigma = \mathrm{id}_M\}$ is the space of sections of s, which is a submanifold of $C^\infty(M, Q)$ by [31, Proposition 3.6].

Proof. Let $(\sigma_0, m_0) \in \Gamma(M \xleftarrow{s} Q) \times M$ be arbitrary, but fixed from now on. Set $q_0 := \sigma_0(m_0)$. Then there exist open neighbourhoods $O \subseteq M$ of m_0 and $P \subseteq Q$ of q_0 such that $\sigma_0(M) \subseteq P$ and $s|_{s^{-1}(O)} \cong \mathrm{pr}_1$. Indeed, there exist open neighbourhoods P' of q_0 and O' of m_0 such that $s|_{P'} \cong \mathrm{pr}_1$ and $s(P') = O'$. After shrinking O' if necessary, we may assume that $\sigma(O') \subseteq P'$. Then we choose O to be an open neighbourhood O with $\overline{O} \subseteq O'$ and set $P := s^{-1}(M \setminus \overline{O}) \cup P'$.

From this it follows that $C^\infty(M, P) \cap \Gamma(M \xleftarrow{s} Q)$ is an open neighbourhood of M. Observe that $s|_P : P \to M$ is also a submersion and that the adapted local addition on Q restricts to an adapted local addition on P. Consequently, the manifold structure on

$$C^\infty(M, P) \cap \Gamma(M \xleftarrow{s} Q) = \Gamma(M \xleftarrow{s|_P} P)$$

that is induced from $\Gamma(M \xleftarrow{s} Q)$ on the left hand side coincides with the manifold structure on the right hand side that is induced from applying [31, Proposition 3.6] to the submersion $s|_P$. Thus it suffices to consider the case where $Q = P$ and $s = s|_P$.

We now reduce the claimed submersion property of (8) to the case of the evaluation in m_0. Since the evaluation map $\mathrm{ev}_{m_0}\colon \operatorname{Diff}(M) \to M$ is also a submersion, there exists a local smooth section of it, i.e., an open neighbourhood $U_0 \subseteq M$ of m_0 and a smooth map $\beta\colon U_0 \to \operatorname{Diff}(M)$ such that $\beta(m_0) = \mathrm{id}_M$ and $\beta(m)(m_0) = m$ for all $m \in U_0$. Moreover, we may assume that $U_0 \subseteq O$ and β takes values in the identity neighbourhood $\varphi_{\mathrm{id}}(\Omega)$ where φ_{id} is the chart of $\operatorname{Diff}(M)$ from Theorem B.9 a).

Consider now $\varphi_{\mathrm{id}} \circ \beta\colon U_0 \to \Gamma(TM)$. We choose a and a compact m_0-neighbourhood K and a neighbourhood $\Omega \Subset TO$ of the zero-section which is mapped by the local addition on M (which was used to define φ_{id}) to O. For later use, we shrink Ω to achieve that $\Omega \cap T_x M$ is convex for each $x \in O$. Then the open set $\lfloor K, \Omega \rfloor := \{X \in \Gamma(TM) \mid X(K) \subseteq \Omega\} \Subset \Gamma(TM)$ is a zero-neighbourhood. Further, $\varphi_{\mathrm{id}}^{-1}$ maps $\lfloor K, \Omega \rfloor \cap (\varphi_{\mathrm{id}}^{-1})^{-1}(\mathrm{Diff}(M))$ to the set of diffeomorphisms which map K into O. Shrinking U_0 we can achieve that $\varphi_{\mathrm{id}} \circ \beta$ takes its image in $\lfloor K, \Omega \rfloor$, i.e. the diffeomorphisms in the image of β map K into O.

Apply now the exponential law [35, Corollary 7.5] (cf. Theorem B.9 c)) to obtain a smooth map

$$\gamma := (\varphi_{\mathrm{id}} \circ \beta)^{\vee} \colon U_0 \times M \to TM, (u, m) \mapsto \varphi_{\mathrm{id}} \circ \beta(u)(m) \text{ with } \gamma(\cdot, m) \in T_m M \quad \forall m \in M.$$

Note that $\gamma(m_0, \cdot)$ coincides with the zero-section as $\beta(m_0) = \mathrm{id}_M$. Choose a smooth cutoff function $\rho \colon M \to [0,1]$ which maps a m_0-neighbourhood $N \subseteq K$ to 1 and vanishes near the boundary ∂K. Multiplying fibre-wise we obtain a smooth map $\tilde{\gamma} \colon U_0 \times M \to TM, (u, m) \mapsto \rho(m) \cdot \gamma(u, m)$ which vanishes near the boundary of K. Apply the exponential law in reverse to obtain a smooth map $\tilde{\gamma}^{\wedge} \colon U_0 \to \Gamma(TM)$.

Now recall that $(\varphi_{\mathrm{id}}^{-1})^{-1}(\mathrm{Diff}(M))$ is an open set in the compact open C^1-topology on $\Gamma(TM)$ (see [17, 4.3] and [13, proof of Theorem 43.1]). Hence we can choose suitable convex zero-neighbourhoods which control only the values of $X \in \Gamma(TM)$ and TX on suitable compact sets, such that their intersection is contained in $(\varphi_{\mathrm{id}}^{-1})^{-1}(\mathrm{Diff}(M))$. Since $\gamma(m_0, \cdot)$ is the zero-section, we can thus shrink U_0 to achieve that $\tilde{\gamma}^{\wedge}$ is still contained in $\lfloor K, \Omega \rfloor \cap (\varphi_{\mathrm{id}}^{-1})^{-1}(\mathrm{Diff}(M))$. As $\tilde{\gamma}^{\wedge}$ vanishes near the boundary of K (and outside of K), we can replace β with a map which satisfies $\beta(m)(O) \subseteq O$ for all $m \in U_0$.

Choose another smooth cutoff function δ which vanishes near the boundary ∂U_0 and takes an open m_0-neighbourhood $U \subseteq N$ to 1. Multiplying (in local charts) $\varphi_{\mathrm{id}} \circ \beta$ with δ, we can assume that $\beta(m) = \mathrm{id}_M$ for all m near the boundary ∂U_0. Thus we can extend β to a smooth function $\beta \colon M \to \mathrm{Diff}(M)$ by setting $\beta(m) = \mathrm{id}_M$ if $m \notin U_0$. By construction this map satisfies $\beta(m)(O) \subseteq O$ for all $m \in M$. Moreover, since ρ and δ take the m_0-neighbourhood $U \cap N$ to 1, β satisfies $\beta(m)(m_0) = m$ for all $m \in U \cap N$.

Choose a diffeomorphism $\xi \colon s^{-1}(O) \to O \times W$ that makes

$$\begin{tikzcd} s^{-1}(O) \arrow[rr, "\xi"] \arrow[dr, "s"'] & & O \times W \arrow[dl, "\mathrm{pr}_1"] \\ & O & \end{tikzcd}$$

commute, we may lift each $\beta(m)$ to a diffeomorphism

$$\widetilde{\beta}_m \colon Q \to Q, \quad q \mapsto \begin{cases} \xi^{-1}(\beta(m)(o), w) & \text{if } q = \xi^{-1}(o, w) \in s^{-1}(O) \\ q & \text{if } q \notin s^{-1}(O) \end{cases}.$$

From this explicit construction it follows in particular, that the map

$$Q \times M \to Q, \quad (q, m) \mapsto \widetilde{\beta}(m)(q)$$

is smooth. This then gives rise to the diffeomorphism

$$B \colon \Gamma(M \xleftarrow{s} Q) \times M \to \Gamma(M \xleftarrow{s} Q) \times M, \quad (\sigma, m) \mapsto ((\widetilde{\beta}(m))^{-1} \circ \sigma \circ \beta(m), m),$$

with inverse $(\sigma, m) \mapsto (\widetilde{\beta}(m) \circ \sigma \circ (\beta(m))^{-1}, m)$. Moreover, $\beta(m_0) = \mathrm{id}_M$ implies $B(\sigma_0, m_0) = (\sigma_0, m_0)$.

For the next step, let $\Sigma \colon TQ \supseteq \Omega \to Q$ be an s-adapted local addition with corresponding open σ_0-neighbourhood $O_{\sigma_0} \subseteq C^{\infty}(M, Q)$ and let $\varphi_{\sigma_0} \colon O_{\sigma_0} \to \Gamma(\sigma_0^*(TQ))$ be the chart from Theorem B.9 a). Recall from the proof of [31, Proposition 3.6] that φ_{σ_0} also is a submanifold chart for $\Gamma(M \xleftarrow{s} Q)$ that maps the open neighbourhood $O_{\sigma_0} \cap \Gamma(M \xleftarrow{s} Q)$ onto an open neighbourhood of the zero section in the closed subspace $\{\tau \in \Gamma(\sigma_0^*(TQ)) \mid \tau(m) \in T_{\sigma_0(m)} s^{-1}(m) \text{ for all } m \in M\}$ of $\Gamma(\sigma_0^*(TQ))$.

Denote $R_0 := \varphi_{\sigma_0}(O_{\sigma_0} \cap \Gamma(M \xleftarrow{s} Q))$. Then we define the diffeomorphism

$$\Xi\colon B^{-1}((O_{\sigma_0} \cap \Gamma(M \xleftarrow{s} Q)) \times O) \to R_0 \times O, \quad (f, m) \mapsto (\varphi_{\sigma_0}((\widetilde{\beta}(m))^{-1} \circ f \circ \beta(m)), m).$$

By shrinking $B^{-1}(O_{\sigma_0} \times O)$ if necessary, we may assume that ev maps the open neighbourhood $(B^{-1}(O_{\sigma_0} \times O))$ of (σ_0, m_0) into the open neighbourhood $s^{-1}(O)$ of q_0. Moreover, the following diagram commutes by the definitions of φ_{σ_0} and of $\widetilde{\beta}(m)$

$$\begin{tikzcd}
B^{-1}(O_{\sigma_0} \times O) \arrow[r, "\mathrm{ev}"] \arrow[d, "\Xi"'] & s^{-1}(O) \arrow[r, "\xi"] & W \times O \\
R_0 \times O \arrow[rr, "\mathrm{ev}_{m_0} \times \mathrm{id}_O"] & & T_{q_0}s^{-1}(m_0) \times O. \arrow[u, "\Sigma \times \mathrm{id}_O"']
\end{tikzcd}$$

Since W is diffeomorphic to $s^{-1}(m_0)$ and Σ restricts to a local diffeomorphism of a zero-neighbourhood in $T_{q_0}s^{-1}(m_0)$ onto an open neighbourhood of q_0 in $s^{-1}(m_0)$, it follows that the arrow on the right is a local diffeomorphism. Hence each morphism except ev and $\mathrm{ev}_{m_0} \times \mathrm{id}_O$ in this diagram is a local diffeomorphism. By [7, Lemma 1.6] ev will thus be a submersion if ev_{m_0} is a submersion.

The map ev_{m_0} is defined on an open zero neighbourhood of the space

$$E_{\sigma_0} := \{\tau \in \Gamma(\sigma_0^*(TQ)) \mid \tau(m) \in T_{\sigma_0(m)}s^{-1}(m) \text{ for all } m \in M\}$$

and takes values in $T_{q_0}s^{-1}(m_0)$.

By using the diffeomorphism $\xi\colon s^{-1}(O) \to W \times O$ we have the following identifications

$$T_{q_0}s^{-1}(m_0) \cong T_{(w_0,m_0)}\xi(s^{-1}(m_0)) \cong F$$

where F is the modelling space of W and $(w_0, m_0) := \xi(q_0)$. Using these and a cutoff function, one can build as in Remark 2.3 a continuous inverse to ev_{m_0} that takes $T_{q_0}s^{-1}(m_0)$ into E_{σ_0}. Thus ev_{m_0} is a submersion, finishing the proof. □

Corollary 2.9. *Let M be a compact manifold and N be a locally metrisable manifold that admits a local addition. Then the joint evaluation map*

$$\mathrm{ev}\colon C^\infty(M, N) \times M \to N, \quad (f, m) \mapsto f(m) \tag{9}$$

is a submersion.

Proof. We have the natural identification $C^\infty(M, N) \cong \Gamma(M \xleftarrow{\mathrm{pr}_2} N \times M)$. Thus Proposition 2.8 shows that

$$C^\infty(M, N) \times M \to N \times M, \quad (f, m) \mapsto (f(m), m)$$

is a submersion. Now (9) is a submersion, since it is the composition to two submersions. □

Corollary 2.10. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid. Then the joint evaluation map*

$$\mathrm{ev}\colon \mathrm{Bis}(\mathcal{G}) \times M \to G, \quad (\sigma, m) \mapsto \sigma(m)$$

is a submersion.

Proof. This is implied by Proposition 2.8 since $\operatorname{Bis}(\mathcal{G})$ is an open submanifold of $\Gamma(M \xleftarrow{\alpha} G)$. $\square$

Corollary 2.11. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid. Then the* division morphism

$$\delta\colon \operatorname{Bis}(\mathcal{G}) \times \operatorname{Bis}(\mathcal{G}) \times M \to G, (\sigma, \tau, m) \mapsto (\sigma \star \tau^{-1})(m)$$

and for $m \in M$ the restricted division $\delta_m\colon \operatorname{Bis}(\mathcal{G}) \times \operatorname{Bis}(\mathcal{G}) \to \alpha^{-1}(m), \delta_m(\sigma, \tau) := \delta(\sigma, \tau, m)$ are submersions.

Proof. Note that we can write $\delta(\sigma, \tau, m) = \operatorname{ev}(\sigma \star \tau^{-1}, m)$ and $\delta_m(\sigma, \tau) = \operatorname{ev}_m(\sigma \star \tau^{-1})$. Since ev and ev_m are submersions by Corollary 2.5 and Corollary 2.10, it suffices to prove that the map

$$f\colon \operatorname{Bis}(\mathcal{G}) \times \operatorname{Bis}(\mathcal{G}) \to \operatorname{Bis}(\mathcal{G}), (\sigma, \tau) \mapsto \sigma \star \tau^{-1}$$

is a submersion. However, as $(\operatorname{Bis}(\mathcal{G}), \star)$ is a Lie group the map f is a submersion. $\square$

We have now established that the evaluation map from the bisections onto the manifold of arrows is a submersion. However, to completely determine the manifold of arrows, we need ev to be surjective. Note that this means that there is a (global) bisection through each point in G. Consider first an easy example

Example 2.12. Let H be a Lie group modelled on a metrisable space which acts on the compact manifold M, i.e. the associated action groupoid $H \ltimes M$ admits an adapted local addition by Remark 2.2. Then the evaluation $\operatorname{ev}\colon \operatorname{Bis}(H \ltimes M) \times M \to H \times M$, $(\sigma, m) \mapsto (\sigma(m), m)$ is a surjective submersion. We already know from Corollary 2.10 that ev is a submersion and thus have to establish only surjectivity. For each pair $(h, m) \in H \times M$ we can define the constant bisection $c_h\colon M \to H \times M, n \mapsto (h, n)$ which is contained in $\operatorname{Bis}(H \ltimes M)$. Hence $\operatorname{ev}(c_h, m) = (h, m)$ and thus ev is surjective. In particular, for each arrow $g \in H \times M$ in the action groupoid $\mathcal{G}$ there is a global bisection σ_g with $\sigma_g(\alpha(g)) = g$.

The structure of $\operatorname{Bis}(H \ltimes M)$ is interesting in its own. Since the second component of a bisection $\sigma\colon M \to H \times M$ has to be the identity, $\operatorname{Bis}(H \ltimes M)$ can be identified with the subset

$$\{\gamma \in C^\infty(M, H) \mid m \mapsto \gamma(m).m \text{ is a diffeomorphism of } M\}$$

of $C^\infty(M, H)$. Since $C^\infty(M, H) \to C^\infty(M, M)$, $\gamma \mapsto (m \mapsto \gamma(m).m)$ is smooth and $\operatorname{Diff}(M) \subseteq C^\infty(M, M)$ is open, it follows that $\operatorname{Bis}(M \ltimes H)$ is an open submanifold of $C^\infty(M, H)$ that contains the constant maps. However, the group structure on the functions from M to H is not given by the pointwise multiplication, but by

$$(\gamma \star \eta)(m) := \gamma(\eta(m).m) \cdot \eta(m).$$

One effect of this is that $\gamma \mapsto (m \mapsto \gamma(m).m)$ is a homomorphism $\operatorname{Bis}(H \ltimes M) \to \operatorname{Diff}(M)$. $\square$

In general there will not be a bisection through each arrow of a given groupoid (see Remark 2.18 b) below). Nevertheless, for source connected finite-dimensional Lie groupoids it is known (see [37]) that bisections through each arrow exist. We will now generalise this result to our infinite-dimensional setting.

Definition 2.13. Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid.

a) Denote for $m \in M$ by C_m the connected component of 1_m in $\alpha^{-1}(m)$. Then we define the subset $C(\mathcal{G}) := \bigcup_{m \in M} C_m$. By [16, Proposition 1.5.1] we obtain a wide subgroupoid $C(\mathcal{G}) \rightrightarrows M$ of $\mathcal{G}$, called the *identity-component subgroupoid* of $\mathcal{G}$.[3]
b) The groupoid $\mathcal{G}$ is called α- or *source connected* if for each $m \in M$ the fibre $\alpha^{-1}(m)$ is connected.

Observe that for an α-connected groupoid $\mathcal{G}$ we have $C(\mathcal{G}) = \mathcal{G}$. □

Note that Proposition 2.7 implies that for α-connected Lie groupoids there is for every arrow a bisection whose image contains the given arrow. However, we give an alternative proof in the following theorem, which also yields more information:

Theorem 2.14. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally metrisable Lie groupoid with compact M that admits an adapted local addition.*

a) *The image of the evaluation* ev *is an open and wide Lie subgroupoid which contains the identity subgroupoid $C(\mathcal{G})$.*
b) *The identity subgroupoid $C(\mathcal{G})$ coincides with $\mathrm{ev}(\mathrm{Bis}(\mathcal{G})_0 \times M)$, where $\mathrm{Bis}(\mathcal{G})_0$ is the identity component of $\mathrm{Bis}(\mathcal{G})$. Hence, $C(\mathcal{G})$ is an open Lie subgroupoid of $\mathcal{G}$.*

Assume in addition that $\mathcal{G}$ is α-connected, then

c) *For each $g \in G$ there is a bisection $\sigma_g \in \mathrm{Bis}(\mathcal{G})_0$ with $\sigma_g(\alpha(g)) = g$. In particular,* ev *is surjective.*

Proof.

a) The image $\mathcal{U} := \mathrm{ev}(\mathrm{Bis}(\mathcal{G}) \times M)$ contains the image of the object inclusion $1\colon M \to G$, i.e. $1_m \in \mathcal{U}$ for all $m \in M$. Define for $m \in M$ the set $\mathcal{U}_m = \mathcal{U} \cap \alpha^{-1}(m)$ and note that $\mathcal{U}_m = \mathrm{ev}_m(\mathrm{Bis}(\mathcal{G}))$. As $\mathrm{ev}_m\colon \mathrm{Bis}(\mathcal{G}) \to \alpha^{-1}(m)$ is a submersion by Corollary 2.5 we infer that $\mathcal{U}_m$ is an open subset of $\alpha^{-1}(m)$. By construction of the group operations of $\mathrm{Bis}(\mathcal{G})$ the set $\mathcal{U}$ yields a wide subgroupoid $\mathcal{U} \rightrightarrows M$ of $\mathcal{G}$ such that $\mathcal{U}_m$ is open in $\alpha^{-1}(m)$. Hence the image of ev is an open and wide Lie subgroupoid. Now [16, Proposition 1.5.7][4] shows that $\mathcal{U}_m$ is also closed in $\alpha^{-1}(m)$. Thus the clopen set $\mathcal{U}_m$ contains the connected component of $1_m \in \alpha^{-1}(m)$. Since this holds for each m, we see that $C(\mathcal{G}) \subseteq \mathcal{U}$.
b) Set $B_0 := \mathrm{Bis}(\mathcal{G})_0$. As B_0 is an open subgroup, an argument as in a) shows that $\mathrm{ev}(B_0 \times M)$ is a subgroupoid of $\mathcal{G}$ which contains $C(\mathcal{G})$.
Furthermore, for each $m \in D_i$ the set $\mathrm{ev}_m(B_0) \subseteq \alpha^{-1}(m)$ is connected and contains 1_m. Thus by definition of the connected set $C_m \subseteq \alpha^{-1}(m)$ we have $\mathrm{ev}_m(B_0) \subseteq C_m \subseteq C(\mathcal{G})$. Hence, $\mathrm{ev}(B_0 \times M) = C(\mathcal{G})$ and since ev is a submersion by Corollary 2.10, $C(\mathcal{G})$ is open in G. In particular, $C(\mathcal{G})$ is an open subgroupoid of $\mathcal{G}$, i.e. it is an open Lie subgroupoid.
c) If $\mathcal{G}$ is α-connected then $C(\mathcal{G}) = G$ whence the assertion follows from b). □

Definition 2.15. We say that a for a Lie groupoid $\mathcal{G} = (G \rightrightarrows M)$ there exists a *bisection through each arrow* if $\mathcal{G}$ satisfies the condition of Theorem 2.14 c), i.e. for each $g \in G$ there exists $\sigma_g \in \mathrm{Bis}(\mathcal{G})$ with $\sigma_g(\alpha(g)) = g$. □

[3] Note that at this stage we do not know that $C(\mathcal{G})$ is a Lie subgroupoid. In particular, the proof for finite dimensional Lie groupoids (see [16, Proposition 1.5.1]) does not carry over to our setting. Compare however Theorem 2.14 b).

[4] Mackenzie [16] considers only finite-dimensional Lie groupoids. However, the proof of this result carries over verbatim to the infinite-dimensional setting.

Note that Part c) of Theorem 2.14 yields [37, Theorem 3.1] as a corollary for (finite-dimensional) Lie groupoids over a compact base.

Corollary 2.16. *In a finite-dimensional source-connected Lie groupoid with compact space of objects there exist bisections through each arrow.*

We also obtain the following well known result on the natural action of $\mathrm{Diff}(M)$ on M (cf. [1,21]).

Corollary 2.17. *If M is a compact and connected manifold, then $\mathrm{Diff}(M)_0$ acts transitively on M.*

Remark 2.18.

a) Note that [37, Theorem 3.1] holds for arbitrary finite-dimensional α-connected Lie groupoids whereas Corollary 2.16 is limited to groupoids over compact base.
b) The assumption on $\mathcal{G}$ to be source connected cannot be dispensed with. For instance, if N, N' are non-diffeomorphic compact manifolds (of the same dimension), then the pair groupoid $\mathcal{P}(M) := (M \times M \rightrightarrows M)$ of $M := N \sqcup N'$ has $\mathrm{Bis}(\mathcal{P}(M)) \cong \mathrm{Diff}(M)$ and the action of $\mathrm{Bis}(\mathcal{P}(M))$ on the source fibre naturally identifies with the natural action of $\mathrm{Diff}(M)$ on M. But since N, N' are not diffeomorphic, there cannot exist a diffeomorphism of M that interchanges the points n and n' if $n \in N$ and $n' \in N'$. Consequently, there cannot exist a bisection through the morphism $((n, n'), (n', n))$ of $\mathcal{P}(M)$.
However, as we have seen in Example 2.12, there exist non-source connected Lie groupoids, for which there exist bisections through each point. For another example consider the gauge groupoid $\mathcal{G} = \mathrm{Gauge}(M \times K)$ of the trivial principal bundle $M \times K \to M$, then $\mathrm{Bis}(\mathrm{Gauge}(M \times K)) \cong \mathrm{Aut}(M \times K) \cong C^\infty(M, K) \rtimes \mathrm{Diff}(M)$. If M is connected, then $\mathrm{Diff}(M)$ acts transitively on M and $C^\infty(M, K)$ always acts transitively on K. Thus there exist a bisection through each arrow of $\mathrm{Gauge}(M \times K)$, even if K is not connected. □

Before we continue with our investigation of the bisection action groupoid, note the following interesting consequences of Corollary 2.5.

Lemma 2.19. *Let $\mathcal{G}$ be locally trivial and denote by $\theta\colon \mathrm{Bis}(\mathcal{G}) \times M \to M, (\sigma, x) \mapsto \beta_{\mathcal{G}} \circ \sigma(m)$ the canonical Lie group action. Then*

a) *θ restricts for each $m \in M$ to a submersion $\theta_m\colon \mathrm{Bis}(\mathcal{G}) \times \{m\} \to M$.*

If in addition $\mathcal{G}$ admits bisections through each arrow or M is connected, then

b) *$\mathcal{B}(\mathcal{G})$ is locally trivial and θ is transitive.*

Proof. As $\mathcal{G}$ is locally trivial $\beta_{\mathcal{G}}|_{\alpha^{-1}(m)} : \alpha^{-1}(m) \to M$ is a surjective submersion. Since the $\alpha_{\mathcal{B}}$-fibre is $\mathrm{Bis}(\mathcal{G})$ and $\mathrm{ev}_m\colon \mathrm{Bis}(\mathcal{G}) \to \alpha^{-1}(m)$ is a submersion by Corollary 2.5 we see that $\beta_{\mathcal{B}}|_{\alpha_{\mathcal{B}}^{-1}(m)} = \beta_{\mathcal{G}} \circ \mathrm{ev}_m$ is a submersion. We claim that $\beta_{\mathcal{B}}$ is surjective. If this is true then $\mathcal{B}(\mathcal{G})$ is locally trivial by [16, Proposition 1.3.3] which carries over verbatim to our infinite-dimensional setting. Moreover, we derive that the canonical action of $\mathrm{Bis}(\mathcal{G})$ is transitive and restricts to a submersion $\mathrm{Bis}(\mathcal{G}) \times \{m\} \to M$ for all $m \in M$.

To prove the claim we have to treat both cases separately. Assume first that $\mathcal{G}$ admits bisections through each arrow, then ev_m is surjective and thus $\beta_{\mathcal{B}}$ is surjective. On the other hand let now M be connected. Then we note that $\beta_{\mathcal{B}} = \widetilde{\mathrm{ev}_m} \circ (\beta_{\mathcal{G}})_*$, where $(\beta_{\mathcal{G}})_*\colon \mathrm{Bis}(\mathcal{G}) \to \mathrm{Diff}(M), \sigma \mapsto \beta_{\mathcal{G}} \circ \sigma$ and $\widetilde{\mathrm{ev}_m}\colon \mathrm{Diff}(M) \to$

$M, \varphi \mapsto \varphi(m)$. Now the image of $(\beta_{\mathcal{G}})_*$ contains the identity component $\mathrm{Diff}(M)_0$ of $\mathrm{Diff}(M)$ (by [31, Example 3.16]) and $\mathrm{Diff}(M)_0$ acts transitively on the connected manifold M by Corollary 2.17. Thus $\beta_{\mathcal{B}}$ is surjective.

We conclude that in both cases the assertion holds. □

Remark 2.20. Quotient constructions for Lie groupoids (and already for Lie groups) are quite tricky. In fact, Lie groupoids are a tool to circumvent badly behaved quotients (for instance for non-free group actions). However, each category carries a natural notion of quotient object for an internal equivalence relation. If $\mathcal{C}$ is a category with finite products and $R \subseteq E \times E$ is an internal equivalence relation, then the quotient $E \to E/R$ in $\mathcal{C}$ (uniquely determined up to isomorphism) is, if it exists, the coequaliser of the diagram

$$R \underset{\mathrm{pr}_2}{\overset{\mathrm{pr}_1}{\rightrightarrows}} E\ . \tag{10}$$

If, in the case that the quotient exists, (10) is also the pull-back of $E \to E/R$ along itself, then the quotient $E \to E/R$ is called *effective* (see [19, Appendix.1] for details). We want to apply this to the category $\mathsf{LieGroupoids}_M$, whose objects are locally convex and locally metrisable Lie groupoids over M and whose morphisms are smooth functors that are the identity on M. Note that the product of two Lie groupoids $(G \rightrightarrows M)$ and $(H \rightrightarrows M)$ is given by restricting the product Lie groupoid $(G \times H \rightrightarrows M \times M)$ to the diagonal $M \cong \Delta M \subseteq M \times M$. □

Theorem 2.21. *If $\mathcal{G} = (G \rightrightarrows M)$ is a Lie groupoid with a bisection through each arrow in G, e.g. $\mathcal{G}$ is source connected, then the morphism $\mathrm{ev}\colon \mathcal{B}(\mathcal{G}) \to \mathcal{G}$ is the quotient of $\mathcal{B}(\mathcal{G})$ in $\mathsf{LieGroupoids}_M$ by*

$$R = \{(\sigma, m), (\tau, m) \in \mathrm{Bis}(\mathcal{G}) \times \mathrm{Bis}(\mathcal{G}) \times M \mid \sigma(m) = \tau(m)\}.$$

Proof. We first note that R is isomorphic to $K \times \mathrm{Bis}(\mathcal{G})$, where

$$K := \{(\sigma, m) \in \mathrm{Bis}(\mathcal{G}) \times M \mid \sigma(m) = 1_m\} = \mathrm{ev}^{-1}(M) \tag{11}$$

is the kernel of ev. To see this note that as $M \subseteq G$ is a closed submanifold and ev is a submersion by Corollary 2.10, it follows that K is a closed submanifold of $\mathrm{Bis}(\mathcal{G}) \times M$. Now

$$\mathrm{Bis}(\mathcal{G}) \times \mathrm{Bis}(\mathcal{G}) \times M \to \mathrm{Bis}(\mathcal{G}) \times M \times \mathrm{Bis}(\mathcal{G}), \quad ((\sigma, m), (\tau, m)) \mapsto (\sigma \star \tau^{-1}, m, \tau)$$

is a diffeomorphism which maps R onto $K \times \mathrm{Bis}(\mathcal{G})$. Consequently, R is a closed submanifold of $\mathrm{Bis}(\mathcal{G}) \times \mathrm{Bis}(\mathcal{G}) \times M$.

We now argue that R is in fact an internal equivalence relation. It is clear that the relation is reflexive and symmetric. After applying the diffeomorphism (11), the second projection $\mathrm{pr}_2\colon R \to \mathrm{Bis}(\mathcal{G}) \times M$, $((\sigma, m), (\tau, m)) \mapsto (\tau, m)$ is an actual projection. So pr_2 is a surjective submersion. Thus the pull-back

$$R * R := R \times_{(\mathrm{Bis}(\mathcal{G}) \times M)} R = \{(((\sigma, m), (\tau, m)), ((\sigma', m'), (\tau', m'))) \mid m = m', \tau = \sigma'\}$$

is a submanifold of $R \times R$ and

$$R * R \to \mathrm{Bis}(\mathcal{G}) \times \mathrm{Bis}(\mathcal{G}) \times M, \quad (((\sigma, m), (\tau, m)), ((\tau, m), (\tau', m))) \mapsto ((\sigma, m), (\tau', m))$$

clearly factors through R. Consequently, R is an internal equivalence relation.

Finally, if $f\colon \mathrm{Bis}(\mathcal{G}) \times M \to H$ is a morphism of Lie groupoids that satisfies $f \circ \mathrm{pr}_1 = f \circ \mathrm{pr}_2$ (for $\mathrm{pr}_i\colon R \to \mathrm{Bis}(\mathcal{G}) \times M$ the canonical maps), then we construct a smooth map $\varphi\colon G \to H$ by taking a local smooth section of ev and composing it with f. Since $f \circ \mathrm{pr}_1 = f \circ \mathrm{pr}_2$, two possible pre-images of an element from G in $\mathrm{Bis}(\mathcal{G}) \times M$ are mapped to the same element in H, and thus φ is well-defined and smooth by construction. One directly checks that it also defines a morphism of Lie groupoids (i.e., φ is compatible with the structure maps, see also [16, Proposition 2.2.3]). □

Remark 2.22. We have seen in Theorem 2.21 that a source connected Lie groupoid $\mathcal{G}$ with compact base is the quotient of its associated bisection action Lie groupoid $\mathcal{B}(\mathcal{G})$. However, Theorem 2.21 already uses that a candidate for the quotient, namely $\mathcal{G}$ exists. Thus the theorem does not provide the existence of the quotient without using $\mathcal{G}$. □

In the proof of Theorem 2.21 it is visible that the quotient of $\mathcal{B}(\mathcal{G})$ was taken with respect to the kernel[5] of the base-preserving morphism ev. We refer to [16, 2.2] for details on the groupoid quotient by a normal subgroupoid. Since ev is a surjective submersion in the situation of Theorem 2.21, its kernel is a Lie subgroupoid of $\mathcal{B}(\mathcal{G})$. Finally, the α-fibre of the kernel over $m \in M$ is given by

$$\mathrm{ev}(\cdot, m)^{-1}\{1_m\} = \{\sigma \in \mathrm{Bis}(\mathcal{G}) \mid \sigma(m) = 1_m\}.$$

In the next section we will study the subgroups of $\mathrm{Bis}(\mathcal{G})$ which arise from this construction. Later on these information will allow us to investigate the groupoid quotients in more detail.

3. Subgroups and quotients associated to the bisection group

In this section we study subgroups of the bisections which are associated to a fixed point in the base manifold. These subgroups are closely related to the reconstruction result outlined in Theorem 2.21 and will become an important tool to study locally trivial Lie groupoids in Sections 4 and 5.

As before, (unless stated explicitly otherwise) we shall assume that $\mathcal{G} = (G \rightrightarrows M)$ is a locally metrisable Lie groupoid over a compact base M which admits an adapted local addition.

Definition 3.1. Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally convex Lie groupoid (we require neither that M is compact or finite dimensional nor that G admits a local addition). Fix $m \in M$ and denote by $\mathrm{Vert}_m(\mathcal{G})$ the vertex subgroup of the groupoid $\mathcal{G}$. There are now two subsets of $\mathrm{Bis}(\mathcal{G})$ whose elements are characterised by their value at m

$$\begin{aligned}
\mathrm{Loop}_m(\mathcal{G}) &:= \{\sigma \in \mathrm{Bis}(\mathcal{G}) \mid \sigma(m) \in \mathrm{Vert}_m(\mathcal{G}) = \alpha^{-1}(m) \cap \beta^{-1}(m)\}, \\
\mathrm{Bis}_m(\mathcal{G}) &:= \{\sigma \in \mathrm{Bis}(\mathcal{G}) \mid \sigma(m) = 1_m\}.
\end{aligned}$$

Clearly $\mathrm{Bis}_m(\mathcal{G}) \subseteq \mathrm{Loop}_m(\mathcal{G})$ and both are subgroups of $\mathrm{Bis}(\mathcal{G})$ by definition of the group operation (see (2) and (3)). □

Note that $\mathrm{Bis}_m(\mathcal{G})$ is a normal subgroup of $\mathrm{Loop}_m(\mathcal{G})$ as for $\sigma \in \mathrm{Bis}_m(\mathcal{G})$ and $\tau \in \mathrm{Loop}_m(\mathcal{G})$ we have

$$\begin{aligned}
\tau \star \sigma \star \tau^{-1}(m) &= \tau \star (\sigma \circ \beta \circ \tau \cdot \tau)(m) = \tau(\underbrace{\beta \circ \sigma \circ \beta \circ \tau(m)}_{=m}) \cdot (\sigma(\underbrace{\beta \circ \tau(m)}_{=m})) \cdot \tau(m) \\
&= \tau(m) \cdot 1_m \cdot \tau^{-1}(m) = 1_m.
\end{aligned}$$

[5] I.e. the set $\{g \in \mathcal{B}(\mathcal{G}) \mid \mathrm{ev}(g) = 1_x \text{ for some } x \in M\}$ which is a wide subgroupoid of the inner subgroupoid, called a normal Lie subgroupoid (see [16, Definition 2.2.2]).

We will now investigate the subgroups from Definition 3.1 in the case that $\operatorname{Bis}(\mathcal{G})$ is a Lie group. Thus M will be assumed to be compact, whence Lemma A.4 implies that the vertex group $\operatorname{Vert}_m(\mathcal{G})$ of $\mathcal{G} = (G \rightrightarrows M)$ is a submanifold of G and in particular a Lie group.

Proposition 3.2. *Fix some $m \in M$.*

a) *The group $\operatorname{Loop}_m(\mathcal{G})$ is a Lie subgroup of $\operatorname{Bis}(\mathcal{G})$ and as a submanifold in $\operatorname{Bis}(\mathcal{G})$ it is of finite codimension.*
b) *The map $\operatorname{ev}_m \colon \operatorname{Bis}(\mathcal{G}) \to \alpha^{-1}(m)$ restricts to a Lie group morphism $\psi_m \colon \operatorname{Loop}_m(\mathcal{G}) \to \operatorname{Vert}_m(\mathcal{G})$ whose kernel is $\operatorname{Bis}_m(\mathcal{G})$. Moreover, ψ_m is a submersion.*
c) *The group $\operatorname{Bis}_m(\mathcal{G})$ is a split Lie subgroup of $\operatorname{Loop}_m(\mathcal{G})$ and a split Lie subgroup of $\operatorname{Bis}(\mathcal{G})$. It is modelled on $\Gamma(1^*T^\alpha G)_m = \{X \in \Gamma(1^*T^\alpha G) \mid X(1_m) = 0_{1_m}\}$.*
d) *If $\mathcal{G}$ is a Banach–Lie groupoid then $\operatorname{Bis}_m(\mathcal{G})$ is a co-Banach submanifold in $\operatorname{Loop}_m(\mathcal{G})$ and also in $\operatorname{Bis}(\mathcal{G})$.*

Proof.

a) Recall that by Corollary 2.5 the map $\operatorname{ev}_m \colon \operatorname{Bis}(\mathcal{G}) \to \alpha^{-1}(m)$ is a submersion. Moreover, the vertex group $\operatorname{Vert}_m(\mathcal{G})$ is a submanifold of finite codimension of $\alpha^{-1}(m)$ by Lemma A.4. Thus $\operatorname{Loop}_m(\mathcal{G}) = \operatorname{ev}_m^{-1}(\operatorname{Vert}_m(\mathcal{G}))$ is a submanifold of finite codimension in $\operatorname{Bis}(\mathcal{G})$ by [7, Theorem C]. Now $\operatorname{Loop}_m(\mathcal{G})$ is a subgroup and a split submanifold of $\operatorname{Bis}(\mathcal{G})$ whence a split Lie subgroup.
b) To see that ψ_m is also a group homomorphism we pick $\sigma, \tau \in \operatorname{Loop}_m(\mathcal{G})$ and compute

$$\psi_m(\sigma \star \tau) = \psi_m((\sigma \circ \beta \circ \tau) \cdot \tau) = \sigma(\beta(\tau(m)) \cdot \tau(m) = \sigma(m) \cdot \tau(m) = \psi_m(\sigma) \cdot \psi_m(\tau).$$

As $\operatorname{ev}_m(\sigma) = 1_m$ if and only if $\sigma(m) = 1_m$, we see that $\operatorname{Bis}_m(\mathcal{G})$ is the kernel of ψ_m. Having applied [7, Theorem C] in part a), we observe that this also entails that ψ_m is a submersion.
c) By part b) the subgroup $\operatorname{Bis}_m(\mathcal{G})$ is the preimage of a singleton under a submersion, whence a split submanifold of $\operatorname{Loop}_m(\mathcal{G})$ by the regular value theorem [7, Theorem D]. In particular, $\operatorname{Bis}_m(\mathcal{G})$ becomes a split Lie subgroup of $\operatorname{Loop}_m(\mathcal{G})$. Since $\operatorname{Loop}_m(\mathcal{G})$ is a split submanifold in $\operatorname{Bis}(\mathcal{G})$, [7, Lemma 1.4] yields that $\operatorname{Bis}_m(\mathcal{G})$ is a split submanifold of $\operatorname{Bis}(\mathcal{G})$ and thus a split Lie subgroup of $\operatorname{Bis}(\mathcal{G})$.
Recall that by the regular value theorem the tangent space of $\operatorname{Bis}_m(\mathcal{G})$ at 1 is the kernel $\ker T_1 \operatorname{ev}_m \subseteq T_1 \operatorname{Bis}(\mathcal{G}) \cong \Gamma(1^*T^\alpha G)$. Taking identifications we compute $\ker T_1 \operatorname{ev}_m$ as a subspace of $\Gamma(1^*T^\alpha G)$. On the level of isomorphism classes of curves the isomorphism $\varphi_{\mathcal{G}} \colon T_1 \operatorname{Bis}(\mathcal{G}) \to \Gamma(1^*T\alpha G)$ is given by

$$\varphi_{\mathcal{G}}([t \mapsto c(t)]) = (m \mapsto [t \mapsto c^\wedge(t,m)])(\text{cf. [31, Remark 4.1]}).$$

Hence $\varphi_{\mathcal{G}}$ identifies $T_1 \operatorname{ev}_m$ with $\Gamma(1^*T^\alpha G) \to G, X \mapsto X(m)$ whose kernel is $\Gamma(1^*T^\alpha G)_m$.
d) If $\mathcal{G}$ is a Banach–Lie groupoid then $\alpha^{-1}(m)$ and thus also $\operatorname{Vert}_m(\mathcal{G})$ are manifolds modelled on Banach-spaces. In this situation the regular value theorem implies that $\operatorname{Bis}_m(\mathcal{G})$ is a co-Banach submanifold of $\operatorname{Loop}_m(\mathcal{G})$. Since $\operatorname{Loop}_m(\mathcal{G})$ is of finite codimension in $\operatorname{Bis}(\mathcal{G})$, we deduce that $\operatorname{Bis}_m(\mathcal{G})$ is a co-Banach submanifold of $\operatorname{Bis}(\mathcal{G})$. $\square$

An important property of the Lie subgroups constructed in Proposition 3.2 is that they are regular as Lie groups (we recall the definition of regularity for Lie groups in Appendix B). Namely, the subgroups

$\mathrm{Bis}_m(\mathcal{G})$ and $\mathrm{Loop}_m(\mathcal{G})$ will be regular Lie groups if $\mathrm{Bis}(\mathcal{G})$ is a regular Lie group. Let us first recall when bisection groups are regular.[6]

Remark 3.3. In [31, Section 5] we established C^k-regularity of $\mathrm{Bis}(\mathcal{G})$ if $\mathcal{G}$ is either

a) a Banach–Lie groupoid (then $\mathrm{Bis}(\mathcal{G})$ is C^0-regular);
b) or a locally-trivial Lie groupoid whose vertex groups are locally exponential C^k-regular Lie groups (then $\mathrm{Bis}(\mathcal{G})$ is C^k-regular). □

We will now prove that $\mathrm{Bis}_m(\mathcal{G})$ and $\mathrm{Loop}_m(\mathcal{G})$ inherits the regularity properties from $\mathrm{Bis}(\mathcal{G})$.

Proposition 3.4. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid and fix $m \in M$. Assume that $\mathrm{Bis}(\mathcal{G})$ is C^k-regular for some $k \in \mathbb{N}_0 \cup \{\infty\}$, e.g. in the situation of Remark 3.3. Let H be either $\mathrm{Bis}_m(\mathcal{G})$ and $\mathrm{Loop}_m(\mathcal{G})$ and consider $\eta \in C^k([0,1], \mathbf{L}(H))$.*

a) *The solution γ_η of the initial value problem*

$$\begin{cases} \gamma'(t) &= \gamma(t).\eta(t) \quad \forall t \in [0,1] \\ \gamma(0) &= 1 \end{cases} \tag{12}$$

in G takes its image in H.

b) *The Lie groups $\mathrm{Bis}_m(\mathcal{G})$ and $\mathrm{Loop}_m(\mathcal{G})$ are C^k-regular*

Proof.

a) Fix $\eta \in C^k([0,1], \mathbf{L}(H))$ together with the evolution $\gamma_\eta \colon [0,1] \to \mathrm{Bis}(\mathcal{G})$ of η. The composition $\mathrm{ev}_m \circ \gamma_\eta$ yields a smooth curve in $\alpha^{-1}(m)$. As γ_η solves (12) we already know that $\mathrm{ev}_m \circ \gamma_\eta(0) = 1_m$. Now we consider both subgroups separately:

Case $H = \mathrm{Bis}_m(\mathcal{G})$. By definition, γ_η will take its image in $\mathrm{Bis}_m(\mathcal{G})$ if $\mathrm{ev}_m \circ \gamma_\eta(t) = 1_m, \forall t \in [0,1]$, i.e. we have to prove that $(\mathrm{ev}_m \circ \gamma_\eta)'(t) = 0 \in T^\alpha_{\gamma_\eta(t)} G,\ \forall t \in [0,1]$. To this end fix $t \in [0,1]$ and a curve $c_{t,\eta} \colon]-\varepsilon, \varepsilon[\to \mathrm{Bis}_m(\mathcal{G})$ such that $c_{t,\eta}(0) = 1$ and $\eta(t)$ (as an element in the tangent space $T_1 \mathrm{Bis}_m(\mathcal{G})$) coincides with the equivalence class $[s \mapsto c_{t,\eta}(s)]$. Now we compute

$$\begin{aligned} (\mathrm{ev}_m \gamma_\eta)'(t) &= T\mathrm{ev}_m(\gamma_\eta'(t)) \overset{(12)}{=} T\mathrm{ev}_m(\gamma_\eta(t).\eta(t)) = T(\mathrm{ev}_m \circ \lambda_{\gamma_\eta(t)})(\eta(t)) \\ &= [s \mapsto \mathrm{ev}_m(\gamma_\eta(t) \star c_{t,\eta}(s))] = [s \mapsto \gamma_\eta(t)(\beta \circ c_{t,\eta}(s)(m)) \cdot c_{t,\eta}(s)(m)] \\ &= [s \mapsto \gamma_\eta(t)(m) \cdot 1_m] = 0 \in T^\alpha_{\gamma_\eta(t)(m)} G \end{aligned} \tag{13}$$

In passing from the second to the last line, we have used $c_{t,\eta}(s) \in \mathrm{Bis}_m(\mathcal{G})$, whence $c_{t,\eta}(s)(m) = 1_m$. We finally conclude that $\gamma_\eta(t) \in \mathrm{Bis}_m(\mathcal{G})$ for all $t \in [0,1]$ if $\eta \in C^k([0,1], \mathbf{L}(\mathrm{Bis}_m(\mathcal{G}))$.

Case $H = \mathrm{Loop}_m(\mathcal{G})$. We need to show that for all $t \in [0,1]$ we have $\gamma_\eta(t) \in \mathrm{Vert}_m(\mathcal{G})$, i.e. that $\beta \circ \mathrm{ev}_m \circ \gamma_\eta(t) = m$. Since $\mathrm{ev}_m(\gamma_\eta(0)) = 1_m$ this will follow from $(\beta \circ \mathrm{ev}_m \circ \gamma_\eta)'(t) = 0$ for all $t \in [0,1]$. To this end fix again $t \in [0,1]$ and a curve $c_{t,\eta} \colon]-\varepsilon, \varepsilon[\to \mathrm{Loop}_m(\mathcal{G})$ with $c_{t,\eta}(0) = 1$ and $\eta(t) = [s \mapsto c_{t,\eta}(s)]$. Computing as in (13) we obtain

[6] At the moment, no example of a non-regular Lie group modelled on a space with suitable completeness properties (i.e. Mackey completeness) is known.

$$(\beta \circ \mathrm{ev}_m \circ \gamma_\eta)'(t) \overset{(13)}{=} [s \mapsto \beta\left(\gamma_\eta(t)(m) \cdot c_{t,\eta}(s)(m)\right)] = [s \mapsto \beta\left(\gamma_\eta(t)(m)\right)] = 0 \in T_{\beta(\gamma_\eta(t)(m))}M.$$

We can thus conclude that $\gamma_\eta(t) \in \mathrm{Loop}_m(\mathcal{G})$ for all $t \in [0,1]$ if $\eta \in C^k([0,1], \mathbf{L}(\mathrm{Loop}_m(\mathcal{G})))$.

b) Proposition 3.2 c) asserts that the subgroups $\mathrm{Bis}_m(\mathcal{G})$ and $\mathrm{Loop}_m(\mathcal{G})$ are closed subgroups of the C^k-regular Lie group $\mathrm{Bis}(\mathcal{G})$. By part (a), the Lie groups $\mathrm{Bis}_m(\mathcal{G})$ and $\mathrm{Loop}_m(\mathcal{G})$ are C^k-semiregular. Thus Lemma B.5 proves that $\mathrm{Bis}_m(\mathcal{G})$ and $\mathrm{Loop}_m(\mathcal{G})$ are C^k-regular. □

Example 3.5. Let M be a compact manifold and consider the pair groupoid $\mathcal{P}(M) = (M \times M \rightrightarrows M)$. The vertex group $\mathrm{Vert}_m(\mathcal{P}(M))$ for $m \in M$ is just $\{(m,m)\}$, whence $\mathrm{Loop}_m(\mathcal{P}(M)) = \mathrm{Bis}(\mathcal{P}(M))_m$. Then the map $(\mathrm{pr}_2)_* \colon \mathrm{Bis}(\mathcal{P}(M)) \to \mathrm{Diff}(M), \sigma \mapsto \mathrm{pr}_2 \circ \sigma$ is an isomorphism of Lie groups. By construction, this restricts to an isomorphism

$$\mathrm{Loop}_m(\mathcal{P}(M)) = \mathrm{Bis}(\mathcal{P}(m))_m \cong \mathrm{Diff}_m(M) := \{\varphi \in \mathrm{Diff}(M) \mid \varphi(m) = m\}.$$

In particular, we infer from Proposition 3.2 and Proposition 3.4 that $\mathrm{Diff}_m(M)$ is a regular and split Lie subgroup of $\mathrm{Diff}(M)$. □

Proposition 3.6. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid. For $m \in M$ we endow the right coset space $\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ with the quotient topology induced by $\mathrm{Bis}(\mathcal{G})$. Then the map ev_m induces homeomorphisms*

a) *a homeomorphism $\widetilde{\mathrm{ev}_m}$ of $\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ onto $\mathrm{ev}_m(\mathrm{Bis}(\mathcal{G})) \Subset \alpha^{-1}(m)$,*
b) *an isomorphism of topological groups e_m of $\Lambda_m := \mathrm{Loop}_m(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ onto an open subgroup of $\mathrm{Vert}_m(\mathcal{G})$,*

Moreover, $\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ and $\mathrm{Loop}_m(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ carry unique manifold structures turning the canonical quotient maps into submersions.

If there is a bisection through every arrow in G, e.g. $\mathcal{G}$ is α-connected, then $\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G}) \cong \alpha^{-1}(m)$ as manifolds and $\mathrm{Loop}_m(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G}) \cong \mathrm{Vert}_m(\mathcal{G})$ as Lie groups.

Proof.

a) By definition, the quotient topology turns $q_m \colon \mathrm{Bis}(\mathcal{G}) \to \mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G}), \sigma \mapsto \sigma\,\mathrm{Bis}_m(\mathcal{G})$ into a quotient map. Recall from Corollary 2.5 that $\mathrm{ev}_m \colon \mathrm{Bis}(\mathcal{G}) \to \alpha^{-1}(m)$ is a submersion, whence its image in the α-fibre is open. For $\tau \in \mathrm{Bis}(\mathcal{G})_m$ we observe $\mathrm{ev}_m(\sigma \star \tau) = \sigma \star \tau(m) = \sigma(\beta(\tau(m)))\tau(m) = \sigma(\beta(1_m))1_m = \sigma(m) = \mathrm{ev}_m(\sigma)$. Hence ev_m is constant on right cosets and ev_m factors through $h_m \colon \mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G}) \to \mathrm{im}\,\mathrm{ev}_m \Subset \alpha^{-1}(m)$, $\sigma\,\mathrm{Bis}(\mathcal{G}) \mapsto \mathrm{ev}_m(\sigma)$. Now h_m is continuous since $\mathrm{ev}_m = h_m \circ q_m$ is continuous.
Let us prove that for $\sigma, \tau \in \mathrm{Bis}(\mathcal{G})$ with $\mathrm{ev}_m(\sigma) = \mathrm{ev}_m(\tau)$ we have $\sigma^{-1} \star \tau \in \mathrm{Bis}_m(\mathcal{G})$. Using the formulae (2) and (3) for the group operations of $\mathrm{Bis}(\mathcal{G})$ we obtain

$$\sigma^{-1} \star \tau(m) = \sigma^{-1}(\beta(\tau(m)))\tau(m) = \sigma^{-1}(\beta(\sigma(m)))\sigma(m) = \iota(\sigma(m))\sigma(m) = 1_m.$$

Hence $\sigma^{-1} \star \tau \in \mathrm{Bis}_m(\mathcal{G})$ if $\sigma(m) = \tau(m)$ and in this case we see $\sigma\,\mathrm{Bis}_m(\mathcal{G}) = \tau\,\mathrm{Bis}_m(\mathcal{G})$. This implies that h_m is a bijection onto the open set $\mathrm{im}\,\mathrm{ev}_m$.
We deduce from Corollary 2.5 that $\mathrm{ev}_m|^{\mathrm{im}\,\mathrm{ev}_m} \colon \mathrm{Bis}(\mathcal{G}) \to \mathrm{im}\,\mathrm{ev}_m \Subset \alpha^{-1}(m)$ is a surjective submersion. In particular, ev_m is open, whence a quotient map and thus $q_m = h_m^{-1} \circ \mathrm{ev}_m$ implies that h_m^{-1} is continuous.

b) By Proposition 3.2 b) we know that ev_m induces a Lie group morphism $\psi_m \colon \mathrm{Loop}_m(\mathcal{G}) \to \mathrm{Vert}_m(\mathcal{G})$ which is a submersion. Its kernel is the normal Lie subgroup $\mathrm{Bis}_m(\mathcal{G})$. Thus $\mathrm{Loop}_m(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ with the quotient topology becomes a topological group such that ψ_m descents to an isomorphism of topological groups onto $\mathrm{ev}_m(\mathrm{Loop}_m(\mathcal{G})) \Subset \mathrm{Vert}_m(\mathcal{G})$. Endow the quotient with the manifold structure turning the isomorphism into an isomorphism of Lie groups. Then the canonical quotient map becomes a submersion as a composition of a diffeomorphism and the submersion ψ_m.

The manifold structure on the open submanifolds $\operatorname{im}\mathrm{ev}_m \Subset \alpha^{-1}(m)$ and $\mathrm{ev}_m(\mathrm{Loop}_m(\mathcal{G})) \Subset \mathrm{Vert}_m(\mathcal{G})$ is uniquely determined up to diffeomorphism by the property that ev_m is a submersion. This is due to [7, Lemma 1.9].

The last assertion follows from part a) and b), since then $\operatorname{im}\mathrm{ev}_m = \alpha^{-1}(m)$ and $\operatorname{im}\psi_m = \mathrm{Vert}_m(\mathcal{G})$ hold. □

Example 3.7. Suppose $\pi\colon P \to M$ is a principal K-bundle for some locally exponential Lie group K with connected P. Then the gauge groupoid $\mathrm{Gauge}(P) := ((P \times P)/K \rightrightarrows M)$ admits an adapted local addition [31, Proposition 3.14] and $\mathrm{Bis}(\mathrm{Gauge}(P))$ is naturally isomorphic to $\mathrm{Aut}(P)$. Assume that K and P are locally metrisable, i.e. $\mathrm{Gauge}(P)$ is locally metrisable. The source fibre $\alpha^{-1}(m) = (P_m \times P)/K$ of $\mathrm{Gauge}(P)$ is diffeomorphic to P by choosing $o \in P_m$ and mapping $\langle p, q\rangle$ to $q.(p^{-1} \cdot o)$. Here we use $p^{-1} \cdot o$ as the suggestive notation for the element $k \in K$ that satisfies $p.k = o$. With respect to these identification the evaluation ev_m turns into the evaluation map

$$\mathrm{ev}_o\colon\ \mathrm{Aut}(P) \to P, \quad \varphi \mapsto \varphi(o).$$

Consequently,

$$\mathrm{Aut}_o(P) := \mathrm{ev}^{-1}(o) = \{f \in \mathrm{Aut}(P) \mid f(o) = o\}$$

is a Lie subgroup of $\mathrm{Aut}(P)$ and by Proposition 3.6, $\mathrm{Aut}(P)/\mathrm{Aut}_o(P)$ carries a unique smooth structure turning the induced map $[\varphi] \mapsto \varphi(p)$ into a diffeomorphism. So we may view P a homogeneous space for its automorphism group. In particular, this applies to the trivial bundle, yielding a smooth structure on $\mathrm{Diff}(M)/\mathrm{Diff}_m(M)$ and a diffeomorphism $\mathrm{Diff}(M)/\mathrm{Diff}_m(M) \cong M$. □

By now, the quotients $\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ and $\mathrm{Loop}_m(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ carry a manifold structure which was derived from the manifold structure of the α-fibre to the quotient. However, if the Lie groupoid $\mathcal{G}$ is a Banach–Lie groupoid then the homogeneous space $\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ already carries a natural manifold structure as a homogeneous space. This is a consequence of Glöckner's inverse function theorem (see the next Lemma for references and details). Again this manifold structure turns the canonical quotient map into a submersion.

Lemma 3.8. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Banach–Lie groupoid and $m \in M$. Then the homogeneous spaces $\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ and $\mathrm{Loop}_m(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ are manifolds and these manifold structures coincide with the ones obtained in Proposition 3.6.*

Proof. Combining Proposition 3.4 and Remark 3.3 we see that $\mathrm{Bis}_m(\mathcal{G})$ is a C^0-regular closed Lie subgroup of $\mathrm{Bis}(\mathcal{G})$ and of $\mathrm{Loop}_m(\mathcal{G})$. Moreover, $\mathrm{Bis}_m(\mathcal{G})$ is a co-Banach submanifold in both $\mathrm{Loop}_m(\mathcal{G})$ and $\mathrm{Bis}(\mathcal{G})$. Thus the homogeneous spaces $\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ and $\mathrm{Loop}_m(\mathcal{G})/\mathrm{Bis}\,\mathcal{G}_m$ carry manifold structures by [7, Theorem G (a)]. Furthermore, since $\mathrm{Bis}_m(\mathcal{G})$ is a normal Lie subgroup, the manifold $\mathrm{Loop}_m(\mathcal{G})/\mathrm{Bis}\,\mathcal{G}_m$ becomes a Lie group. In both cases this manifold structure turns the canonical quotient map into a submersion. By the uniqueness assertion in Proposition 3.6 the manifold structures on the homogeneous spaces must coincide. □

The interesting feature of the manifold structure obtained on the homogeneous spaces $\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ and $\mathrm{Loop}_m(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ for Banach–Lie groupoids is that it exactly coincides with the structure induced by the fibre. Hence, under some assumptions, we can endow the quotient $\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}(\mathcal{G})_*$ with a manifold structure which does not a priori use the manifold structure on the α-fibre. We will apply these results in the next section after we compile some more facts on natural group actions on the quotient $\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$.

Lemma 3.9. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid, fix $m \in M$ and define $\Lambda_m := \mathrm{Loop}_m(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$. Then the quotient $\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ admits left- and right group actions*

$$\lambda_{\mathrm{Bis}(\mathcal{G})}\colon \mathrm{Bis}(\mathcal{G}) \times (\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})) \to \mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G}), (\sigma, \tau\,\mathrm{Bis}_m(\mathcal{G})) \mapsto (\sigma \star \tau)\,\mathrm{Bis}_m(\mathcal{G})$$

$$\rho_{\Lambda_m}\colon (\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})) \times \Lambda_m \to \mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G}), (\sigma\,\mathrm{Bis}_m(\mathcal{G}), \gamma\,\mathrm{Bis}_m(\mathcal{G})) \mapsto (\sigma \star \gamma)\,\mathrm{Bis}_m(\mathcal{G})$$

which commute, i.e. $\lambda_{\mathrm{Bis}(\mathcal{G})}(\sigma, \cdot) \circ \rho_{\Lambda_m}([\tau], \cdot) = \rho_{\Lambda_m}([\tau], \cdot) \circ \lambda_{\mathrm{Bis}(\mathcal{G})}(\sigma, \cdot)$ for all $\sigma \in \mathrm{Bis}(\mathcal{G})$ and $[\tau] \in \Lambda_m$.

Proof. Observe that $\mathrm{Bis}(\mathcal{G})$ acts via left translation on itself and this action descents to a group action on the quotient $\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$. Moreover, $\mathrm{Loop}_m(\mathcal{G})$ acts via right translation on $\mathrm{Bis}(\mathcal{G})$ and this action descents to a $\Lambda_m = \mathrm{Loop}_m(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$-action on the quotient. The left action by left translation on $\mathrm{Bis}(\mathcal{G})$ commutes with the right translation with elements in $\mathrm{Loop}_m(\mathcal{G})$, whence the induced actions on the quotient commute. □

Lemma 3.10. *Let $\mathcal{G}$ be a transitive Lie groupoid which admits bisections through each arrow. Then $\lambda_{\mathrm{Bis}(\mathcal{G})}$ is an effective group action, i.e. $\lambda_{\mathrm{Bis}(\mathcal{G})}(\sigma, \cdot) = \mathrm{id}_{\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})}$ implies $\sigma = 1$.*

Proof. The prerequisites imply that ev_m is a surjective map. Consider $\sigma \in \mathrm{Bis}(\mathcal{G}) \setminus \{1\}$ and choose $n \in M$ such that $\sigma(n) \neq 1_n$. Now $\mathcal{G}$ is transitive, whence there is $g_n \in \alpha^{-1}(m)$ with $\beta(g) = n$. As ev_m is surjective we can choose $\tau \in \mathrm{Bis}(\mathcal{G})$ with $\tau(m) = g_n$. Arguing indirectly, we assume that $[\sigma \star \tau] = [\tau]$, i.e. there is $s \in \mathrm{Bis}_m(\mathcal{G})$ with $\sigma \star \tau = \tau \star s$. Evaluating in m, we use $s \in \mathrm{Bis}_m(\mathcal{G})$ to obtain

$$\sigma(n) \cdot g = \sigma(\beta(\tau(m)) \cdot \tau(m) = (\sigma \star \tau)(m) = (\tau \star s)(m) = \tau(m) \cdot 1_m = g$$

Hence, $\sigma(n) = g \cdot g^{-1} = 1_n$ follows, contradicting our choice of n. We conclude that $\lambda_{\mathrm{Bis}(\mathcal{G})}$ is effective. □

Remark 3.11. In general, the left action $\Lambda_{\mathrm{Bis}(\mathcal{G})}$ will not be effective. To see this, we return to the example given in Remark 2.18 b):

Let $M = N \sqcup N'$ be the disjoint union of two non-isomorphic smooth manifolds. Then the pair groupoid $\mathcal{P}(M)$ is locally trivial, but there are arrows which are not contained in the image of any bisection. Fix $m \in N$ and recall that

$$\mathrm{Bis}(\mathcal{G}) \cong \mathrm{Diff}(M) \cong \mathrm{Diff}(N) \times \mathrm{Diff}(N') \text{ and } \mathrm{Bis}_m(\mathcal{G}) \cong \mathrm{Diff}_m(N) \times \mathrm{Diff}(N').$$

Thus $\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G}) \cong \mathrm{Diff}(N)/\mathrm{Diff}_m(N)$. Hence, for $\varphi \in \mathrm{Diff}(N')$ the bisection $\mathrm{id}_N \times \varphi$ acts trivially on $\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$. If N' is not the singleton manifold, choose $\varphi \neq \mathrm{id}_{N'}$, to see that $\lambda_{\mathrm{Bis}(\mathcal{G})}$ can not be effective. □

Lemma 3.12. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a Lie groupoid, fix $m \in M$ and define $\Lambda_m := \mathrm{Loop}_m(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$. Then the map*

$$\widetilde{\mathrm{ev}_m}\colon \mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G}) \to \widetilde{\mathrm{ev}_m}(\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})) \underset{\text{open}}{\subseteq} \alpha^{-1}(m), \sigma\,\mathrm{Bis}_m(\mathcal{G}) \mapsto \sigma(m)$$

(cf. Proposition 3.6) is equivariant with respect to the right Λ_m-action ρ_{Λ_m} and the right $\mathrm{Vert}_m(\mathcal{G})$-action, i.e. for $\tau \in \mathrm{Loop}_m(\mathcal{G})$ and $\sigma \in \mathrm{Bis}(\mathcal{G})$ we obtain the formula

$$\widetilde{\mathrm{ev}_m}((\sigma \star \tau)\,\mathrm{Bis}_m(\mathcal{G}))) = \widetilde{\mathrm{ev}_m}(\rho_{\Lambda_m}(\sigma\,\mathrm{Bis}_m(\mathcal{G}), \tau\,\mathrm{Bis}_m(\mathcal{G}))) = \widetilde{\mathrm{ev}_m}(\sigma\,\mathrm{Bis}_m(\mathcal{G}))) \cdot \tau(m)$$

Proof. Fix $\sigma \in \mathrm{Bis}\,\mathcal{G}$ and $\tau \in \mathrm{Loop}_m(\mathcal{G})$ and compute

$$\widetilde{\mathrm{ev}_m}((\sigma \star \tau)\,\mathrm{Bis}_m(\mathcal{G}))) = (\sigma \star \tau)(m) = \sigma(\underbrace{\beta(\tau(m))}_{=m}) \cdot \tau(m) = \sigma(m) \cdot \tau(m) = \widetilde{\mathrm{ev}_m}(\sigma\,\mathrm{Bis}_m(\mathcal{G}))\tau(m). \qquad \square$$

4. Locally trivial Lie groupoids and transitive group actions

Our aim is now to study the construction of groupoids from their groups of bisections for locally trivial Lie groupoids. Again we consider in this section only Lie groupoids over a compact manifold M that admit an adapted local addition. Moreover, we choose and fix a point $m \in M$.

For locally trivial Lie groupoids the α-fibre over any point already determines the manifold of arrows. Hence, the groupoid quotient discussed in Theorem 2.21 of $\mathcal{B}(\mathcal{G})$ is determined by a quotient of the α-fibre. To construct the quotient, one needs to construct the fibre over a point and the vertex group from the group action of $\mathrm{Bis}(\mathcal{G})$ on M and the subgroup $\mathrm{Bis}_m(\mathcal{G})$. Following Proposition 3.6 these objects can be obtained as certain quotients of the group of bisections. The idea is now to study similar situation for abstract Lie groups and relate these Lie groups to groups of bisections. To this end, we define the central notion of this section:

Definition 4.1. Let $\theta \colon K \times M \to M$ be a transitive (left-)Lie group action of a Lie group K modelled on a metrisable space and H be a subgroup of K.

Then we call (θ, H) a *transitive pair* (over M with base point m) if the following conditions are satisfied:

(P1) the action is smoothly transitive, i.e., the orbit map $\theta_m := \theta(\cdot, m)$ is a surjective submersion,
(P2) H is a normal Lie subgroup of the stabiliser K_m of m and this structure turns H into a regular Lie group which is co-Banach as a submanifold in K_m.

The largest subgroup of H which is a normal subgroup of K is called *kernel* of the transitive pair.[7] If the action of K on M is also n-fold transitive, then we call (θ, H) an *n-fold transitive pair.* $\square$

Transitive pairs are closely related to Klein geometries [29, Chapter 3]. Indeed, they can be understood as infinite-dimensional Klein geometries for principal bundles. This view motivates the notion of the kernel of a transitive pair. We will come back to this perspective in Remark 4.19.

A transitive pair will allow us to construct a locally trivial Lie groupoid which is related to the group action on M. Before we begin with this construction, let us first exhibit two examples of transitive pairs.

Example 4.2.

a) Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally trivial Banach–Lie groupoid over a compact manifold M. By [31, Proposition 3.12] $\mathcal{G}$ admits an adapted local addition, whence $\mathrm{Bis}(\mathcal{G})$ becomes a Lie group. Then the Lie group action $\theta \colon \mathrm{Bis}(\mathcal{G}) \times M \to M, \theta := \beta \circ \mathrm{ev}$ induces a submersion $\theta_m = \beta|_{\alpha^{-1}(m)} \circ \mathrm{ev}_m$ by

[7] We will see in Proposition 4.16 that there exists a kernel for each transitive pair. By standard arguments for topological groups, the kernel is a closed subgroup. In general this will not entail that the kernel is a closed Lie subgroup (of the infinite-dimensional Lie group K).

Lemma 2.19. Observe that $K_m = \mathrm{Loop}_m(\mathcal{G})$ and $\mathrm{Bis}_m(\mathcal{G}) \subseteq \mathrm{Loop}_m(\mathcal{G})$ is a normal subgroup. Combining Proposition 3.2 and Proposition 3.4 we see that $(\theta, \mathrm{Bis}_m(\mathcal{G}))$ is a transitive pair if the action θ is transitive.

In general θ will not be transitive. However, under some mild assumptions, e.g., M being connected or if $\mathcal{G}$ admits bisections through each arrow (see Lemma 2.19), the action will be transitive and we obtain a transitive pair.

The preceding example motivated the definition of a transitive pair. However, one has considerable freedom in choosing the ingredients for such a pair (see also Remark 4.3):

b) Consider the diffeomorphism group $\mathrm{Diff}(M)$ of a compact and connected manifold M. Choose a Lie group B modelled on a Banach space and define $K := \mathrm{Diff}(M) \times B$. Then K becomes a Lie group which acts transitively via $\theta \colon K \times M \to M, ((\varphi, b), m) \mapsto \varphi(m)$. Fix $m \in M$ and observe that θ_m is a submersion as $\theta_m = \mathrm{ev}_m \circ \mathrm{pr}_1$ and $\mathrm{ev}_m \colon \mathrm{Diff}(M) \to M$ is a submersion. By construction $K_m = \mathrm{Diff}_m(M) \times B$ and $H := K_m$ is a regular (and normal) Lie subgroup of K_m by Proposition 3.4.[8] We conclude that $(\mathrm{ev}_m \circ \mathrm{pr}_1, \mathrm{Diff}_m(M) \times B)$ is a transitive pair. □

Remark 4.3. The conditions (P1) and (P2) in Definition 4.1 are quite weak. We illustrate this by rewriting the conditions for finite-dimensional Lie groups:

If K is a finite-dimensional Lie group, the conditions (P1) and (P2) are equivalent to

(Pfin) H is a normal closed subgroup of the stabiliser K_m of m under the action θ.

Proof. To see this note that $K_m = \theta_m^{-1}(m)$ is a closed subgroup of the finite-dimensional Lie group K, whence it is a Lie subgroup of K. Then θ_m factors through $K/K_m \cong M$ and thus (P1) holds as $K \to K/K_m$ is a submersion. Note that H is a closed subgroup of K_m and every Lie subgroup of a finite-dimensional Lie group is co-Banach as a submanifold and a regular Lie group. Hence (Pfin) implies (P2). □

Remark 4.4. From a transitive pair $(\theta \colon K \times M \to M, H)$ we can construct the following normal subgroupoid $N(\theta, H)$ of the action groupoid $K \ltimes_\theta M$. For each $n \in M$, we choose some $k_n \in K$ with $\theta(k_n, m) = n$ and set $H_n := k_n \cdot H \cdot k_n^{-1}$. Then H_n is a normal subgroup of K_m that does not depend on the choice of k_n. Indeed, if $\theta(k_n', m) = n$, then we have $k_n^{-1} k_n' \in K_m$ and thus

$$k_n H k_n^{-1} = k_n k_n^{-1} k_n' H (k_n')^{-1} k_n k_n^{-1} = k_n' H (k_n')^{-1},$$

since H is normal in K_m. Moreover,

$$N(\theta, H) := \bigcup_{n \in M} H_n \times \{n\}$$

is a closed submanifold of $K \times M$, which can be seen as follows: by (P1) there exist for each $n \in M$ an open neighbourhood $U \subseteq M$ of n such that we can choose k_p to depend smoothly on p for $p \in U$. Then

$$G \times U \to G \times U, \quad (g, p) \mapsto (k_p^{-1} g k_p, p)$$

is a diffeomorphism that maps $N(\theta, N) \cap (G \times U)$ to the submanifold $H \times U$ of $G \times M$. Thus $N(\theta, N)$ is a normal Lie subgroupoid of $K \ltimes_\theta M$.

[8] Here we have used that for the pair groupoid $\mathcal{P}(M)$ the Lie group $\mathrm{Diff}(M) = \mathrm{Bis}(\mathcal{P}(M))$ is regular and $\mathrm{Bis}_m(\mathcal{P}(M)) = \mathrm{Diff}_m(M)$. Moreover, B is regular as a Banach Lie group and $\mathbf{L}(\mathrm{Diff}(M) \times B) \cong \mathbf{L}(\mathrm{Diff}(M)) \times \mathbf{L}(B)$.

On the other hand, given a transitive action of K on M, each normal Lie subgroupoid of $K \ltimes M$ gives rise to a normal subgroup H of K_m, and one easily sees that these two constructions are inverse to each other. Thus normal Lie subgroupoids of action groupoids are the equivalent reformulation of transitive pairs that do not require to fix a point $m \in M$. However, it will be analytically much easier to work with transitive pairs (see Remark 4.10) and to have in mind that the choice of a base point does not matter. □

Now we associate a locally trivial Lie groupoid to a transitive pair. To this end, we will first construct a principal bundle which will then give rise to the desired locally trivial Lie groupoid.

Proposition 4.5. *Let (θ, H) be a transitive pair. Then the quotients K/H and $\Lambda_m := K_m/H$ are Banach manifolds. Moreover, the map θ_m induces a Λ_m-principal bundle $\pi\colon K/H \to M, kH \mapsto \theta(k,m)$.*

To prove Proposition 4.5, the following lemma deals with some needed technical details first (compare [35, Example D.4]).

Lemma 4.6. *Let (θ, H) be a transitive pair. The group action θ induces a $K_m = \theta_m^{-1}(m)$-principal bundle $\theta_m\colon K \to M$. Canonical bundle trivialisations for this bundle are given by*

$$\theta_m^{-1}(U_i) \to U_i \times K_m, \quad g \mapsto (\theta_m(g), \sigma_i(\theta_m(g))^{-1} \cdot g) \tag{14}$$

where $(\sigma_i\colon U_i \to \theta_m^{-1}(U_i))_{i\in I}$ is a section atlas of θ_m.

Proof. Note that θ_m is a surjective submersion. Thus [7, Theorem D] implies that K_m is a split Lie subgroup in K and θ_m descents to a homeomorphism $K/K_m \cong M$. Identify K/K_m with the manifold M. Clearly by conjugation $K_m \cong \theta_m^{-1}(n)$ for all $n \in M$. It is now trivial to check that (14) yields bundle trivialisations whose trivialisation changes are K_m-torsor isomorphisms. □

Proof of Proposition 4.5. The manifold M is finite-dimensional, whence K_m is a Lie subgroup of finite codimension (as a submanifold). By (P2) H is a co-Banach submanifold in K_m and thus H is also a co-Banach submanifold of K by [7, Lemma 1.4]. In particular, H is a Lie subgroup of K. As H is a regular Lie group by (P2), we can apply [7, Theorem G (a)] to obtain a manifold structure on the quotients

$$p_m\colon K \to K/H \text{ and } q_m\colon K_m \to K_m/H =: \Lambda_m$$

turning the projections into submersions. Moreover, we deduce from [7, Theorem G] that K/H is a Banach manifold, Λ_m is a Banach–Lie group, q_m is a morphism of Lie groups and Λ_m acts on K/H via

$$\rho_{\Lambda_m}\colon K/H \times \Lambda_m \to K/H, (gH, \lambda H) \mapsto (g \cdot \lambda)H.$$

The subgroup H is contained in K_m, whence θ_m induces a map $\pi\colon K/H \to M$ which satisfies $\pi \circ p_m = \theta_m$. Now p_m and θ_m are submersions, whence π is smooth with surjective tangent map at every point (cf. [7, p.2 and Lemma 1.8]). Since M is finite-dimensional, [7, Theorem A] implies that π is a submersion. The action θ is transitive and thus π is surjective submersion. Note that the π-fibre over a point $n \in M$ is given by

$$\pi^{-1}(n) = p_m(\theta_m^{-1}(n)) = \{gH \in K/H \mid \theta(g,m) = n\}.$$

Now it is easy to see that the π-fibres coincide with the orbits of the action ρ_{Λ_m} and the ρ_{Λ_m}-action on the fibres is free, i.e. $\pi^{-1}(n)$ is a Λ_m-torsor for each $n \in M$.

To turn $\pi\colon K/H \to M$ into a Λ_m-principal bundle we will now prove that the change of trivialisations induce Λ_m-torsor isomorphisms. Recall that by Lemma 4.6 the bundle $\theta_m\colon K \to M$ is a K_m-principal bundle. The trivialisations (14) descent to K/H via

$$\kappa_i\colon \pi^{-1}(U_i) \to U_i \times \Lambda_m, \quad (gH) \mapsto (\pi(gH), ((\sigma_i(\pi(gH)))^{-1} \cdot g)H). \tag{15}$$

For each $i \in I$ we obtain a commutative diagram

$$\begin{tikzcd}
K \arrow[ddd, "\theta_m"'] \arrow[dr, "p_m"] & & \theta_m^{-1}(U_i) \arrow[ll, "\supseteq"'] \arrow[rr, "(14)"] \arrow[d] & & U_i \times K_m \arrow[dl, "\mathrm{id}_{U_i} \times q_m"] \arrow[ddd, "\mathrm{pr}_1"] \\
& K/H \arrow[ddl, "\pi"] & \pi^{-1}(U_i) \arrow[l, "\supseteq"'] \arrow[r, "\kappa_i"] \arrow[dd] & U_i \times \Lambda_i \arrow[ddr, "\mathrm{pr}_1"] & \\
& & & & \\
M & & U_i \arrow[ll, "\supseteq"'] \arrow[rr, equal] & & U_i
\end{tikzcd}$$

and the sets $\pi^{-1}(U_i), i \in I$ cover K/H. Note that the trivialisation changes descent to Λ_m-torsor isomorphisms, whence the κ_i form an atlas of Λ_m-principal bundle trivialisations for K/H. Summing up, we have constructed a principal Λ_m-bundle $\pi\colon K/H \to M$. $\square$

Example 4.7. Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally trivial Banach–Lie groupoid such that there is a bisection through each element in G. Consider the transitive pair $(\beta \circ \mathrm{ev}_m, \mathrm{Bis}_m(\mathcal{G}))$ discussed in Example 4.2 a). The m-stabiliser of $\beta \circ \mathrm{ev}$ coincides with $\mathrm{Loop}_m(\mathcal{G})$ and Lemma 4.6 yields the $\mathrm{Loop}_m(\mathcal{G})$-bundle $\beta \circ \mathrm{ev}_m\colon \mathrm{Bis}(\mathcal{G}) \to M$. Moreover, the Λ_m-principal bundle constructed in Proposition 4.5 is $\pi\colon \mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G}) \to M, \sigma\,\mathrm{Bis}_m(\mathcal{G}) \mapsto \beta(\sigma(m))$ with $\Lambda_m = \mathrm{Loop}_m(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$. Then the α-fibre through m yields a $\mathrm{Vert}_m(\mathcal{G})$-principal bundle $\beta|_{\alpha^{-1}(m)}\colon \alpha^{-1}(m) \to M$. Now Proposition 3.6 allows us to identify Λ_m and $\mathrm{Vert}_m(\mathcal{G})$ and Lemma 3.12 shows that $\mathrm{ev}_m\colon \mathrm{Bis}(\mathcal{G}) \to \alpha^{-1}(m)$ descends to a Λ_m-principal bundle isomorphism

$$\begin{tikzcd}
\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G}) \arrow[rr, "\widetilde{\mathrm{ev}_m}"] \arrow[dr, "\pi"'] & & \alpha^{-1}(m) \arrow[dl, "\beta|_{\alpha^{-1}(m)}"] \\
& M &
\end{tikzcd} . \quad \square$$

Definition 4.8. Let (θ, H) be a transitive pair with associated Λ_m-principal bundle $\pi\colon K/H \xrightarrow{\Lambda_m} M$. As principal bundles correspond to locally trivial groupoids, this allows us to construct a gauge groupoid

$$\mathcal{R}(\theta, H) := \left(\begin{array}{c} \frac{K/H \times K/H}{\Lambda_m} \\ \alpha_{\mathcal{R}} \downarrow\,\downarrow \beta_{\mathcal{R}} \\ M. \end{array} \right)$$

with $\alpha_{\mathcal{R}}(\langle gH, kH\rangle) = \pi(kH)$ and $\beta_{\mathcal{R}}(\langle gH, kH\rangle) = \pi(gH)$. $\square$

Note that we will work with the gauge groupoid $\mathcal{R}(\theta, H)$ associated to the transitive pair (θ, H) and not with the principal bundle (although many constructions will be carried out in the context of principal bundles). The reason for this is that many interesting maps considered later can not be described as morphisms of principal bundles (with fixed structure group). However, one can treat these maps as morphisms of (locally trivial) Lie groupoids over the fixed base M. Hence we prefer the groupoid perspective.

Before we continue, let us record some technical details on the construction of the Lie groupoid $\mathcal{R}(\theta, H)$.

Remark 4.9. Let (θ, H) be a transitive pair with $\theta\colon K \times M \to M$.

a) Observe that as K/H is a Banach manifold, the gauge groupoid $\mathcal{R}(\theta, H)$ is a Banach–Lie groupoid and thus $\mathcal{R}(\theta, H)$ admits an adapted local addition by [31, Proposition 3.12]. Moreover, the gauge groupoid $\mathcal{R}(\theta, H)$ is source connected if and only if K/H is connected.

b) Choose a section atlas $(\sigma_i, U_i)_{i \in I}$ of $\theta_m\colon K \xrightarrow{K_m} M$ as in (14). This atlas induces a section atlas $s_i := p_m \circ \sigma_i\colon U_i \to \pi^{-1}(U_i) \subseteq K/H$ of the bundle $\pi\colon K/H \xrightarrow{\Lambda_m} M$ (cf. the proof of Proposition 4.5). Using these section, we identify the bisections of $\mathcal{R}(\theta, H)$ with bundle automorphisms via the Lie group isomorphism from [31, Example 3.16]

$$\operatorname{Aut}(\pi\colon K/H \to M) \to \operatorname{Bis}(\mathcal{R}(\theta, H)), \quad f \mapsto (m \mapsto \langle f(s_i(m)), s_i(m)\rangle, \text{ if } m \in U_i. \tag{16}$$

For later use we recall that the bundle trivialisations (15) induce charts for the manifold $\frac{K/H \times K/H}{\Lambda_m}$ via

$$\frac{\pi^{-1}(U_i) \times \pi^{-1}(U_j)}{\Lambda_m} \to U_i \times U_j \times \Lambda_m, \quad \langle p_1, p_2\rangle \mapsto (\pi(p_1), \pi(p_2), \delta(\sigma_i(\pi(p_1)), p_1)\delta(\sigma_j(\pi(p_2), p_2)^{-1}), \tag{17}$$

where $\delta\colon K/H \times_\pi K/H \to \Lambda_m$ is the smooth map mapping a pair (p, q) to the element $p^{-1} \cdot q \in \Lambda_m$ which maps p to q (via the Λ_m-right action). □

Remark 4.10. In the base point free version of transitive pairs from Remark 4.4, one constructs the groupoid $\mathcal{R}(\theta, H)$ by taking the quotient

$$(K \ltimes_\theta M)/N(\theta, H).$$

Indeed, we have the isomorphisms of Lie groupoids over M

$$K \ltimes_\theta M := \left(\begin{array}{c} K \times M \\ \downarrow\downarrow \\ M \end{array}\right) \cong \left(\begin{array}{c} \frac{(K \times K)}{K_m} \\ \downarrow\downarrow \\ M \end{array}\right) \cong \left(\begin{array}{c} \frac{(K \times K)}{H} \Big/ \frac{H}{K_m} \\ \downarrow\downarrow \\ M \end{array}\right).$$

Then $N(\theta, H)$ corresponds exactly to the normal subgroupoid $\frac{(H \times H)}{H} \Big/ \frac{H}{K_m}$ on the right hand side.

However, it is much harder to construct the smooth structure on the quotient $(K \ltimes_\theta M)/N(\theta, H)$ directly. For instance, in the finite-dimensional case (or also in the case of K being a Banach–Lie group) one can use Godement's criterion for the existence of the quotient $(K \ltimes_\theta M)/N(\theta, H)$ in the category of smooth manifolds (cf. [16, Theorem 2.2.4]). It is not known to the authors whether Godement's criterion extends beyond Banach manifolds, whereas the construction of $\mathcal{R}(\theta, H)$ as in 4.8 is possible in the full generality of our definition of a transitive pair. □

So far we have constructed a locally trivial groupoid $\mathcal{R}(\theta, H)$ associated to a transitive pair (θ, H). Let us now analyse how the Lie group K (i.e. the group acting via θ on M) is related to the Lie group $\operatorname{Bis}(\mathcal{R}(\theta, H))$. To this end, we study a natural Lie group morphism $K \to \operatorname{Bis}(\mathcal{R}(\theta, H))$ which is closely related to the action of the transitive pair.

Lemma 4.11. *Let (θ, H) be a transitive pair. Then the action of K on K/H by left multiplication gives rise to a group homomorphism $K \to \operatorname{Aut}(\pi\colon K/H \xrightarrow{\Lambda_m} M)$. With respect to the canonical isomorphism $\operatorname{Aut}(\pi\colon K/H \xrightarrow{\Lambda_m} M) \cong \operatorname{Bis}(\mathcal{R}(\theta, H))$ of Lie groups from (16) this gives rise to the group homomorphism*

$$a_{\theta,H}\colon K \to \operatorname{Bis}(\mathcal{R}(\theta,H)), \quad k \mapsto (x \mapsto \langle k\cdot s_i(x), s_i(x)\rangle, \textit{ for } x \in U_i),$$

where $s_i = p_m \circ \sigma_i, i \in I$ are the sections from 4.9 b). Moreover, $a_{\theta,H}$ is smooth and thus a morphism of Lie groups.

Proof. Consider the smooth group action $\lambda_K\colon K \times K/H \to K/H, (k, gH) \mapsto \lambda_k(gH) := (kg)H$. By Lemma 3.9 this group action commutes with the right action ρ_{Λ_m} on K/H. Hence for each $k \in K$ the map $\lambda_K(k)\colon K/H \to K/H$ is a bundle automorphism of the Λ_m-principal bundle. Now $a_{\theta,H}(k)$ is the image of the bundle automorphism $\lambda_K(k)$ under the Lie group isomorphism (16). Since $\lambda_K(kk') = \lambda_K(k)\lambda_K(k')$, we derive that $a_{\theta,H}$ is a group homomorphism.

Let us now prove that $a_{\theta,H}$ is smooth. To this end recall that K and $\frac{K/H\times K/H}{\Lambda_m}$ are modelled on metrisable spaces and $\operatorname{Bis}(\mathcal{R}(\theta,H)) \subseteq C^\infty(M, \frac{K/H\times K/H}{\Lambda_m})$. Since M is compact, we apply the exponential law Theorem B.9 c) to see that $a_{\theta,H}$ will be smooth if the map

$$a^\vee_{\theta,H}\colon K\times M \to \frac{K/H\times K/H}{\Lambda_m}, (k,x) \mapsto \langle k\cdot s_i(x), s_i(x)\rangle, \text{ for } x\in U_i$$

is smooth. We work locally around $(k,x) \in K\times M$. Fix $i \in I$ such that $x \in U_i$ and recall that $s_i = p_m \circ \sigma_i$ and $\theta_m(\sigma_i) = \mathrm{id}_{U_i}$ hold. Then we have

$$\beta_{\mathcal{R}}(\langle k\cdot s_i(x), s_i(x)\rangle) = \pi(k\cdot s_i(x)) = \theta_m(k\cdot\sigma_i(x)) = \theta(k, \theta_m(\sigma_i(x)) = \theta(k,x). \tag{18}$$

Now we choose $j \in I$ with $\theta(k,x) \in U_j$ and denote by κ_{ji} the manifold charts (17) defined in Remark 4.9. Then the composition $\kappa_{ji}\circ a_{\theta,H}$ which is defined at least on the pair (k,x) and we compute:

$$\kappa_{ji}\circ a^\vee_{\theta,H}(k,x) \overset{(17)}{=} (\pi(k\cdot s_i(x)), \pi(s_i(x)), \delta(s_j(\pi(k\cdot s_i(x))), k\cdot s_i(x))\delta^{-1}(s_i\circ \underbrace{\pi\circ s_i(x)}_{=x}, s_i(x)))$$

$$\overset{(18)}{=} (\theta(k,x), x, \delta(s_j(\theta(k,x)), k\cdot s_i(x))) = (\theta(k,x), x, \delta(s_j(\theta(k,x)), \lambda_K(k, s_i(x))))$$

Note that the above formula did not depend on (k,x), whence it is valid for all (g,y) with $a^\vee_{\theta,H}(g,y) \in \frac{\pi^{-1}(U_j)\times\pi^{-1}(U_i)}{\Lambda_m}$. In particular, we see that $\kappa_{ji}\circ a^\vee_{\theta,H}$ is smooth as a composition of the smooth maps θ, δ and λ_K. Since $\frac{K/H\times K/H}{\Lambda_m}$ carries the identification topology with respect to the atlas $(K_{ji})_{i,j\in I}$, we deduce that $a^\vee_{\theta,H}$ is smooth. Summing up, this proves that $a_{\theta,H}$ is smooth and thus a Lie group morphism. □

Before we clarify the relation of $a_{\theta,H}$ and θ let us return briefly to the problem of (re-)constructing a Lie groupoid from its group of bisections (see Theorem 2.21 and Remark 2.22). To obtain a construction principle for Lie groupoids, we would like $a_{\theta,H}$ to be an isomorphism of Lie groups. Then $a_{\theta,H}$ would identify the Lie group with the group of bisections of $\mathcal{R}(\theta,H)$ and thus transitive pairs would induce (up to isomorphism) unique locally trivial Lie groupoids. However, in general for an arbitrary transitive pair (θ,H) the Lie group morphism $a_{\theta,H}$ will neither be injective nor surjective. We illustrate this with two examples:

Example 4.12.

a) Let K be a compact finite-dimensional Lie group. Then K acts on itself transitively via left multiplication $\lambda\colon K\times K \to K$. Take $m = 1_K$ and $H = \{1_K\}$ to obtain the principal bundle $\mathrm{id}_K\colon K \xrightarrow{H} K$. The associated gauge groupoid is the pair groupoid $K\times K \rightrightarrows K$ whose bisections are given by $\operatorname{Diff}(K)$. Taking this identification, $a_{\theta,H}$ becomes the map $K \to \operatorname{Diff}(K), k\mapsto \lambda(k,\cdot)$ which will only be surjective in trivial cases.

b) We return to Example 4.2 b): Let B be a Banach Lie group, M a compact connected manifold and $m \in M$. Then $(\mathrm{ev}_m \circ \mathrm{pr}_1, \mathrm{Diff}_m(M) \times B)$ is a transitive pair. Set $H := \mathrm{Diff}_m(M) \times B$ and observe

$$(\mathrm{Diff}(M) \times B)/H \cong \mathrm{Diff}(M)/\mathrm{Diff}_m(M) \cong M.$$

Moreover, since H is the m-stabiliser of the action $\mathrm{ev}_m \circ \mathrm{pr}_1$, we deduce that $\mathcal{R}(\mathrm{ev}_m \circ \mathrm{pr}_1, H)$ is isomorphic to the pair groupoid $\mathcal{P}(M)$. With respect to these identifications, the map $a_{\mathrm{ev}_m \circ \mathrm{pr}_1, H}$ becomes

$$\mathrm{Diff}(M) \times B \to \mathrm{Bis}(\mathcal{P}(M)) \cong \mathrm{Diff}(M), (\varphi, b) \mapsto \varphi$$

which is surjective but can not be injective for non-trivial B. Note that this example arose from enlarging $\mathrm{Diff}(M) \cong \mathrm{Bis}(\mathcal{P}(M))$. Moreover, we record that the action of $\mathrm{Diff}(M) \times B$ by left multiplication on $(\mathrm{Diff}(M) \times B)/H$ is not effective and this causes $a_{\mathrm{ev}_m \circ \mathrm{pr}_1, H}$ to be not injective (cf. Lemma 4.15 below). $\square$

As we have already pointed out, transitive pairs are quite flexible and more general than groups of bisections (of locally trivial Lie groupoids) However, transitive pairs are a source of Lie group morphisms from Lie groups with transitive actions on M into the bisections of suitable locally trivial Lie groupoids over M. In particular, the morphism $a_{\theta,H}$ is closely related to the action θ as the following lemma shows.

Lemma 4.13. *For a transitive pair (θ, H) the Lie group morphism $a_{\theta,H}$ makes the diagram*

$$\begin{tikzcd} K \arrow[r, "a_{\theta,H}"] \arrow[dr, "\theta^\wedge"'] & \mathrm{Bis}(\mathcal{R}(\theta, H)) \arrow[d, "(\beta_{\mathcal{R}})_*"] \\ & \mathrm{Diff}(M) \end{tikzcd}$$

commutative. If $a_{\theta,H}$ is an isomorphism of Lie groups, then the map $\theta^\wedge$ is a submersion.

Proof. Observe first that for $x \in M$ we have (after choosing an appropriate section s_i) the formula (18)

$$\beta_{\mathcal{R}} \circ a_{\theta,H}(k)(x) = \beta_{\mathcal{R}}(\langle k \cdot s_i(x), s_i(x)\rangle) = \theta(k, x) = \theta^\wedge(k)(x).$$

Hence $(\beta_{\mathcal{R}})_* \circ a_{\theta,H} = \theta^\wedge$ and the diagram commutes. This also entails $a_{\theta,H}(H) \subseteq \mathrm{Loop}_m(\mathcal{R}(\theta, H))$.

If $a_{\theta,H}$ is a Lie group isomorphism, $\theta^\wedge$ is a submersion if $(\beta_{\mathcal{R}})_* \colon \mathrm{Bis}(\mathcal{R}(\theta, H)) \to \mathrm{Diff}(M)$ is a submersion. However, since $\mathcal{R}(\theta, H)$ is locally trivial, the map $(\beta_{\mathcal{R}})_*$ is a submersion by [31, Example 3.16]. $\square$

Remark 4.14. That $\theta^\wedge$ must be a submersion if $a_{\theta,H}$ is an isomorphism of Lie groups can be understood as the statement that the Lie group K needs to be large enough to be eligible to be the bisection group. In particular, if M is not a zero-dimensional manifold, this condition rules out every transitive pair which arises by a group action of a finite-dimensional Lie group. $\square$

Building on the observation in Example 4.12 b) we will now develop a simple criterion which ensures that for a given transitive pair (θ, H) the morphism $a_{\theta,H}$ is injective.

Lemma 4.15. *Let (θ, H) be a transitive pair. The Lie group morphism*

$$a_{\theta,H} \colon K \to \mathrm{Bis}(\mathcal{R}(\theta, H)), g \mapsto (kH \mapsto (g \cdot k)H)$$

is injective if and only if the action $\lambda_H \colon H \times K/H \to K/H, (h, gH) \mapsto (hg)H$ is effective.

Proof. The isomorphism $\mathrm{Aut}(\pi\colon K/H \xrightarrow{\Lambda_m} M) \cong \mathrm{Bis}(\mathcal{R}(\theta, H))$ allows us to rewrite $a_{\theta,H}$ as the left multiplication $\lambda_K\colon K \to \mathrm{Aut}(\pi\colon K/H \xrightarrow{\Lambda_m} M)$, where $\lambda_K(k)\colon K/H \to K/H, gH \mapsto (kg)H$. Assume first that $a_{\theta,H}$ (and thus also λ_K) is injective. Now consider $k \in H$ such that $(hg)H = gH$ for all $g \in G$. This implies $\lambda_K(h)(gH) = (hg)H = gH = \mathrm{id}_{K/H}(gH)$, whence $h = 1_K$ as the group homomorphism λ_K is injective.

Conversely assume that the action $H \times K/H \to K/H, (h, gH) \mapsto (hg)H$ is effective and consider $g \in \ker \lambda_K$, i.e. $\lambda_K(g) = \mathrm{id}_{G/H}$. Then $(g \cdot k)H = kH$ holds for all $kH \in G/H$. As this entails $gH = 1_G H$, we deduce $g \in H$. Now the left action of H on G/H by multiplication is effective, forcing $g \in H$ to be the identity 1_K. □

Proposition 4.16. *Consider a transitive pair (θ, H) and denote by λ_H the left action on the quotient K/H as in Lemma 4.15. Then the kernel of $a_{\theta,H}$ is given by*

$$\ker a_{\theta,H} = \{h \in H \mid \lambda_H(h, \cdot) \equiv \mathrm{id}_{K/H}\}$$

and coincides with the kernel of the transitive pair (θ, H). In particular every transitive pair admits a unique kernel.

Proof. Recall from the proof of Lemma 4.15 that $\ker a_{\theta,H}$ is contained in H and consists of all elements of H which act trivially by left multiplication on K/H. Thus we obtain the first description of $\ker a_{\theta,H}$.

As a kernel of a Lie group morphism, $\ker a_{\theta,H}$ is a closed and normal subgroup of K. Let us now prove that every subgroup G of H which is normal in K is contained in $\ker a_{\theta,H}$. Then for $k \in K$ and $g \in G \subseteq H$ we derive from G being normal in K that $gk = kg'$ for $g' \in H$, i.e. $gkH = kH$ for all $k \in K$. Thus elements in G act trivially on K/H, whence $G \subseteq \ker a_{\theta,H}$. We conclude that $\ker a_{\theta,H}$ is the kernel of the transitive pair (θ, H). □

Definition 4.17. A transitive pair (θ, H) is called *effective* if H acts effectively on the quotient K/H by left multiplication, i.e. the kernel of the transitive pair is trivial. □

The characterisation of the kernel of a transitive pair in Proposition 4.16 can be used to compute it. For the examples considered in this section we obtain:

Example 4.18. The transitive pairs in Example 4.2 a) and 4.12 a) are effective, whence the kernel is trivial. For the transitive pair $(\mathrm{ev}_m \circ \mathrm{pr}_1, \mathrm{Diff}(M)_m \times B)$ from Example 4.2 b) the kernel is $\{\mathrm{id}_M\} \times B$ (by Example 4.12 b)). In Theorem 5.15 below the kernel of a class of transitive pairs arising from extensions of diffeomorphism groups is computed. □

Although the criterion for the injectivity of $a_{\theta,H}$ gives rise to a very natural condition on the transitive pair, the question of surjectivity is much more subtle.

Remark 4.19. We now describe the relation between Klein geometries [29, Chapter 3] and transitive pairs. First note that our setting is infinite-dimensional, and the notion of a transitive pair takes the additional analytical issues caused by this into account.

Recall that a *Klein geometry* is a pair (K, H), where K is a finite-dimensional Lie group (called the *principal group*) and H is a closed subgroup such that the manifold K/H is connected. The *kernel* of a Klein geometry is the largest subgroup L of H which is normal in K. A Klein geometry is called *effective* if L is trivial. Klein geometries are constructed to model geometry via the principal H-bundle $K \to K/H$.

Note that the principal group of a Klein geometry is finite-dimensional. Hence the quotient K/H inherits a canonical manifold structure turning $K \to K/H$ into a submersion. In our infinite-dimensional setting the quotient does not automatically inherit a manifold structure, whence a transitive pair has to guarantee this behaviour via extra assumptions (cf. Proposition 4.5).

In studying a transitive pair (θ, H), we are interested in the principal K_m/H-bundle $K/H \to K/K_m = M$. Thus a transitive pair encodes more information than a Klein geometry, as the principal H-bundle $K \to K/H$ is obtained as additional information. To some extent, one can interpret a transitive pair as a "Klein geometry for principal bundles". In particular, the notion of a transitive pair also covers the concept of an (infinite-dimensional) Klein geometry in the case that $K_m = H$ and the quotient K/H is connected.

Finally, there is a close connection between effective transitive pairs and effective Klein geometries. Namely, the kernel of the transitive pair (i.e. the kernel of the Lie group morphism $a_{\theta,H}\colon K \to \mathcal{R}(\theta, H)$) is by definition the largest closed subgroup of H which is normal in K. Hence the kernel of the transitive pair (θ, H) corresponds to the kernel of a Klein geometry. Let us stress again that contrary to the finite dimensional case, the kernel of a transitive pair will only be a closed subgroup and not automatically a closed Lie subgroup. Summing up, if the pair (θ, H) is effective and $H = K_m$ and K/H is connected, then the transitive pair corresponds to an (infinite-dimensional) effective Klein geometry. $\square$

Let us now return to Example 4.2 a) and consider $\mathcal{R}(\theta, H)$ and $a_{\theta,H}$ for the action of a bisection group on M. We will see that the constructions given in this section are in a certain sense inverse to computing the bisections of a locally trivial groupoid.

Example 4.20. Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally trivial Banach–Lie groupoid such that there is a bisection through every $g \in G$. Consider the group action $\beta \circ \mathrm{ev}\colon \operatorname{Bis}(\mathcal{G}) \times M \to M, (\tau, m) \mapsto \beta \circ \tau(m)$. We have seen in Example 4.2 a) that $(\beta \circ \mathrm{ev}, \operatorname{Bis}_m(\mathcal{G}))$ is a transitive pair. Moreover, since every locally trivial Lie groupoid is transitive, Lemma 3.10 shows that $(\beta \circ \mathrm{ev}, \operatorname{Bis}_m(\mathcal{G}))$ is an effective transitive pair. For this special effective transitive pair we note the following consequences (a detailed verification for the claims made in this example can be found in Lemma 4.21 below):

The groupoid $\mathcal{G}$ is isomorphic to $\mathcal{R}(\beta \circ \mathrm{ev}, \operatorname{Bis}_m(\mathcal{G}))$. We have already seen in Example 4.7 that the vertex bundles associated to the locally trivial Lie groupoids $\mathcal{G}$ and $\mathcal{R}(\beta \circ \mathrm{ev}, \operatorname{Bis}_m(\mathcal{G}))$ are isomorphic. Moreover, from Proposition 3.6 we recall that $\operatorname{Bis}(\mathcal{G})/\operatorname{Bis}_m(\mathcal{G}) \cong \alpha^{-1}(m)$ and $\operatorname{Loop}_m(\mathcal{G})/\operatorname{Bis}_m(\mathcal{G}) \cong \operatorname{Vert}_m(\mathcal{G})$. Hence the gauge groupoid $\mathcal{R}(\beta \circ \mathrm{ev}, \operatorname{Bis}_m(\mathcal{G}))$ is canonically isomorphic to the gauge groupoid of a vertex bundle of $\mathcal{G}$. Consider the canonical map over M from the gauge groupoid $\mathcal{R}(\beta \circ \mathrm{ev}, \operatorname{Bis}_m(\mathcal{G}))$ to the locally trivial Lie groupoid $\mathcal{G}$

$$\chi_{\mathcal{G}}\colon \frac{\operatorname{Bis}(\mathcal{G})/\operatorname{Bis}_m(\mathcal{G}) \times \operatorname{Bis}(\mathcal{G})/\operatorname{Bis}_m(\mathcal{G})}{\operatorname{Loop}_m(\mathcal{G})/\operatorname{Bis}_m(\mathcal{G})} \to G, \langle \sigma \operatorname{Bis}_m(\mathcal{G}), \tau \operatorname{Bis}_m(\mathcal{G})\rangle \mapsto \sigma(m) \cdot (\tau(m))^{-1}. \tag{19}$$

Since $\mathcal{G}$ admits bisections through each arrow and $\mathcal{G}$ is locally trivial, Proposition 3.6 shows that the image of $\chi_{\mathcal{G}}$ coincides with G. One then proves that $\chi_{\mathcal{G}}$ is an isomorphism of Lie groupoids over M. We can thus recover the locally trivial Lie groupoid $\mathcal{G}$ (up to isomorphism depending on m) as the groupoid $\mathcal{R}(\beta \circ \mathrm{ev}, \operatorname{Bis}_m(\mathcal{G}))$.

The map $a_{\beta\circ\mathrm{ev},\operatorname{Bis}_m(\mathcal{G})}\colon \operatorname{Bis}(\mathcal{G}) \to \operatorname{Bis}(\mathcal{R}(\beta \circ \mathrm{ev}, \operatorname{Bis}_m(\mathcal{G})))$ is a Lie group isomorphism. For this special transitive pair, an inverse of $a_{\beta\circ\mathrm{ev},\operatorname{Bis}_m(\mathcal{G})}$ is given by $\operatorname{Bis}(\chi_{\mathcal{G}})\colon \operatorname{Bis}(\mathcal{R}(\theta, H)) \to \operatorname{Bis}(\mathcal{G}), \sigma \mapsto \chi_{\mathcal{G}} \circ \sigma$. $\square$

Lemma 4.21. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally trivial Banach–Lie groupoid such that the action $\beta \circ \mathrm{ev}$ of the bisections on M is transitive, e.g. $\mathcal{G}$ is source-connected. Then*

a) *the Lie groupoid morphism $\chi_{\mathcal{G}}$ from (19) induces an isomorphism onto the open subgroupoid $\mathrm{ev}(\operatorname{Bis}(\mathcal{G}) \times M)$ of $\mathcal{G}$. Hence ev is surjective if and only if $\chi_{\mathcal{G}}\colon \mathcal{R}(\beta \circ \mathrm{ev}, \operatorname{Bis}_m(\mathcal{G})) \to \mathcal{G}$ is an isomorphism;*

b) *the Lie group morphism* $\mathrm{Bis}(\chi_{\mathcal{G}})\colon \mathrm{Bis}(\mathcal{R}(\beta \circ \mathrm{ev}, \mathrm{Bis}_m(\mathcal{G}))) \to \mathrm{Bis}(\mathcal{G})$ *is an isomorphism with inverse* $a_{\beta\circ\mathrm{ev},\mathrm{Bis}_m(\mathcal{G})}$.

Proof.

a) Clearly $\chi_{\mathcal{G}}$ is injective and after restricting it to its image, it becomes a bijection. Now consider the commutative diagram

$$\begin{tikzcd} \mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G}) \times \mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G}) \arrow[r, "\widetilde{\mathrm{ev}_m}\times\widetilde{\mathrm{ev}_m}"] \arrow[d, "q"] & \alpha^{-1}(m) \times \alpha^{-1}(m) \arrow[d, "(\xi,\eta)\mapsto \xi\cdot\eta^{-1}"] \\ \frac{\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})\times \mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})}{\mathrm{Loop}_m(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})} \arrow[r, "\chi_{\mathcal{G}}"] & G \end{tikzcd} . \tag{20}$$

Here q is the canonical quotient map which is a submersion. The map $\widetilde{\mathrm{ev}_m} \times \widetilde{\mathrm{ev}_m}$ is the map induced on the quotient via $\widetilde{\mathrm{ev}_m} \times \widetilde{\mathrm{ev}_m} \circ (p_m \times p_m) = \mathrm{ev}_m \times \mathrm{ev}_m$ where $p_m\colon \mathrm{Bis}(\mathcal{G}) \to \mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ is the canonical quotient map. Now $\mathrm{ev}_m \times \mathrm{ev}_m$ is a submersion by [7, Lemma 1.6] and Corollary 2.5). Since $\mathcal{G}$ is a Banach–Lie groupoid and q_m is a submersion, we deduce with [7, Lemma 1.10] that $\widetilde{\mathrm{ev}_m} \times \widetilde{\mathrm{ev}_m}$ is a submersion. Finally, [16, Proposition 1.3.3] shows that the division map $\alpha^{-1}(m)\times\alpha^{-1}(m) \to G, (\xi,\eta) \mapsto \xi\cdot\eta^{-1}$ is a surjective submersion as $\mathcal{G}$ is a locally trivial Lie groupoid.
Note that the division map restricts to a map $\mathrm{im}(\mathrm{ev}_m \times \mathrm{ev}_m) \to \mathrm{ev}(\mathrm{Bis}(\mathcal{G})\times M)$ because for $\sigma,\tau \in \mathrm{Bis}(\mathcal{G})$ we have $\sigma(m)\cdot(\tau(m))^{-1} = (\sigma\star\tau^{-1})(\beta(\tau(m))) \in \mathrm{ev}(\mathrm{Bis}(\mathcal{G})\times M)$. In particular $\mathrm{im}(\chi_{\mathcal{G}}) \subseteq \mathrm{ev}(\mathrm{Bis}(\mathcal{G})\times M)$ We claim that the image of $\chi_{\mathcal{G}}$ coincides with $\mathrm{ev}(\mathrm{Bis}(\mathcal{G})\times M)$. If the claim is true, then the proof can be finished as follows: The set $\mathrm{ev}(\mathrm{Bis}(\mathcal{G})\times M)$ is an open and wide subgroupoid of $\mathcal{G}$ by Theorem 2.14. Hence (20) proves that $\chi_{\mathcal{G}}$ induces a Lie groupoid isomorphism from $\mathcal{R}(\beta\circ\mathrm{ev},\mathrm{Bis}_m(\mathcal{G}))$ onto the subgroupoid $\mathrm{ev}(\mathrm{Bis}(\mathcal{G})\times M)$ of $\mathcal{G}$.

Proof of the claim. Choose a bisection $\gamma \in \mathrm{Bis}(\mathcal{G})$ and $x \in M$ and let us show that $\gamma(x) \in \mathrm{im}\,\chi_{\mathcal{G}}$. The action $\beta\circ\mathrm{ev}\colon \mathrm{Bis}(\mathcal{G}) \times M \to M, (\sigma, y) \mapsto \beta(\sigma(y))$ is transitive. Hence there are $\sigma,\tau \in \mathrm{Bis}(\mathcal{G})$ with $\beta(\tau(m)) = x$ and $\beta(\sigma(m)) = \beta(\gamma(x))$. We compute

$$(\gamma(x))^{-1}\cdot\sigma(m)\cdot(\tau(m))^{-1} = (\gamma(x))^{-1}\cdot(\sigma\star\tau^{-1})(x) = \underbrace{\gamma^{-1}\star\sigma\star\tau^{-1}}_{=:l_x\in\mathrm{Bis}(\mathcal{G})}(x).$$

By construction $\gamma\star l_x(x) = \sigma(m)\cdot(\tau(m))^{-1}$ and thus $\gamma(x) = \sigma(m)\cdot((l_x\star\tau)(m))^{-1} \in \mathrm{im}\,\chi_{\mathcal{G}}$. As γ and x were arbitrary this establishes $\mathrm{ev}(\mathrm{Bis}(\mathcal{G})\times M) = \mathrm{im}\,\chi_{\mathcal{G}}$.

b) Set $\Lambda_m := \mathrm{Loop}_m(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$. To compute on the Λ_m-principal bundle $\beta\circ\mathrm{ev}_m\colon \mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G}) \to M$ we choose a section atlas $(s_i, U_i)_{i\in I}$ and obtain

$$\begin{aligned} \mathrm{Bis}(\chi_{\mathcal{G}})\circ a_{\beta\circ\mathrm{ev},\mathrm{Bis}_m(\mathcal{G})}(\sigma) &= \chi_{\mathcal{G}}\circ a_{\beta\circ\mathrm{ev},\mathrm{Bis}_m(\mathcal{G})}(\sigma) = \chi_{\mathcal{G}}\circ(x\mapsto\langle\sigma\star s_i(x), s_i(x)\rangle, x\in U_i) \\ &= (x\mapsto(\sigma\star s_i(x))(m)\cdot(s_i(x)(m))^{-1}, x\in U_i) \\ &= (x\mapsto\sigma(\underbrace{\beta(s_i(x)(m))}_{=x})\cdot(s_i(x)(m))\cdot(s_i(x)(m))^{-1}, x\in U_i) = \sigma. \end{aligned}$$

Hence $\mathrm{Bis}(\chi_{\mathcal{G}})\circ a_{\beta\circ\mathrm{ev},\mathrm{Bis}_m(\mathcal{G})} = \mathrm{id}_{\mathrm{Bis}(\mathcal{G})}$. Let us show $a_{\beta\circ\mathrm{ev},\mathrm{Bis}_m(\mathcal{G})}\circ\mathrm{Bis}(\chi_{\mathcal{G}}) = \mathrm{id}_{\mathrm{Bis}(\mathcal{R}(\theta,H))}$. Denote by $\psi\colon \mathrm{Aut}(\beta\circ\mathrm{ev}_m\colon \mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G}) \to M) \to \mathrm{Bis}(\mathcal{R}(\theta,H))$ the isomorphism from (16). We fix a Λ_m-principal bundle automorphism f and compute the image of $\psi(f)$ under $a_{\beta\circ\mathrm{ev},\mathrm{Bis}_m(\mathcal{G})}\circ\mathrm{Bis}(\chi_{\mathcal{G}})$ as

$$\begin{aligned} a_{\beta\circ \mathrm{ev},\mathrm{Bis}_m(\mathcal{G})}\circ \mathrm{Bis}(\chi_{\mathcal{G}})(\psi(f)) &= a_{\beta\circ \mathrm{ev},\mathrm{Bis}_m(\mathcal{G})}(\chi_{\mathcal{G}}(x\mapsto \langle f(s_i(x)), s_i(x)\rangle, x\in U_i)\\ &= \psi\circ\psi^{-1}\circ a_{\beta\circ \mathrm{ev},\mathrm{Bis}_m(\mathcal{G})}(x\mapsto (f(s_i(x))(m))\cdot (s_i(x)(m))^{-1}, x\in U_i)\\ &= \psi(\tau\,\mathrm{Bis}_m(\mathcal{G})\mapsto \big(((f(s_i(\bullet))(m))\cdot (s_i(\bullet)(m))^{-1})\star\tau\big)\,\mathrm{Bis}_m(\mathcal{G})). \end{aligned} \tag{21}$$

In passing from the second to the third line we have used that ψ^{-1} takes $a_{\theta,H}$ to the left action $\lambda_{\mathrm{Bis}(\mathcal{G})}$ discussed in Lemma 3.9. We will now prove that the argument of ψ in (21) coincides with the principal bundle automorphism f. If this is true, then $\mathrm{Bis}(\chi_{\mathcal{G}})$ is an isomorphism with inverse $a_{\beta\circ\mathrm{ev},\mathrm{Bis}_m(\mathcal{G})}$.
The equivariant map $\widetilde{\mathrm{ev}_m}\colon \mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})\to\alpha^{-1}(m)$ from Lemma 3.12 identifies $\mathrm{Bis}(\mathcal{G})/\mathrm{Bis}_m(\mathcal{G})$ with the open subset $\mathrm{ev}_m(\mathrm{Bis}(\mathcal{G}))$ of $\alpha^{-1}(m)$. Recall from Proposition 3.6 that Λ_m is isomorphic to an open subgroup $e_m(\Lambda_m)\Subset \mathrm{Vert}_m(\mathcal{G})$ and $e_m(\Lambda_m)=\{\tau(m)\mid \tau\in\mathrm{Loop}_m(\mathcal{G})\}$. Since $\beta\circ\mathrm{ev}$ is transitive, the $\mathrm{Vert}_m(\mathcal{G})$-principal bundle $\beta|_{\alpha^{-1}(m)}\colon \alpha^{-1}(m)\to M$ restricts on $\mathrm{ev}_m(\mathrm{Bis}(\mathcal{G}))$ to a $e_m(\Lambda_m)$-principal bundle. Identifying the groups, we obtain a Λ_m-principal bundle $\beta|_{\mathrm{ev}_m(\mathrm{Bis}(\mathcal{G}))}\colon \mathrm{ev}_m(\mathrm{Bis}(\mathcal{G}))\to M$ and $\widetilde{\mathrm{ev}_m}$ induces an isomorphism of Λ_m-principal bundles (cf. Example 4.7).
The principal bundle isomorphism $\widetilde{\mathrm{ev}_m}$ allows us to associate to the automorphism f a $e_m(\Lambda_m)-$bundle automorphism $\tilde f\colon \mathrm{ev}_m(\mathrm{Bis}(\mathcal{G}))\to\mathrm{ev}_m(\mathrm{Bis}(\mathcal{G}))$ via $\tilde f\circ\widetilde{\mathrm{ev}_m}=\widetilde{\mathrm{ev}_m}\circ f$. Evaluating the argument of ψ from (21) at $\tau\,\mathrm{Bis}_m(\mathcal{G})$, our preparations allow us to compute as follows.

$$\begin{aligned} \widetilde{\mathrm{ev}_m}^{-1}(((f(s_i(\bullet))(m))\cdot(s_i(\bullet)(m))^{-1})\star\tau(m)) &= \widetilde{\mathrm{ev}_m}^{-1}((f(s_i(\beta(\tau(m))(m))\cdot(s_i(\beta(\tau(m))(m))^{-1}\cdot\tau(m))\\ &= \widetilde{\mathrm{ev}_m}^{-1}(\tilde f(s_i(\beta(\tau(m))(m)))\cdot\underbrace{(s_i(\beta(\tau(m))(m))^{-1}\cdot\tau(m))}_{\in e_m(\Lambda_m)\Subset\mathrm{Vert}_m(\mathcal{G})})\\ &= \widetilde{\mathrm{ev}_m}^{-1}(\tilde f(\underbrace{s_i(\beta(\tau(m))(m))\cdot(s_i(\beta(\tau(m))(m))^{-1}}_{=1_{\beta(\tau(m))}}\cdot\tau(m)))\\ &= \widetilde{\mathrm{ev}_m}^{-1}(\tilde f(\tau(m))=f(\tau\,\mathrm{Bis}_m(\mathcal{G})). \qquad\square \end{aligned}$$

Problem 4.22.

a) It would be interesting to develop a notion/theory of infinitesimal transitive pairs, i.e., an infinitesimal action $\mathfrak{k}\to\mathcal{V}(M)$ together with an ideal $\mathfrak{h}$ of $\mathfrak{k}_m$. In particular, a derivation of the Lie algebra morphism $\mathbf{L}(a_{\theta,H})$ directly from these data in case of an effective transitive pair would be a valuable tool.
b) It would also be interesting to develop the theory of this section for not necessarily locally trivial Lie groupoids (analogously perhaps to the notion of Lie–Rinehart algebras in the infinitesimal setting, see [11]). $\square$

5. Integrating extensions of Lie groups to transitive pairs

We now study the application of the previously developed theory to the integration theory of extensions of Lie algebroids and Lie groupoids. Throughout this section, M denotes a compact and *1-connected* manifold for which we choose some fixed base-point $m\in M$. We will heavily use the integration theory of abelian extensions of infinite-dimensional Lie groups, for which we refer to [23] (see also the Appendix in [23] for some of the terminology that we are using).

Throughout this section, K will be a connected and C^∞-regular Lie subgroup of $\mathrm{Diff}(M)$ such that (θ,K_m) is a 2-fold transitive pair, where $\theta\colon K\times M\to M$, is the natural action of $K\leq\mathrm{Diff}(M)$ on M and K_m is the stabiliser of the base-point m in K. In particular, $\mathrm{ev}_m\colon K\to M$ then factors through a diffeomorphism $K/K_m\to M$ and K_m acts transitively on $M\setminus\{m\}$.

The Lie group $\mathrm{Diff}(M)$ acts naturally and smoothly on $C^\infty(M) := C^\infty(M, \mathbb{R})$ via $\varphi.f := f \circ \varphi^{-1}$. This turns $\mathfrak{a} := C^\infty(M)$ into a K-module containing the constant functions $\mathfrak{a}_0 = \mathfrak{a}^K \cong \mathbb{R}$. If we set $\mathfrak{k} = \mathbf{L}(K)$ and $\mathfrak{k}_m = \mathbf{L}(K_m)$, then $\mathfrak{a}$ is also a $\mathfrak{k}$-module for the derived action

$$(X.f)(n) := df_n(X(n)).$$

Moreover, $\mathfrak{a}_m := C^\infty_m(M)$ is a $\mathfrak{k}_m$-submodule for

$$C^\infty_m(M) := \{f \in C^\infty(M) \mid f(m) = 0\}.$$

Clearly, $\mathfrak{a}_0$ is a module complement for this $\mathfrak{k}_m$-submodule, so that we have $\mathfrak{a} \cong \mathfrak{a}_0 \oplus \mathfrak{a}_m$ as $\mathfrak{k}_m$-modules.

The last piece of information that we choose is a closed 2-form $\omega \in \Omega^2(M) := \Omega^2(M, \mathbb{R})$. This gives rise to the abelian cocycle

$$\overline{\omega} \colon \mathfrak{k} \times \mathfrak{k} \to \mathfrak{a}, \quad (X, Y) \mapsto (n \mapsto \omega_n(X(n), Y(n))),$$

i.e.,

$$[(f, X), (g, Y)] := (X.g - Y.f + \overline{\omega}(X, Y), [X, Y])$$

defines on $\mathfrak{a} \oplus \mathfrak{k}$ the structure of a Lie algebra. We denote this Lie algebra by $\mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k}$, and the canonical maps give rise to an abelian extension

$$\mathfrak{a} \to \mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k} \to \mathfrak{k}$$

of topological Lie algebras. Moreover, as elements of $\mathfrak{k}_m$ vanish when they are evaluated in m, we have $\overline{\omega}(\mathfrak{k}_m \times \mathfrak{k}_m) \subseteq \mathfrak{a}_m$. Hence there is a subalgebra

$$\mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}_m \leq \mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k},$$

where $\overline{\omega}_m \colon \mathfrak{k}_m \times \mathfrak{k}_m \to \mathfrak{a}_m$ denotes the restriction (and corestriction) of $\overline{\omega}$ to $\mathfrak{k}_m \times \mathfrak{k}_m$ (and $\mathfrak{a}_m$).

Before we go on we give the two examples of the above situations that we have in mind.

Example 5.1.

a) The easiest example is that where $K = \mathrm{Diff}(M)_0$ and $\mathfrak{a} = C^\infty(M)$. Then we have seen in Example 3.5 that $\mathrm{Diff}_m(M)$ is a regular Lie subgroup of $\mathrm{Diff}(M)$. Moreover, $\mathrm{Diff}(M)$ acts smoothly transitively on M by Corollary 2.17 and Corollary 2.5. Finally, $\mathrm{Diff}(M)$ acts 2-fold transitively on M [21]. In this case $\omega \in \Omega^2(M)$ can be an arbitrary closed 2-form on M (cf. [23, Section 9]).

b) For the next example, let $\omega \in \Omega^2(M)$ be symplectic and $\dim(M) = 2d$. Then

$$\mathrm{Symp}(M, \omega) := \{\varphi \in \mathrm{Diff}(M) \mid \varphi^*\omega - \omega = 0\}$$

is a closed Lie subgroup of $\mathrm{Diff}(M)$ [13, Theorem 43.12] with Lie algebra

$$\mathfrak{symp}(M, \omega) := \{X \in \mathcal{V}(M) \mid \mathcal{L}_X\omega = d(i_X\omega) = 0\}$$

the symplectic vector fields on M. This Lie algebra has the Hamiltonian vector fields

$$\mathfrak{ham}(M, \omega) := \{X \in \mathfrak{symp}(M) \mid i_X\omega \text{ is exact}\}$$

as closed subalgebra. Moreover, the chart constructed in the proof of [13, Theorem 43.12] maps

$$\mathfrak{k}_m := \mathfrak{symp}_m(M,\omega) := \{X \in \mathfrak{symp}(M,\omega) \mid X(m) = 0\}$$

to

$$\operatorname{Symp}_m(M,\omega) := \{\varphi \in \operatorname{Symp}(M,\omega) \mid \varphi(m) = m\}.$$

Thus $K := \operatorname{Symp}(M,\omega)_0$ has $K_m = \operatorname{Symp}_m(M,\omega)_0$ as Lie subgroup with Lie algebra $\mathfrak{k}_m$. That (θ, K_m) is indeed a transitive pair will follow from the following two propositions. Finally, K acts 2-fold transitively by [21]. □

Proposition 5.2. *Let (M,ω) be a compact symplectic manifold and $x \in M$. Then the evaluation map*

$$\operatorname{ev}_x \colon \operatorname{Symp}(M,\omega)_0 \to M, \quad \varphi \mapsto \varphi(x)$$

is a submersion. If, moreover, M is connected, then ev_x is also surjective.

Proof. We show that ev_x is a submersion by showing that $T_{\mathrm{id}} \operatorname{ev}_x \colon T_{\mathrm{id}} \operatorname{Symp}(M,\omega) \to T_x M$ is surjective. This suffices by [7, Theorem A]. To this end, note that $T_{\mathrm{id}} \operatorname{ev}_x$ is given with respect to the identification $\mathfrak{symp}(M,\omega) \cong T_{\mathrm{id}} \operatorname{Symp}(M,\omega)$ by

$$\operatorname{ev}_x \colon \mathfrak{symp}(M,\omega) \to T_x M, \quad X \mapsto X(x)$$

(cf. [13, Corollary 42.18] or [31, Theorem 7.9]). Thus the claim follows from observing that for each $v \in T_x M$ there exists a function $H \in C^\infty(M)$ with $dH_x = \omega_x(v,\cdot)$, and thus a Hamiltonian vector field X (which is then in particular symplectic) with $X(x) = v$.

Since the point $x \in M$ in the above argument was arbitrary, this shows in particular that each orbit of the natural action of $\operatorname{Symp}(M,\omega)$ on M is open and thus consist of unions of path components. In particular, ev_x is surjective if M is connected. □

Proposition 5.3. *Let (M,ω) be a compact symplectic manifold and $k \in \mathbb{N}_0 \cup \{\infty\}$. Then* $\operatorname{Symp}(M,\omega)$ *and*

$$\operatorname{Symp}_m(M,\omega) := \{\varphi \in \operatorname{Symp}(M,\omega) \mid \varphi(m) = m\}$$

are C^k-regular Lie subgroups of $\operatorname{Diff}(M)$.

Proof. By [13, Theorem 43.12] the Lie group $\operatorname{Symp}(M,\omega)$ is a closed Lie subgroup of $\operatorname{Diff}(M)$. Recall from [31, Theorem 5.5] that $\operatorname{Diff}(M)$ is C^k-regular. Arguing as in the proof of [13, Theorem 43.12] (where C^∞-(semi)regularity for $\operatorname{Symp}(M,\omega)$ was established), one proves that $\operatorname{Symp}(M,\omega)$ is C^k-semiregular. Hence Lemma B.5 implies that $\operatorname{Symp}(M,\omega)$ is C^k-regular.

Fix some $\eta \in C^k([0,1], \mathfrak{symp}_m(M,\omega))$ and let γ_η be a solution in $\operatorname{Symp}(M,\omega)$ of the corresponding initial value problem from Definition B.3. Note that γ_η is also a solution of the corresponding initial value problem in $\operatorname{Diff}(M)$. By Proposition 3.4, η_γ takes its image in $\operatorname{Symp}_m(M,\omega)$, whence $\operatorname{Symp}_m(M,\omega)$ is C^k-semiregular. Again by Lemma B.5, $\operatorname{Symp}_m(M,\omega)$ is also C^k-regular. □

For later reference we also record the following fact (see also [12, Corollary 3.5]).

Proposition 5.4. *If (M,ω) is a compact and connected symplectic manifold, then the restriction of the abelian cocycle*

$$\overline{\omega}\colon \mathcal{V}(M)\times\mathcal{V}(M)\to C^\infty(M),\quad (X,Y)\mapsto \omega(X,Y)$$

to $\mathfrak{ham}(M,\omega)$ *is a coboundary. This also applies to the restriction of* $\overline{\omega}$ *to* $\mathfrak{symp}(M,\omega)$ *if* M *is 1-connected.*

Proof. We consider the continuous linear surjection $\xi\colon C^\infty(M)\to\mathfrak{ham}(M,\omega)$, $f\mapsto X_f$, where X_f is the unique vector field such that $i_{X_f}\omega=df$. Since $\ker(\xi)\cong\mathbb{R}$ is finite-dimensional and thus complemented, ξ has a continuous linear section $\sigma\colon\mathfrak{ham}(M,\omega)\to C^\infty(M)$. We claim that

$$b\colon \mathfrak{ham}(M,\omega)\to C^\infty(M),\quad X\mapsto \int_M \sigma(X)\,\omega^d-\sigma(X)$$

satisfies $d_{\mathrm{CE}}b=\omega$, where $d=\frac{\dim(M)}{2}$ and we consider $\mathbb{R}$ as constant functions on M. To this end we note that we have the equality

$$\omega(X,Y)-\int_M \omega(X,Y)\,\omega^d=\sigma([X,Y])-\int_M\sigma([X,Y])\omega^d. \tag{22}$$

Indeed, both sides of (22) are uniquely determined by the property of being mapped to $[X,Y]$ under ξ and having vanishing integral over M. If $X=X_f$ and $Y=X_g$ for some $f,g\in C^\infty(M)$, then we also have

$$\int_M\omega(X,Y)\,\omega^d=\int_M\{f,g\}\,\omega^d=\int_M(\mathcal{L}_{X_f}g)\,\omega^d=\int_M\mathcal{L}_{X_f}(g\,\omega^d)=\int_M d(i_{X_f}g\,\omega^n)=0. \tag{23}$$

Thus we conclude from (23) and (22) that

$$d_{\mathrm{CE}}b(X,Y)=-X.\sigma(Y)+Y.\sigma(X)-\Big(\int_M\sigma([X,Y])\omega^d-\sigma([X,Y])\Big)=\omega(X,Y).$$

If M is also simply connected, then $\mathfrak{ham}(M,\omega)=\mathfrak{symp}(M,\omega)$ and the assertion follows. □

Remark 5.5. We shortly fix our conventions about the periods and prequantisation of presymplectic manifolds and integration of abelian extensions of Lie algebras to Lie group extensions. Let N be an arbitrary manifold, V be a locally convex space and $\omega\in\Omega^2(M,V)$ be a V-valued closed 2-form. Then the *period homomorphism* associated to the cohomology class $[\omega]\in H^2_{\mathrm{dR}}(N,V)$ is the homomorphism

$$\mathrm{per}_{[\omega]}\colon\pi_2(N)\to V,\quad [\sigma]\mapsto\int_{S^2}\sigma^*\omega,$$

where $\sigma\colon S^2\to N$ is a smooth representative of $[\sigma]$ (cf. [22, Appendix A.3] or [34, Corollary 14]). Moreover, let $\Gamma\subseteq\mathbb{R}$ be an arbitrary but fixed discrete subgroup and set $A_\Gamma:=\mathfrak{a}/\Gamma$ denote by $q_\Gamma\colon\mathfrak{a}\to A_\Gamma$ the canonical quotient homomorphism. In addition, we assume that V comes along with a distinguished embedding $\mathbb{R}\hookrightarrow V$.

We now consider two special cases of this. At first, assume that $V=\mathbb{R}$ and that N is finite-dimensional (or more generally smoothly paracompact) and 1-connected. Then the subgroup $\mathrm{per}_{[\omega]}(\pi_2(M))$ is called the group of *periods* (or shortly just the periods) of $(M,[\omega])$. Moreover, we say that (M,ω) is Γ-*prequantisable* if there a principal $\mathbb{T}_\Gamma:=\mathbb{R}/\Gamma$-bundle $P\to N$ that admits a connection with curvature ω. By the general theory (see, e.g., [36, Chapter 8] or [14]) this is the case if and only if the periods $\mathrm{per}_{[\omega]}(\pi_2(M))$ are contained in Γ (or more generally if $[\omega]$ is contained in the image of $H^2(N,\Gamma)\to H^2(M,V)$ if N is not simply connected).

The other case is that $N = K$ and $V = \mathfrak{a}$ for K and $\mathfrak{a}$. Then we consider the equivariant extension $\overline{\omega}^{\mathrm{eq}} \in \Omega^2(K, \mathfrak{a})$ of $\overline{\omega}$, which is given by

$$(\overline{\omega}^{\mathrm{eq}})_k \colon T_kK \times T_kK \to \mathfrak{a}, \quad (X,Y) \mapsto k.\overline{\omega}(X \cdot k^{-1}, Y \cdot k^{-1}).$$

Note that we have here used the right trivialisation of the tangent bundle of a Lie group, which is more adapted to our setting than the left trivialisation (cf. [31, Remarks 2.4 and 2.6]). Then we call $\mathrm{per}_{[\overline{\omega}^{\mathrm{eq}}]}(\pi_2(K))$ the group of *primary periods* (or shortly just the primary periods) of $(K, [\omega])$. Note that they really only depend on the cohomology class of $[\omega]$ in $H^2_{\mathrm{dR}}(M, \mathbb{R})$ by the following lemma. Moreover, the primary periods $\mathrm{per}_{[\overline{\omega}^{\mathrm{eq}}]}(\pi_2(K))$ are contained in the fixed points $\mathfrak{a}^K$ [23, Lemma 4.2]. If they are also contained in Γ, then there exists an extension of Lie groups

$$A_\Gamma \to K^\sharp \xrightarrow{q^\sharp} \widetilde{K}, \tag{24}$$

whose underlying extension of Lie algebras is equivalent to $\mathfrak{a} \to \mathfrak{a} \oplus_\omega \mathfrak{g} \to \mathfrak{g}$ [23, Theorem 6.7]. Here and in the sequel, $q_{\pi_1(K)} \colon \widetilde{K} \to K$ denotes the universal covering morphism. The extension (24) is uniquely determined (up to equivalence) by the associated Lie algebra extension [23, Theorem 7.2]. Moreover, $K^\sharp \to \widetilde{K}$ is a principal A_Γ-bundle that admits a connection with curvature ω^{eq}, so ω^{eq} is Γ-prequantisable on $\widetilde{K}$ in this case (cf. the proof of Lemma 5.13). The question, whether $A_\Gamma \to K^\sharp \to \widetilde{K}$ factors to an extension of K by A_Γ is then controlled by the flux homomorphism

$$F_{[\overline{\omega}]} \colon \pi_1(K) \to H^1_{\mathrm{dR}}(M, \mathbb{R}) \subseteq H^1_c(\mathfrak{k}, \mathfrak{a})$$

(cf. [23, Lemma 6.2, Proposition 6.3 and Proposition 9.13]). Since M is assumed to be 1-connected, $F_{[\overline{\omega}]}$ vanishes automatically. Almost all flux phenomena will be irrelevant for this paper since we only work with 1-connected M. The flux will only occur shortly in Remark 5.12. One main observation of this section is that the integration of the Lie algebra extension $\mathfrak{a} \to \mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k} \to \mathfrak{k}$ to a transitive pair is *not* governed by the flux, but rather by the *secondary* periods that we will introduce in Remark 5.10. $\square$

Lemma 5.6. *The map*

$$H^2_{\mathrm{dR}}(M, \mathbb{R}) \to H^2_c(\mathcal{V}(M), C^\infty(M)), \quad [\omega] \mapsto [\overline{\omega}] \tag{25}$$

is injective.

Proof. We may assume without loss of generality that $\dim(M) \geq 2$. Let $H^2_{c,\Delta}(\mathcal{V}(M), C^\infty(M))$ be the diagonal cohomology of $\mathcal{V}(M)$ with coefficients in $C^\infty(M)$, i.e., the cohomology of the subcomplex of $\mathbb{R}$-linear alternating cochains

$$\xi \colon \mathcal{V}(M) \times \mathcal{V}(M) \to C^\infty(M)$$

for which $\xi(X,Y)(m)$ only depends on the germs of X and Y at m. Since $\omega \in \Omega^2(M)$ is C^∞-linear in both arguments, $\overline{\omega}$ is a cocycle of this kind, so that the map (25) factors as

$$H^2_{\mathrm{dR}}(M, \mathbb{R}) \to H^2_{c,\Delta}(\mathcal{V}(M), C^\infty(M)) \to H^2_c(\mathcal{V}(M), C^\infty(M)). \tag{26}$$

Now the first map in (26) is injective by [15, Corollary 2] and the second map in (26) is injective by [4, Theorem 2.4.10], and the assertion follows. $\square$

The following is the only result on the flux homomorphism that we shall need in the sequel.

Lemma 5.7. *The flux*

$$F_{[\overline{\omega}_m]}\colon \pi_1(K_m) \to H^1_c(\mathfrak{k}_m, \mathfrak{a}_m)$$

also vanishes. More generally, if M is only assumed to be compact, then $F_{[\overline{\omega}_m]}$ vanishes if $F_{[\overline{\omega}]}$ does so.

Proof. This follows immediately from the commuting diagram

$$\begin{tikzcd}
\widetilde{K_m} \arrow[r] \arrow[d, equal] & \widetilde{K} \arrow[r, "F_{[\overline{\omega}]}"] & H^1_{\mathrm{dR}}(M,\mathbb{R}) \arrow[r, hook] & H^1_c(\mathfrak{k},\mathfrak{a}) \arrow[r, "\mathrm{res}_{\mathfrak{k}_m}"] & H^1_c(\mathfrak{k}_m,\mathfrak{a}) \arrow[d, "\mathrm{pr}_m"] \\
\widetilde{K_m} \arrow[rrrr, "F_{[\overline{\omega}_m]}"] & & & & H^1_c(\mathfrak{k}_m,\mathfrak{a}_m),
\end{tikzcd}$$

where $\widetilde{K_m} \to \widetilde{K}$ is induced by the inclusion $K_m \hookrightarrow K$, $\mathrm{res}_{\mathfrak{k}_m}$ is induced by the inclusion $\mathfrak{k}_m \hookrightarrow \mathfrak{k}$ and pr_m is induced by the morphism $\mathfrak{a} \to \mathfrak{a}_m$, $f \mapsto f - f(m)$ of $\mathfrak{k}_m$-modules. □

The previous remark introduces the most important concepts from the integration theory of abelian extensions of Lie groups that occur in the sequel, for the rest see [23]. From Proposition 5.4 we immediately obtain the following

Corollary 5.8. *If $K = \mathrm{Symp}(M,\omega)_0$ for $\omega \in \Omega^2(M)$ symplectic, then $\mathrm{per}_{[\overline{\omega}]}(\pi_2(K)) = 0$.*

Remark 5.9. Keeping the notation as in Remark 5.5, we can also integrate the restricted extension

$$\mathfrak{a}_m \to \mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}_m \to \mathfrak{k}_m$$

to a unique extension

$$\mathfrak{a}_m \to K_m^\sharp \to \widetilde{K_m}.$$

Indeed, $\mathrm{per}_{\overline{\omega}_m}(\pi_2(K_m))$ is contained in the fixed-points of K_m. Since K_m is assumed to act transitively on $M \setminus \{m\}$, if follows that

$$(\mathfrak{a}_m)^{K_m} = \{f \in C^\infty(M) \mid f \text{ is constant on } M \setminus \{m\} \text{ and } f(m) = 0\} = \{0\}$$

by the continuity of f. Consequently, $\mathrm{per}_{\overline{\omega}_m}(\pi_2(K_m)) = \{0\}$ and [23, Theorem 6.7] implies that there exists an extension $\mathfrak{a}_m \to K_m^\sharp \to \widetilde{K_m}$ whose derived extension is equivalent to $\mathfrak{a}_m \to \mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}_m \to \mathfrak{k}_m$. □

Now we have the extensions $K^\sharp \to \widetilde{K}$ from Remark 5.9 and $K_m^\sharp \to \widetilde{K_m}$ from Remark 5.9 at hand, and we want to build our transitive pair from them. The question is now how these two extensions are relate to one another. The crucial information about this in contained in the modification of the period homomorphism $\mathrm{per}^\flat_{[\omega]}$ and the associated secondary periods that we introduce and analyse now.

Remark 5.10. From the transitive action of K on M we obtain the principal K_m-bundle $\mathrm{ev}_m\colon K \to M$. This gives in particular rise to the long exact sequence

$$\cdots \to \pi_2(K_m) \to \pi_2(K) \xrightarrow{\mathrm{ev}_m} \pi_2(M) \xrightarrow{\delta} \pi_1(K_m) \to \pi_1(K) \to \cdots$$

At $\pi_2(M)$, this induces a short exact sequence

$$0 \to \Lambda \xrightarrow{\mathrm{ev}_m} \pi_2(M) \xrightarrow{\delta} \Delta \to 0$$

with $\Delta := \ker(\pi_1(K_m) \to \pi_1(K))$, $\Lambda := \mathrm{im}(\pi_2(K) \to \pi_2(M))$. From [25, Theorem 3.18] it follows that $\mathrm{per}_{[\varpi]}([\gamma]) = \mathrm{per}_{[\omega]}([\mathrm{ev}_m \circ \gamma])$ holds for $\gamma \in C^\infty_*(\mathbb{S}^2, K)$ and thus $\mathrm{per}_{[\varpi]}$ factors through $\mathrm{per}_{[\omega]}\big|_\Lambda$.

If now $\Gamma \leq \mathbb{R}$ is a (not necessarily discrete) subgroup that contains $\mathrm{per}_{[\varpi]}(\pi_2(K))$ and $q_\Gamma \colon \mathbb{R} \to \mathbb{R}/\Gamma$ is the quotient map, then $q_\Gamma \circ \mathrm{per}_{[\omega]}$ factors through δ and a homomorphism $\mathrm{per}^\flat_{[\omega]} \colon \Delta \to \mathbb{R}/\Gamma$. This gives rise to a morphism

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r] \arrow[d, "\mathrm{per}_{[\varpi]}"] & \pi_2(M) \arrow[r] \arrow[d, "\mathrm{per}_{[\omega]}"] & \Delta \arrow[r] \arrow[d, "\mathrm{per}^\flat_{[\omega]}"] & 0 \\
0 \arrow[r] & \Gamma \arrow[r] & \mathbb{R} \arrow[r] & \mathbb{R}/\Gamma \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences. □

Definition 5.11. If, in the setting of the previous remark, $\Gamma := \mathrm{per}_{[\varpi]}(\pi_2(K))$ is the group of primary periods (cf. Remark 5.5), then we call

$$\mathrm{per}^\flat_{[\omega]}(\pi_1(K_m)) \subseteq \mathbb{R}/\Gamma$$

the group of *secondary periods* (or shortly the secondary periods) of $(K, [\omega])$. □

Remark 5.12. We keep the notation from Remark 5.5 and Remark 5.9. The inclusion $\mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}_m \hookrightarrow \mathfrak{a} \oplus_\omega \mathfrak{k}$ induces a unique homomorphism

$$\varphi_m \colon K^\sharp_m \to K$$

(cf. [18, Theorem 8.1]) that makes the diagram

$$\begin{tikzcd}
\mathfrak{a}_m \arrow[r] \arrow[d] & K^\sharp_m \arrow[r, "q^\sharp_m"] \arrow[d, "\varphi_m"] & \widetilde{K_m} \arrow[d, "q_\Delta"] \\
A_\Gamma \arrow[r] & K^\sharp \arrow[r] & \widetilde{K}
\end{tikzcd} \tag{27}$$

commute. Here, $\mathfrak{a}_m \hookrightarrow A_\Gamma = \mathfrak{a}/\Gamma$ is the morphism induced from the embedding $\mathfrak{a}_m \hookrightarrow \mathfrak{a}$ and the quotient $\mathfrak{a} \to \mathfrak{a}/\Gamma$ (note that $\mathfrak{a}_m \cap \Gamma = \{0\}$). The morphism $q_\Delta \colon \widetilde{K_m} \to \widetilde{K}$ in (27) is induced from the inclusion $K_m \hookrightarrow K$. The kernel $\ker(q_\Delta)$ can be identified with $\Delta = \ker(\pi_1(K_m) \to \pi_1(K))$ and the image with $\widetilde{K_m}/\Delta$. Then the latter is a closed Lie subgroup of $\widetilde{K}$. This gives rise to an extension

$$A^\sharp_m \longrightarrow K^\sharp_m \xrightarrow{q_\Delta \circ q^\sharp_m} \widetilde{K_m}/\Delta$$

of $\widetilde{K_m}/\Delta$ by $A^\sharp_m := (q^\sharp_m)^{-1}(\Delta)$. Now $F_{[\omega_m]}$ vanishes by Lemma 5.7, whence there exists a homomorphism $\sigma \colon \Delta \to Z(K^\sharp_m)$ with $q_m \circ \sigma = \mathrm{id}_\Delta$ [23, Corollary 6.6]. □

Lemma 5.13. *In the situation of Remark 5.12 and Remark 5.10 we have*

$$\varphi_m(\sigma(x)) + \mathfrak{a}_m = \mathrm{per}^\flat_{[\omega]}(x) + \mathfrak{a}_m$$

for each $x \in \Delta$.

Proof. We first recall the following fact about abelian extensions of Lie groups. If $A \to \widehat{G} \xrightarrow{q} G$ is such an extension and if $\tau\colon \mathfrak{g} := \mathbf{L}(G) \to \widehat{\mathfrak{g}} := \mathbf{L}(\widehat{G})$ is a continuous linear splitting of the induced extension $\mathbf{L}(A) \to \mathbf{L}(\widehat{G}) \to \mathbf{L}(G)$, then we obtain a connection on the principal right A-bundle $q\colon \widehat{G} \to G$, induced for each $\widehat{g} \in \widehat{G}$ by the horizontal lift of tangent vectors

$$\tau_{\widehat{g}}\colon T_gG \to T_{\widehat{g}}\widehat{G}, \quad v \mapsto \tau(v \cdot g^{-1}) \cdot \widehat{g}$$

where $g := q(\widehat{g})$. The curvature of this connection is given by $\omega_\tau^{\mathrm{eq}}$, where $\omega_\tau\colon \mathfrak{g} \times \mathfrak{g} \to \mathbf{L}(A)$ is the abelian Lie algebra cocycle $(x, y) \mapsto \tau([x, y]) - [\tau(x), \tau(y)]$. Note again that we have here used the right trivialisation of the tangent bundle.

For each $x \in \Delta$ we now choose a representative $\gamma_x \in C^\infty_*(\mathbb{S}^1, K_m)$ in $\pi_1(K_m)$ such that there exists a smooth map $F_x\colon [0, 1] \times \mathbb{S}^1 \to K$ with $F_x(0, \cdot) \equiv e_K$, $F_x(1, \cdot) = \gamma_x$ and $F_x(t, *) = \mathrm{id}_M$ for all $t \in [0, 1]$. If $\gamma_x^\sharp$ denotes the horizontal lift of γ_x, induced by the canonical linear splitting $\mathfrak{k}_m \to \mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}_m$ and the Lie algebra isomorphism $\mathbf{L}(K_m^\sharp) \cong \mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}_m$, then there exists some $b_\sigma\colon \Delta \to \mathfrak{a}_m$ such that $\sigma(x) = \gamma_x^\sharp(1) + b_\sigma(x)$ (cf. [23, Corollary 6.6]).

By the construction of the morphism φ_m (cf. [18, Theorem 8.1]),

$$\varphi_m(\sigma(x)) = \varphi_m(\gamma_x^\sharp(1)) + \varphi_m(b_\sigma(x)) = \Gamma_x(1) + \varphi_m(b_\sigma(x)),$$

where $\Gamma_x \in C^\infty([0, 1], K^\sharp)$ is the solution of the initial value problem in $K^\sharp$ with $\Gamma_x(0) = e_{K^\sharp}$ and $\delta^l(\Gamma_x) = \iota \circ \delta^l(\gamma_x^\sharp)$ for $\iota\colon \mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}_m \to \mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k}$ the canonical embedding. Note that Γ_x exists as $K^\sharp$ is C^∞-regular as an extension of C^∞-regular Lie groups (cf. [26, Appendix B]). This implies in particular that Γ_x is a horizontal lift of the smooth loop $\gamma'_x := q_\Delta \circ \gamma_x$ for the connection on the principal A_Γ-bundle $K^\sharp \to \widetilde{K}$ that is induced by the canonical linear splitting $\mathfrak{k} \to \mathfrak{a} \oplus \mathfrak{k}$ and the Lie algebra isomorphism $\mathbf{L}(K^\sharp) \cong \mathfrak{a} \oplus_\omega \mathfrak{k}$. Consequently, $\varphi_m(\gamma_x^\sharp(1)) = \Gamma_x(1)$ equals the holonomy of the loop γ'_x for this connection.

If we take the restriction $K^\sharp|_{\widetilde{K_m}/\Delta}$ of the A_Γ-bundle $K^\sharp \to K$ to the Lie subgroup $\widetilde{K_m}/\Delta$, then the curvature of the above connection takes values in the subspace $\mathfrak{a}_m$ of $A_\Gamma \cong \mathfrak{a}_m \times \mathbb{T}_\Gamma$. Consequently, the connection is flat modulo $\mathfrak{a}_m$ and thus the $\mathbb{T}_\Gamma$-component of the holonomy can be computed as the integral of the curvature over any filler of the loop γ'_x. This implies

$$\Gamma_x(1) + \mathfrak{a}_m = \mathrm{hol}(\gamma'_x) + \mathfrak{a}_m = \int\limits_F \omega^{\mathrm{eq}} + \mathfrak{a}_m = \mathrm{per}^\flat_{[\omega]}(x) + \mathfrak{a}_m.$$

Since $\varphi_m(b_\sigma(x)) \subseteq \varphi_m(\mathfrak{a}_m) \subseteq \mathfrak{a}_m$, this establishes the claim. □

The following example illustrates the rôle of $\mathrm{per}^\flat_{[\omega]}$ pretty well.

Example 5.14. Let ω be the standard volume form on $M := \mathbb{S}^2$ with total volume 1. Then we consider the subgroup $K := \mathrm{Diff}(\mathbb{S}^2)_0$. The action of $\mathrm{SO}_3(\mathbb{R})$ on $\mathbb{S}^2$ by rotations induces a map $\mathrm{SO}_3(\mathbb{R}) \to \mathrm{Diff}(\mathbb{S}^2)_0$, which is a homotopy equivalence [30]. Consequently, $\pi_2(K) = 0$ and the primary periods vanish. Thus we may take $\Gamma = 0$ to integrate the extension

$$C^\infty(\mathbb{S}^2) \to C^\infty(\mathbb{S}^2) \oplus_{\overline{\omega}} \mathcal{V}(\mathbb{S}^2) \to \mathcal{V}(\mathbb{S}^2)$$

to an extension

$$C^\infty(\mathbb{S}^2) \to K^\sharp \to \widetilde{K}$$

of Lie groups (cf. Remark 5.5). From the five lemma and the long exact sequence in homotopy groups of the fibrations $K_m \to K \to \mathbb{S}^2$ and $\mathrm{SO}_2(\mathbb{R}) \to \mathrm{SO}_3(\mathbb{R}) \to \mathbb{S}^2$ it follows that the induced map $\mathrm{SO}_2(\mathbb{R}) \to K_m$ is also a homotopy equivalence. From this it follows that the exact sequence

$$\pi_2(K) \to \pi_2(\mathbb{S}^2) \to \pi_1(K_m) \to \pi_1(K) \to \pi_1(\mathbb{S}^2)$$

identifies with

$$0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z} \to 0$$

and with respect to this identification we have $\Delta = 2\mathbb{Z}$. Since $\mathrm{per}_{[\omega]}$ is given by the natural embedding $\mathbb{Z} \hookrightarrow \mathbb{R}$, it follows that $\mathrm{per}^\flat_{\overline{\omega}}$ is given by

$$\mathrm{per}^\flat_{[\omega]} \colon \Delta \cong 2\mathbb{Z} \to \mathbb{R}, \quad 2x \mapsto x.$$

Thus the secondary periods coincide with $\mathbb{Z}$ and $\varphi_m(A^\sharp_m) = C^\infty_m(\mathbb{S}^2) \times \mathbb{Z}$. $\square$

We now put all the bits and pieces that we have collected so far together.

Theorem 5.15. *Let M be a compact and 1-connected manifold, $\omega \in \Omega^2(M)$ be closed and let $K \leq \mathrm{Diff}(M)$ be a connected and C^∞-regular Lie subgroup such that $\mathrm{ev}_m \colon K \to M$ is a surjective submersion and K acts 2-fold transitively on M. Let Δ be the kernel of the map $\widetilde{K_m} \to \widetilde{K}$ induced on the universal cover by the inclusion $K_m \hookrightarrow K$. Suppose $\Gamma, \Pi \subseteq \mathbb{R}$ are discrete subgroups with $\mathrm{per}_{[\overline{\omega}]} \subseteq \Gamma \subseteq \Pi$, set $A_\Gamma := C^\infty(M,\mathbb{R})/\Gamma$ and identify A_Γ with $C^\infty_m(M) \times \mathbb{T}_\Gamma$ (as abelian Lie groups or as K_m-modules). Then the following assertions are equivalent:*

a) $\mathrm{per}^\flat_{[\omega]}(\Delta) \subseteq \Pi/\Gamma$.
b) *Let*

$$A_\Gamma \to K^\sharp \xrightarrow{q^\sharp} \widetilde{K}$$

be the unique extension of $\widetilde{K}$ by A_Γ whose Lie algebra extension is equivalent to $\mathfrak{a} \to \mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k} \to \mathfrak{k}$ with $\mathfrak{a} := C^\infty(M)$. If $\mathfrak{a}_m := C^\infty_m(M)$, then the closed Lie subalgebra $\mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}_m$ of $\mathfrak{a} \oplus \mathfrak{k}$ integrates to a closed Lie subgroup I_m of $K^\sharp$ such that $I_m \cap A_\Gamma \subseteq \mathfrak{a}_m \times \Pi/\Gamma$.
c) *Let*

$$\mathfrak{a}_m \to K^\sharp_m \xrightarrow{q^\sharp_m} \widetilde{K_m}$$

be the unique extension of $\widetilde{K_m}$ by $\mathfrak{a}_m$ whose Lie algebra extension is equivalent to $\mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}_m$ and set $A^\sharp_m := (q^\sharp_m)^{-1}(\Delta)$. Then the image of the Lie group morphism $\varphi_m \colon K^\sharp_m \to K^\sharp$ induced by the canonical embedding $\mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}_m \to \mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k}$ is a closed Lie subgroup and $\varphi_m(A^\sharp_m) \subseteq C^\infty_m(M,\mathbb{R}) \times \Pi/\Gamma$.

If one (and thus all) of these conditions is satisfied, then the composition of the maps

$$\theta^\wedge \colon K^\sharp \xrightarrow{q^\sharp} \widetilde{K} \xrightarrow{q_{\pi_1(K)}} K \hookrightarrow \mathrm{Diff}(M)$$

gives rise to a transitive pair $(\theta, (\varphi_m(K^\sharp_m) \cdot \Pi/\Gamma))$ with kernel Π/Γ. Moreover, the associated principal $\mathbb{T}_\Pi$-bundle $K^\sharp/(\varphi_m(K^\sharp_m) \cdot \Pi/\Gamma) \to M$ admits a connection whose curvature equals ω and thus is (together with the choice of such a connection) a Π-prequantisation of ω.

Proof. We will use throughout that C^∞-regularity is an extension property [26, Appendix B], so that all extensions that appear will automatically be C^∞-regular (which suffices to integrate Lie algebra morphisms to 1-connected Lie groups).

a)⇒c): Consider $\widetilde{K_m}/\Delta$ as a closed Lie subgroup of $\widetilde{K}$ and obtain from this the extension

$$A_\Gamma \to K^\sharp|_{\widetilde{K_m}/\Delta} \to \widetilde{K_m}/\Delta.$$

Since $A_\Gamma \cong \mathfrak{a}_m \times \mathbb{T}_\Gamma$ is a decomposition of A_Γ as a $\widetilde{K_m}/\Delta$-module and $\sigma\colon \Delta \to K^\sharp$ induces the isomorphism $A^\sharp_m \cong \mathfrak{a}_m \times \Delta$, we obtain from Remark 5.12 and Lemma 5.13 the morphism

$$\begin{tikzcd}
\mathfrak{a}_m \times \Delta \arrow[r] \arrow[d, "\mathrm{id}_{\mathfrak{a}_m} \times \mathrm{per}^\flat_{[\omega]}"'] & K^\sharp_m \arrow[r] \arrow[d, "\varphi_m"] & \widetilde{K_m}/\Delta \arrow[d, hook] \\
A_\Gamma \arrow[r] & K^\sharp \arrow[r] & \widetilde{K}
\end{tikzcd}$$

of extensions of Lie groups. Since $\widetilde{K_m}/\Delta \hookrightarrow \widetilde{K}$ is injective and $\mathrm{im}(\mathrm{per}^\flat_{[\omega]}) \subseteq \Pi/\Gamma$ is discrete in $\mathbb{T}_\Gamma$, this implies that $K^\sharp|_{\widetilde{K_m}/\Delta}$ reduces to an extension of $\widetilde{K_m}/\Delta$ by the closed Lie subgroup $\mathfrak{a}_m \times \mathrm{im}(\mathrm{per}^\flat_{[\omega]})$ of $\mathfrak{a}_m \times \mathbb{T}_\Gamma$. This reduction is itself a closed Lie subgroup which equals $\varphi_m(K^\sharp_m)$ by construction. Moreover, $\varphi_m(A^\sharp_m) \subseteq C^\infty_m(M) \times \Pi/\Gamma$ follows from $\mathfrak{a}_m \times \mathrm{im}(\mathrm{per}^\flat_{[\omega]}) \subseteq \mathfrak{a}_m \times \Pi/\Gamma$.

c)⇒b): As above we see that $\varphi_m(K^\sharp_m) \subseteq K^\sharp$ is a reduction of $K^\sharp|_{\widetilde{K_m}/\Delta}$ to an extension of $\widetilde{K_m}/\Delta$ by $\mathfrak{a}_m \times \mathrm{im}(\mathrm{per}^\flat_{[\omega]})$, and thus in particular a closed Lie subgroup. Thus we may take $I_m := \varphi_m(K^\sharp_m)$.

b)⇒a): By construction we have that $\widetilde{I_m} \cong K^\sharp_m$, so that φ_m factors through the inclusion $I_m \hookrightarrow K^\sharp$ and the universal covering map $K^\sharp_m \to I_m$. Thus $\mathrm{per}^\flat_{[\omega]}(\Delta) \subseteq \Pi/\Gamma$ follows from $I_m \cap A_\Gamma \subseteq \mathfrak{a}_m \times \Pi/\Gamma$.

It remains to show the assertion that if c) is satisfied, then $(\theta, \varphi_m(K^\sharp_m)\cdot \Pi/\Gamma)$ is a transitive pair with kernel Π/Γ and that $K^\sharp/(\varphi_m(K^\sharp_m)\cdot\Pi/\Gamma) \to M$ is a Π-prequantisation of ω.

Set $H_m := \varphi_m(K^\sharp_m)\cdot\Pi/\Gamma$. To show that (θ, H_m) is a transitive pair, we first show that H_m is in fact a normal subgroup of $(\theta^\wedge)^{-1}(K_m)$. In fact, each $g \in (\theta^\wedge)^{-1}(K_m)$ may be written as a product $a_0 \cdot g_m$ for $g_m \in \varphi_m(K^\sharp_m)$ and $a_0 \in \mathbb{T}_\Gamma$. This is due to the fact that $\varphi_m(K^\sharp_m)$ is a reduction of $K^\sharp|_{\widetilde{K_m}/\Delta}$ to an extension by $C^\infty_m(M,\mathbb{R}) \times \mathrm{im}(\mathrm{per}^\flat_{[\omega]})$ and that $C^\infty_m(M,\mathbb{R})\cdot\mathbb{T}_\Gamma \cong C^\infty(M,\mathbb{R})/\Gamma = A_\Gamma$. Since the action of $k \in \widetilde{K}$ on $C^\infty(M,\mathbb{T}_\Gamma)$ coincides with conjugation action of an arbitrary lift of k to $K^\sharp$ it follows that $a_0 \in \mathbb{T}_\Gamma = C^\infty(M,T_\Gamma)^{\widetilde{K}} \subseteq Z(K^\sharp)$ and thus $\mathrm{Ad}(a_0) = \mathrm{id}_{K^\sharp}$. Consequently, $\mathrm{Ad}(g) = \mathrm{Ad}(g_m)$ and thus $\mathrm{Ad}(g)$ preserves the subalgebra $\mathbf{L}(\varphi_m(K^\sharp_m))$. Furthermore, $\mathrm{Ad}(g) = \mathbf{L}(c_g)$ (where c_g is conjugation by g and the groups $K^\sharp$ and $\varphi_m(K^\sharp_m)$ are connected and regular Lie groups. Thus Lemma B.6 implies that conjugation by g preserves $\varphi_m(K^\sharp_m)$. Since $H_m = \varphi_m(K^\sharp_m)\cdot\Pi/\Gamma$ and $\Pi/\Gamma \subseteq \mathbb{T}_\Gamma \subseteq Z(K^\sharp)$ follows as above, it also follows that conjugation by g preserves H_m.

Since $g \in (\theta^\wedge)^{-1}(K_m)$ was arbitrary, this shows that H_m is normal in $(\theta^\wedge)^{-1}(K_m)$. To conclude that (θ, H_m) is a transitive pair it thus suffices to observe that $\theta(\cdot, m) = \mathrm{ev}_m \circ q_{\pi_1(K)} \circ q^\sharp$ clearly is a submersion, and $\mathbf{L}(\widehat{G})/\mathbf{L}(H_m)$ is finite-dimensional, so H_m in particular co-Banach.

It remains to show that $P := K^\sharp/(H_m) \to M$ is a Π-prequantisation. First note that H_m is an extension of $\widetilde{K_m}/\Delta$ by $\mathfrak{a}_m \times \Pi/\Gamma$ and $(\theta^\wedge)^{-1}(K_m)$ is an extension of $\widetilde{K_m}/\Delta$ by $\mathfrak{a}_m \times \mathbb{T}_\Gamma$, so that the morphism of Lie groups

$$\mathbb{T}_\Pi \to ((\theta^\wedge)^{-1}(K_m))/H_m, \tag{28}$$

induced by mapping an element of $\mathbb{T}_\Pi$ to the respective constant function, is an isomorphism. With respect to this isomorphism we endow $P \to M$ with the structure of a $\mathbb{T}_\Pi$-bundle over M.

We now construct a connection on $P \to M$ with curvature ω as follows. Let $H \leq T\widetilde{K}$ be a horizontal distribution on the bundle $\mathrm{ev}_m \colon \widetilde{K} \to M$, i.e., we have each $k \in \widetilde{K}$ a subspace $\widetilde{H}_k \leq T_k K$ such that $\widetilde{H}_{k \cdot k'} = \widetilde{H}_k \cdot k'$ for $k' \in \widetilde{K_m}/\Delta$ and that $T\,\mathrm{ev}_m \colon \widetilde{H}_k \to T_{k(m)}M$ is a linear isomorphism. Denote by $\widetilde{\sigma} \colon \mathcal{V}(M) \to \mathcal{V}(\widetilde{K})^{\widetilde{K_m}/\Delta}$ the corresponding horizontal lift of vector fields. On the bundle $q^\sharp \colon K^\sharp \to \widetilde{K}$ we have the connection which is induced by the isomorphisms $TR_{k^{-1}} \colon T_k\widetilde{K} \to T_e\widetilde{K} \cong \mathfrak{k}$, $TR_{\overline{k}^{-1}} \colon T_{\overline{k}}K^\sharp \to T_eK^\sharp \cong \mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k}$ and the canonical linear splitting $\sigma \colon \mathfrak{k} \to \mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k}$. Denote the corresponding horizontal lift by $\sigma_{\overline{\omega}} \colon \mathcal{V}(\widetilde{K}) \to \mathcal{V}(K^\sharp)^{A_\Gamma}$. If we now set

$$H^\sharp_{\overline{k}} := \sigma(\widetilde{H}_k \cdot k^{-1}) \cdot \overline{k}$$

for $\overline{k} \in \widetilde{K}$ and $k := q^\sharp(\overline{k})$ (where we suppressed the isomorphisms $T_k\widetilde{K} \cong \mathfrak{k}$ and $T_{\overline{k}}K^\sharp \cong \mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k}$), then $T_{\overline{k}}(\mathrm{ev}_m \circ q^\sharp)$ also restricts to a linear isomorphism on $H^\sharp_{\overline{k}}$ and we have

$$H^\sharp_{\overline{kk'}} = \sigma(\widetilde{H}_{kk'}(kk')^{-1}) \cdot \overline{kk}' = H^\sharp_{\overline{k}} \cdot \overline{k}'$$

for each $\overline{k}' \in (q^\sharp)^{-1}(\widetilde{K}_m/\Delta)$ and $k' := q^\sharp(\overline{k}')$. Consequently, $H^\sharp$ defines a horizontal distribution on the bundle $\mathrm{ev}_m \circ q^\sharp \colon K^\sharp \to M$. If $\sigma^\sharp \colon \mathcal{V}(M) \to \mathcal{V}(K^\sharp)^{(\theta^\wedge)^{-1}(K_m)}$ denotes the corresponding horizontal lift of vector fields, then we clearly have $\sigma^\sharp = \sigma_{\overline{\omega}} \circ \widetilde{\sigma}$.

From this we obtain a connection

$$\sigma_P \colon \mathcal{V}(M) \to \mathcal{V}(P)^{\mathbb{T}_\Pi}, \quad X \mapsto TQ_*(\sigma^\sharp(X))$$

on $P \to M$, where $Q \colon K^\sharp \to P = K^\sharp/(H_m)$ is the canonical quotient morphism. For the curvature of the connection σ_P we then have

$$F_{\sigma_P}(X,Y) := \sigma_P([X,Y]) - [\sigma_P(X), \sigma_P(Y)] = TQ_*(\sigma^\sharp([X,Y]) - [\sigma_P(X), \sigma_P(Y)]) = TQ_*(F_{\sigma^\sharp}(X,Y))$$

for $X, Y \in \mathcal{V}(M)$, and, furthermore

$$\begin{aligned} F_{\sigma^\sharp}(X,Y) = \sigma_{\overline{\omega}}(\widetilde{\sigma}([X,Y])) - [\sigma_{\overline{\omega}}(\widetilde{\sigma}(X)), \sigma_{\overline{\omega}}(\widetilde{\sigma}(Y))] = \sigma_{\overline{\omega}}(\widetilde{\sigma}([X,Y])) - \sigma_{\overline{\omega}}([\widetilde{\sigma}(X), \widetilde{\sigma}(Y)]) +{}& \\ F_{\sigma_{\overline{\omega}}}(\widetilde{\sigma}(X), \widetilde{\sigma}(Y)) = \sigma_{\overline{\omega}}(F_{\widetilde{\sigma}}(X,Y)) + F_{\sigma_{\overline{\omega}}}(\widetilde{\sigma}(X), \widetilde{\sigma}(Y)).& \end{aligned}$$

Since $F_{\sigma_{\overline{\omega}}} = \overline{\omega}^{\mathrm{eq}}$ (cf. the proof of Lemma 5.13) and since $\sigma_{\overline{\omega}}(F_{\widetilde{\sigma}}(X,Y))$ is at each point tangential to the fibre H_m of Q, it follows that

$$\begin{aligned} TQ(F_{\sigma^\sharp}(X,Y)(k(m))) = TQ(F_{\sigma_{\overline{\omega}}}(\widetilde{\sigma}(X)(k), \widetilde{\sigma}(Y)(k))) = TQ(\overline{\omega}^{\mathrm{eq}}(\widetilde{\sigma}(X)(k), \widetilde{\sigma}(Y)(k)))& \\ = \mathrm{ev}_m(k.\overline{\omega}(\widetilde{\sigma}(X)(k) \cdot k^{-1}, \widetilde{\sigma}(Y)(k) \cdot k^{-1})) = \omega_{k(m)}(X(k(m)), X(k(m)))& \end{aligned}$$

for each $\overline{k} \in K^\sharp$ and $k := q^\sharp(\overline{k}) \in \widetilde{K}$. Thus $F_{\sigma^P} = \omega$.

It remains to check that the kernel actually coincides with Π/Γ. To this end we define $q := q_{\pi_1(K)} \circ q^\sharp$ and consider the diagram

$$\begin{tikzcd}
\ker(q^\sharp) \arrow[r, "a_{\theta,H_m}|_{\ker(q^\sharp)}"] \arrow[d, hook] & \ker((\beta_{\mathcal{R}})_*) \arrow[d, hook] \\
K^\sharp \arrow[r, "a_{\theta,H_m}"] \arrow[d, "q^\sharp"] \arrow[dr, "\theta^\wedge"] & \mathrm{Bis}(\mathcal{R}(\theta, H_m)) \arrow[d, "(\beta_{\mathcal{R}})_*"] \\
K \arrow[r, hook] & \mathrm{Diff}(M),
\end{tikzcd}$$

which commutes by Lemma 4.13 and the construction of $\theta^\wedge$. From this it follows that the kernel of a_{θ,H_m} is contained in $\ker(q^\sharp)$. Moreover, we have $\ker(q^\sharp) = A_\Gamma = C^\infty(M, \mathbb{T}_\Gamma)$ by definition. To determine $a_{\theta,H_m}(\gamma)$ for $\gamma \in \ker(q)$, we first note that the element in $\mathrm{Aut}(K^\sharp/H_m \to M)$ corresponding to $a_{\theta,H_m}(\gamma)$ is given by $\overline{k}H_m \mapsto (\gamma \cdot \overline{k})H_m$ (cf. Lemma 4.11). On the other hand, an element $\eta \in \ker((\beta_{\mathcal{R}})_*) \cong C^\infty(M, \mathbb{T}_\Pi)$ acts on $K^\sharp/H_m$ by

$$\overline{k}H_m \mapsto \overline{k}H_m \cdot \eta(\theta(\overline{k}, m)) = (\overline{k} \cdot \eta(\theta(\overline{k}, m)))H_m,$$

since the bundle projection $K^\sharp/H_m \to M$ is given by $\overline{k}H_m \mapsto \theta(\overline{k}, m)$ and $\mathbb{T}_\Gamma$ is contained in $Z(K^\sharp)$ (cf. (28)). From this it follows that the value of $a_{\theta,H_m}(\gamma)$ in $\theta(\overline{k}, m)$ has to satisfy

$$(\overline{k}^{-1} \cdot \gamma \cdot \overline{k})H_m = a_{\theta,H_m}(\gamma)(\theta(\overline{k}, m))H_m.$$

Since $\overline{k}^{-1} \cdot \gamma \cdot \overline{k} = \gamma \circ \theta^\wedge(\overline{k})$ follows from the fact that $K^\sharp \to K$ is an abelian extension for the natural action of K on $C^\infty(M, \mathbb{T}_\Gamma)$, we conclude that $a_{\theta,H_m}|_{\ker(q)}$ coincides with the map that is induced by the projection $\mathbb{T}_\Gamma \to \mathbb{T}_\Pi = \mathbb{T}_\Gamma/(\Pi/\Gamma)$ and the isomorphisms $C^\infty(M, \mathbb{T}_\Gamma) \cong \ker(q^\sharp)$ and $C^\infty(M, \mathbb{T}_\Pi) \cong \ker((\beta_{\mathcal{R}})_*)$. Consequently, $\ker(a_{\theta,H_m}) = \ker(a_{\theta,H_m}) \cap \ker(q^\sharp) \cong \Pi/\Gamma$ (cf. Proposition 4.16). $\square$

Corollary 5.16. *With the notation and under the assumptions of Theorem 5.15 the following assertions are equivalent:*

a) *The primary periods* $\mathrm{per}_{\overline{\omega}}(\pi_2(M)) \subseteq \mathbb{R}$ *are discrete.*
b) *The extension* $\mathfrak{a} \to \mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k} \to \mathfrak{k}$ *integrates to an extension* $A_\Gamma \to K^\sharp \to \widetilde{K}$ *of Lie groups.*

If one (and thus both) of these conditions is satisfied, then the following assertions are equivalent:

i) *The secondary periods* $\mathrm{per}_{[\omega]}(\pi_1(K_m)) \subseteq \mathbb{R}$ *are discrete.*
ii) *The closed subalgebra* $\mathfrak{a}_m \oplus_{\overline{\omega}_m} \mathfrak{k}$ *of* $\mathfrak{a} \oplus_{\overline{\omega}} \mathfrak{k}$ *integrates to a closed Lie subgroup of* $K^\sharp$.

Corollary 5.17. *If M is a compact and 1-connected manifold and $\omega \in \Omega^2(M)$ is closed, then the extension $\mathbb{R} \to \mathbb{R} \oplus_\omega TM \to TM$ of Lie algebroids integrates to an extension of Lie groupoids if and only if the extension of Lie algebras $C^\infty(M) \to C^\infty(M) \oplus_{\overline{\omega}} \mathcal{V}(M) \to \mathcal{V}(M)$ integrates to an extension $A \to \widehat{K} \to \mathrm{Diff}(M)_0$ of Lie groups and $C^\infty_m(M) \oplus_{\overline{\omega}_m} \mathcal{V}_m(M)$ integrates to a closed Lie subgroup in $\widehat{K}$.*

Example 5.18. This is a continuation of Example 5.14. Of course, $(\mathbb{S}^2, \omega)$ is $\mathbb{Z}$-prequantisable, a (suitably normalised) prequantisation is the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$, viewed as a $U(1) := T_{\mathbb{Z}}$-principal bundle (together with the standard contact form on $\mathbb{S}^3$). From this we obtain the extension

$$C^\infty(\mathbb{S}^2, U(1)) \to \mathrm{Aut}(\mathbb{S}^3 \to \mathbb{S}^2) \to \mathrm{Diff}(\mathbb{S}^2)_0$$

(cf. Example 3.7) whose Lie algebra extension is equivalent to

$$C^\infty(\mathbb{S}^2,\mathbb{R}) \to C^\infty(\mathbb{S}^2,\mathbb{R}) \oplus_{\overline{\omega}} \mathcal{V}(\mathbb{S}^2) \to \mathcal{V}(\mathbb{S}^2). \tag{29}$$

Since the primary periods vanish, the extension (29) integrates to an extension (unique up to equivalence)

$$C^\infty(\mathbb{S}^2,\mathbb{R}) \to K^\sharp \to \widetilde{\mathrm{Diff}(\mathbb{S}^2)}_0$$

and the identity on $C^\infty(\mathbb{S}^2,\mathbb{R}) \oplus_{\overline{\omega}} \mathcal{V}(\mathbb{S}^2)$ integrates to a Lie group morphism $\psi\colon K^\sharp \to \mathrm{Aut}(\mathbb{S}^3 \to \mathbb{S}^2)$. This morphism makes

$$\begin{tikzcd}
C^\infty(\mathbb{S}^2,\mathbb{R}) \arrow[r] \arrow[d, "q_{\mathbb{Z}}"] & K^\sharp \arrow[r] \arrow[d, "\psi"] & \widetilde{\mathrm{Diff}(\mathbb{S}^2)}_0 \arrow[d, "q_{\pi_1}"] \\
C^\infty(\mathbb{S}^2,U(1)) \arrow[r] & \mathrm{Aut}(\mathbb{S}^3 \to \mathbb{S}^2) \arrow[r] & \mathrm{Diff}(\mathbb{S}^2)_0
\end{tikzcd}$$

commute, where $q_{\mathbb{Z}}$ is induced by the quotient map $\mathbb{R} \to U(1) = \mathbb{R}/\mathbb{Z}$ and q_{π_1} is the universal covering morphism of $\mathrm{Diff}(\mathbb{S}^2)_0$. If now $o \in \mathbb{S}^3$ is mapped to the base-point $m \in \mathbb{S}^2$, then $\mathrm{Aut}_o(\mathbb{S}^3 \to \mathbb{S}^2)$ is a closed Lie subgroup of $\mathrm{Aut}(\mathbb{S}^3 \to \mathbb{S}^2)$ (cf. Example 3.7) and since q_{π_1} is a covering morphism, $\psi^{-1}(\mathrm{Aut}_o(\mathbb{S}^3 \to \mathbb{S}^2))$ is also a closed Lie subgroup of $K^\sharp$. Since $\psi|_{C^\infty(\mathbb{S}^2,\mathbb{R})} = q_{\mathbb{Z}}$, we have

$$\psi^{-1}(C^\infty(\mathbb{S}^2,U(1)) \cap \mathrm{Aut}_o(\mathbb{S}^3 \to \mathbb{S}^2)) = C^\infty_m(\mathbb{S}^2,\mathbb{R}) \times \mathbb{Z},$$

and thus $\psi^{-1}(\mathrm{Aut}_o(\mathbb{S}^3 \to \mathbb{S}^2))$ gives rise to an extension

$$C^\infty_m(\mathbb{S}^2,\mathbb{R}) \times \mathbb{Z} \to \psi^{-1}(\mathrm{Aut}_o(\mathbb{S}^3 \to \mathbb{S}^2)) \to q_{\pi_1}^{-1}(\mathrm{Diff}_m(\mathbb{S}^2)_0).$$

If we identify $q_{\pi_1}^{-1}((\mathrm{Diff}(\mathbb{S}^2)_0)_m)$ with $\widetilde{\mathrm{Diff}_m(\mathbb{S}^2)}_0/\Delta$ (for Δ as in Remark 5.10), then we deduce from Example 5.14 and Theorem 5.15 that $\psi^{-1}(\mathrm{Aut}_o(\mathbb{S}^3 \to \mathbb{S}^2))$ is precisely the Lie subgroup $\varphi_m((\mathrm{Diff}(\mathbb{S}^2)_0)^\sharp_m)$. $\square$

Example 5.19. An example where the conditions of Theorem 5.15 are not fulfilled is the following (cf. [33, Example 1]). Let η be the standard volume form on $\mathbb{S}^2$ with total volume 1. On $M := \mathbb{S}^2 \times \mathbb{S}^2$, consider the form $\omega \in \Omega^2(\mathbb{S}^2 \times \mathbb{S}^2)$

$$\omega_{(p,q)}\colon T_{(p,q)}\mathbb{S}^2 \times \mathbb{S}^2 \cong T_p\mathbb{S}^2 \times T_q\mathbb{S}^2 \to \mathbb{R}, \quad (x,y) \mapsto \eta(x) + \lambda\eta(y).$$

Then $\pi_2(\mathbb{S}^2 \times \mathbb{S}^2) = \mathbb{Z} \times \mathbb{Z}$ and we have $\mathrm{per}_{[\omega]}((1,0)) = 1$ and $\mathrm{per}_{[\omega]}((0,1)) = \lambda$. Thus $\mathrm{per}_{[\omega]}(\pi_2(S^2 \times S^2))$ is the subgroup of $\mathbb{R}$ which is generated by 1 and λ. If $\lambda \notin \mathbb{Q}$, then this is not contained in any discrete subgroup.

If we take $K = \mathrm{Symp}(M,\omega)$, then the primary periods vanish by Corollary 5.8 and we may take $\Gamma = \{0\}$ in Theorem 5.15. Consequently, the secondary periods $\mathrm{per}^\flat_{[\omega]}(\pi_1(K_m))$ are not contained in any discrete subgroup of $\mathbb{R}$ and the subalgebra $C^\infty_m(M) \oplus_{\overline{\omega}_m} \mathfrak{k}_m$ does not integrate to a closed Lie subgroup in the extension

$$C^\infty(M,\mathbb{R}) \to K^\sharp \to \widetilde{K}. \qquad \square$$

Problem 5.20. If one takes the results of this section, then the following questions seem to be natural and interesting.

a) How do the primary and secondary periods $\mathrm{per}_{[\overline{\omega}]}(\pi_2(K))$ and $\mathrm{per}^\flat_{[\omega]}(\pi_1(K_m))$ vary if one varies the subgroup K? It is clear that for K, K' with $K \leq K'$ we have $\mathrm{per}_{[\overline{\omega}]}(\pi_2(K)) \leq \mathrm{per}_{[\overline{\omega}]}(\pi_2(K'))$ and $\mathrm{per}^\flat_{[\omega]}(\pi_1(K_m)) \leq \mathrm{per}^\flat_{[\omega]}(\pi_1(K'_m))$, but under which assumptions does one have equality here? In particular, it would be interesting to have a symplectic manifold with non-vanishing primary periods for $K = \mathrm{Diff}(M)_0$ (since for $K = \mathrm{Symp}(M, \omega)$ the primary periods always vanish by Proposition 5.8).
b) It would be interesting to develop an integration theory for infinitesimal transitive pairs (cf. Problem 4.22). In particular, this should shed some further light on the precise relation between the integration theory of Lie algebroids, Lie algebras (of sections) and the associated obstructions.
c) What is the interplay between the primary and secondary periods and the flux group

$$F_{[\omega]}(\pi_1(\mathrm{Symp}(M, \omega))) \subseteq H^1(M, \mathbb{R})$$

in the case that M is only assumed to be connected? Conjecturally, there might be a relation of the long exact homotopy sequence of the evaluation fibration and the one induced by $\Gamma \to \mathbb{R} \to \mathbb{R}/\Gamma$

$$\begin{tikzcd}
\pi_2(\mathrm{Symp}(M,\omega)) \arrow[r] \arrow[d, "\mathrm{per}_{[\overline{\omega}]}"] & \pi_2(M) \arrow[r] \arrow[d, "\mathrm{per}_{[\omega]}"] & \pi_1(\mathrm{Symp}_m(M,\omega)) \arrow[r] \arrow[d, dashed, "?"] & \pi_1(\mathrm{Symp}(M,\omega)) \arrow[d, "F_{[\omega]}"] \\
H^0(M,\Gamma) \arrow[r] & H^0(M,\mathbb{R}) \arrow[r] & H^0(M,\mathbb{R}/\Gamma) \arrow[r] & H^1(M,\Gamma)
\end{tikzcd}$$

in case that the primary periods $\mathrm{per}_{[\overline{\omega}]}(\pi_2(\mathrm{Symp}(M, \omega))) \leq \mathbb{R}$ are contained in the discrete subgroup $\Gamma \leq \mathbb{R}$ and the flux group is contained in $H^1(M, \Gamma)$. Note that the flux group is known to be discrete by the proof of the flux conjecture [27] and that both, the secondary periods and the flux subgroup are related to the integrability of Lie subalgebras to closed Lie subgroups (cf. [20, Proposition 10.20]). The conjectural homomorphism $\pi_1(\mathrm{Symp}_m(M, \omega)) \dashrightarrow H^0(M, \mathbb{R}/\Gamma)$ should be related to the fluxes $F_{[\overline{\omega}]}$, $F_{[\overline{\omega}_m]}$ and the homomorphism $\varphi_m \colon K^\sharp_m \to K^\sharp$, restricted to the pre-image $(q^\sharp_m)^{-1}(\pi_1(\mathrm{Symp}_m(M, \omega)))$. However, this involves the (continuous) Lie algebra cohomology of $\mathfrak{k}_m$ with coefficients in $C^\infty_m(M)$, a topic that goes beyond the scope of the present paper. □

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Appendix A. Local bisections for infinite-dimensional Lie groupoids

In this appendix we prove that (infinite-dimensional) Lie groupoids over a finite-dimensional manifold admit local bisections through each point. Consequently, we are able to derive that their vertex groups are in a natural way Lie groups. These results are standard for finite-dimensional Lie groupoids. We repeat them here for the readers convenience since some details of proofs need to be adapted for our infinite-dimensional setting.

Definition A.1. Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally convex Lie groupoid. For $U \Subset M$, a *local bisection of $\mathcal{G}$ on U* is a smooth map $\sigma \colon U \to G$ such that $\alpha \circ \sigma = \mathrm{id}_U$ and $\beta \circ \sigma \colon U \to (\beta \circ \sigma)(U) \Subset M$ is a diffeomorphism. □

Lemma A.2 *([16, Proposition 1.4.9]). Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally convex Lie groupoid such that M is a finite dimensional manifold. For each $g \in G$ there exists an $\alpha(g)$-neighbourhood $U \Subset M$ and a local bisection σ of $\mathcal{G}$ on U such that $\sigma(\alpha(g)) = g$.*

Proof. Note that in [16] only finite-dimensional Lie groupoids are discussed. However, to prove the assertion, one can copy the proof of [16, Proposition 1.4.9] verbatim to our setting, since the Lie groupoid is assumed to have a finite-dimensional base. The crucial point here is that the algebraic argument used in the proof of [16, Proposition 1.4.9] carries over to subspaces of finite codimension of arbitrary locally convex spaces. Assuming that the base M is finite-dimensional ensures exactly this property. $\square$

For mappings into finite-dimensional manifolds (whose domain is an infinite-dimensional manifold), one can define maps of constant rank analogously to the finite-dimensional case (cf. [7, Theorem F]).

Corollary A.3 *([16, Corollary 1.4.10]). Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally convex Lie groupoid over a finite dimensional manifold M. Then for each $m \in M$ the maps*

$$\beta|_{\alpha^{-1}(m)} \colon \alpha^{-1}(m) \to M, g \mapsto \beta(g) \text{ and } \alpha|_{\beta^{-1}(m)} \colon \beta^{-1}(m) \to M, g \mapsto \alpha(g)$$

are maps of constant rank.

The next proof follows [16, Corollary 1.4.11] but we need to adapt the arguments.

Lemma A.4. *Let $\mathcal{G} = (G \rightrightarrows M)$ be a locally convex Lie groupoid over a finite dimensional manifold M. Then for all $m, n \in M$, $\alpha^{-1}(m) \cap \beta^{-1}(n)$ is a split submanifold (of finite codimension) of $\alpha^{-1}(m)$, of $\beta^{-1}(n)$ and of G. In particular, each vertex group $\mathrm{Vert}_m(\mathcal{G}) := \alpha^{-1}(m) \cap \beta^{-1}(m)$ is a closed submanifold of G and this structure turns it into a Lie group.*

Proof. The set $\alpha^{-1}(m) \cap \beta^{-1}(n)$ is the preimage of n under the constant rank map $\beta|_{\alpha^{-1}(m)}$. As M is a finite dimensional manifold we can apply Glöckner's constant rank theorem [7, Theorem F]. Thus $\alpha^{-1}(m) \cap \beta^{-1}(n)$ is a split submanifold of finite codimension in $\alpha^{-1}(m)$. Moreover, $\alpha^{-1}(m)$ is a split submanifold of G (by the regular value theorem [7, Theorem D]) and since M is finite-dimensional, $\alpha^{-1}(m)$ is even of finite-codimension in G. Thus [7, Lemma 1.4] implies that also $\alpha^{-1}(m) \cap \beta^{-1}(n)$ is a split submanifold (of finite-codimension) of G. Analogously one shows that $\alpha^{-1}(m) \cap \beta^{-1}(n)$ is a split submanifold (of finite-codimension) of $\beta^{-1}(n)$.

Groupoid multiplication and inversion induce a group structure on $\mathrm{Vert}_m(\mathcal{G})$. By the universal property of the pullback, the inclusion $\mathrm{Vert}_m(\mathcal{G}) \times \mathrm{Vert}_m(\mathcal{G}) \to G \times_{\alpha,\beta} G$ is smooth, whence this group structure turns $\mathrm{Vert}_m(\mathcal{G})$ into a Lie group. $\square$

Appendix B. Locally convex manifolds, Lie groups and spaces of smooth maps

In this appendix we collect the necessary background on the theory of manifolds and Lie groups that are modelled on locally convex spaces and how spaces of smooth maps can be equipped with such a structure. Let us first recall some basic facts concerning differential calculus in locally convex spaces. We follow [5,2].

Definition B.1. Let E, F be locally convex spaces, $U \subseteq E$ be an open subset, $f \colon U \to F$ a map and $r \in \mathbb{N}_0 \cup \{\infty\}$. If it exists, we define for $(x, h) \in U \times E$ the directional derivative

$$df(x, h) := D_h f(x) := \lim_{t \to 0} t^{-1}(f(x + th) - f(x)).$$

We say that f is C^r if the iterated directional derivatives

$$d^{(k)}f(x, y_1, \dots, y_k) := (D_{y_k} D_{y_{k-1}} \cdots D_{y_1} f)(x)$$

exist for all $k \in \mathbb{N}_0$ such that $k \leq r$, $x \in U$ and $y_1, \dots, y_k \in E$ and define continuous maps $d^{(k)}f \colon U \times E^k \to F$. If f is C^∞ it is also called smooth. We abbreviate $df := d^{(1)}f$. From this definition of smooth map there is an associated concept of locally convex manifold, i.e., a Hausdorff space that is locally homomorphic to open subsets of locally convex spaces with smooth chart changes. Accordingly, a locally convex Lie group is a manifold, equipped with a group structure such that all group operations are smooth. See [35,24,5] for more details. □

Definition B.2. Let M be a smooth manifold. Then M is called *Banach* (or *Fréchet*) manifold if all its modelling spaces are Banach (or Fréchet) spaces. The manifold M is called *locally metrisable* if the underlying topological space is locally metrisable (equivalently if all modelling spaces of M are metrisable). It is called *metrisable* if it is metrisable as a topological space (equivalently locally metrisable and paracompact). □

Definition B.3. Let H be a Lie group modelled on a locally convex space, with identity element $\mathbf{1}$, and $r \in \mathbb{N}_0 \cup \{\infty\}$. We use the tangent map of the left translation $\lambda_h \colon H \to H$, $x \mapsto hx$ by $h \in H$ to define $h.v := T_{\mathbf{1}}\lambda_h(v) \in T_h H$ for $v \in T_{\mathbf{1}}(H) =: \mathbf{L}(H)$. Following [8], H is called *C^r-semiregular* if for each C^r-curve $\eta \colon [0,1] \to \mathbf{L}(H)$ the initial value problem

$$\begin{cases} \gamma'(t) & = \gamma(t).\eta(t) \\ \gamma(0) & = \mathbf{1} \end{cases}$$

has a (necessarily unique) C^{r+1}-solution $\mathrm{Evol}(\eta) := \gamma \colon [0,1] \to H$. If in addition the map

$$\mathrm{evol} \colon C^r([0,1], \mathbf{L}(H)) \to H, \quad \eta \mapsto \mathrm{Evol}(\eta)(1)$$

is smooth, we call H *C^k-regular.* □

Remark B.4. If H is C^r-regular and $r \leq s$, then H is also C^s-regular. A C^∞-regular Lie group H is called *regular (in the sense of Milnor)* – a property first defined in [18]. Every finite dimensional Lie group is C^0-regular (cf. [24]). Several important results in infinite-dimensional Lie theory are only available for regular Lie groups (see [18,24,7], cf. also [13] and the references therein). □

Lemma B.5. *Let G be a C^k-regular Lie group for $k \in \mathbb{N}_0 \cup \{\infty\}$ and H be a closed Lie subgroup of G. If H is C^k-semiregular, then H is C^k-regular.*

Proof. Denote by $i_H \colon H \to G$ and $I_H \colon C^{k+1}([0,1], H) \to C^{k+1}([0,1], G)$ the canonical inclusions. Then $\mathbf{L}(i_H) \colon \mathbf{L}(H) \to \mathbf{L}(G)$ allows us to identify curves $\eta \in C^k([0,1], \mathbf{L}(H))$ with C^k-curves $\mathbf{L}(i_H) \circ \eta$ in $\mathbf{L}(G)$. As G and H are C^k-semiregular, we obtain maps $\mathrm{Evol}_J \colon C^k([0,1], \mathbf{L}(J)) \to C^{k+1}([0,1], J)$ for $J \in \{G, H\}$ which map a curve to the solution of the initial value problem

$$\begin{cases} \gamma'(t) & = \gamma(t).\eta(t) \quad \forall t \in [0,1] \\ \gamma(0) & = \mathbf{1} \end{cases}$$

in the respective group. Consider the map $\mathbf{L}(i_H)_* \colon C^k([0,1], \mathbf{L}(H)) \to C^k([0,1], \mathbf{L}(G)), \eta \mapsto \mathbf{L}(i_H) \circ \eta$, which is smooth by [9, Lemma 1.2]. By [8, 1.16] we have

$$\mathrm{Evol}_G \circ \mathbf{L}(i_H) = I_H \circ \mathrm{Evol}_H$$

As H is a closed subgroup of G, the same holds for $C^{k+1}([0,1], H) \subseteq C^{k+1}([0,1], G)$ (cf. [8, 1.8 and 1.10]) Hence Evol_H is smooth if $I_H \circ \mathrm{Evol}_H$ is smooth. Observe that since G is C^k-regular, Evol_G and thus $\mathrm{Evol}_G \circ \mathbf{L}(i_H)_* = I_H \circ \mathrm{Evol}_H$ is smooth. We deduce that Evol_H is smooth and [8, Lemma 3.1] shows that H is C^k-regular. □

Lemma B.6. *Suppose H, K are Lie groups with Lie subgroups $H_* \leq H$, $K_* \leq K$ such that K and K_* are regular and H_* is connected. Let $\varphi\colon H \to K$ be a morphism of Lie groups. If $\mathbf{L}(\varphi)(\mathbf{L}(H_*)) \subseteq \mathbf{L}(K_*)$, then $\varphi(H_*) \subseteq K_*$.*[9]

Proof. Since H_* is connected, it is contained in the identity component of H. Restricting to that component we may assume that also H is connected. Then [24, Proposition II.4.1] implies that $\mathbf{L}(\varphi)$ has at most one integration to a morphism of Lie groups, which thus has to coincide with φ. Recall that the left logarithmic derivative $\delta^l(\gamma)\colon [0,1] \to \mathbf{L}(H)$ of a C^1-curve $\gamma\colon [0,1] \to H$ in a Lie group is defined via $\delta^l(\gamma)(t) := \gamma(t)^{-1}.\gamma'(t)$. Thus $\mathrm{Evol}(\eta) = \gamma$ if and only if $\delta^l(\gamma) = \eta$ and $\gamma(0) = \mathbf{1}_H$. The integration of $\mathbf{L}(\varphi)$ is constructed by taking a smooth path $\gamma\colon [0,1] \to H$ with $\gamma(0) = \mathbf{1}_H$, applying $\mathbf{L}(\varphi)$ to $\delta^l(\gamma)$, solving the initial value problem

$$\begin{cases} \delta^l(\eta) & = \mathbf{L}(\varphi) \circ \delta^l(\gamma) \\ \eta(0) & = \mathbf{1}_K \end{cases} \tag{30}$$

in K and setting $\varphi(\gamma(1)) = \eta(1)$ (which is possible since K is regular, cf. [18, Theorem 8.1]). Under the assumptions made, this only depends on $\gamma(1)$.

Since H_* is assumed to be connected, we can choose for each $h \in H_*$ a path γ with $\gamma(0) = e_H$ and $\gamma(1) = h$ such that $\gamma(t) \in H_*$ for all $t \in [0,1]$. Consequently, $\delta^l(\gamma)(t) \in \mathbf{L}(H_*)$, and thus $\mathbf{L}(\varphi)(\delta^l(\gamma)(t)) \in \mathbf{L}(K_*)$ for all $t \in [0,1]$. Now K_* is regular and $\mathbf{L}(\varphi) \circ \delta^l(\gamma)$ takes its image in $\mathbf{L}(K_*)$ by assumption. Thus we can solve (30) in K_*. From [8, Lemma 10.1], we deduce that the solution to the initial value problem (30) for η in K coincides with the solution in K_*, and thus takes its values in K_*. Summing up, $\varphi(h) = \eta(1) \in K_*$ for each $h \in H_*$. □

Definition B.7. Suppose M is a smooth manifold. Then a *local addition* on M is a smooth map $\Sigma\colon U \Subset TM \to M$, defined on an open neighbourhood U of the submanifold $M \subseteq TM$ such that

a) $\pi \times \Sigma\colon U \to M \times M$, $v \mapsto (\pi(v), \Sigma(v))$ is a diffeomorphism onto an open neighbourhood of the diagonal $\Delta M \subseteq M \times M$ and
b) $\Sigma(0_m) = m$ for all $m \in M$.

We say that M *admits a local addition* if there exist a local addition on M. □

Definition B.8. (cf. [17, 10.6]) Let $s\colon Q \to N$ be a surjective submersion. Then a *local addition adapted to* s is a local addition $\Sigma\colon U \Subset TQ \to Q$ such that the fibres of s are additively closed with respect to Σ, i.e. $\Sigma(v_q) \in s^{-1}(s(q))$ for all $q \in Q$ and $v_q \in T_q s^{-1}(s(q))$ (note that $s^{-1}(s(q))$ is a submanifold of Q). □

An important tool will be the following excerpt from [35, Theorem 7.6].

[9] The authors believe that this result is well-known to experts in the field but were unable to locate a reference.

Theorem B.9. *Let M be a compact manifold and N be a locally convex and locally metrisable manifold that admits a local addition $\Sigma\colon U \Subset TN \to N$. Set $V := (\pi \times \Sigma)(U)$, which is an open neighbourhood of the diagonal ΔN in $N \times N$. For each $f \in C^\infty(M,N)$ we set*

$$O_f := \{g \in C^\infty(M,N) \mid (f(x), g(x)) \in V\}.$$

Then the following assertions hold.

a) *The set O_f contains f, is open in $C^\infty(M,N)$ and the formula $(f(x), g(x)) = (f(x), \Sigma(\varphi_f(g)(m)))$ determines a homeomorphism*

$$\varphi_f\colon O_f \to \{h \in C^\infty(M, TN) \mid \pi(h(x)) = f(x)\} \cong \Gamma(f^*(TN))$$

from O_f onto the open subset $\{h \in C^\infty(M,TN) \mid \pi(h(x)) = f(x)\} \cap C^\infty(M,U)$ of $\Gamma(f^*(TN))$.

b) *The family $(\varphi_f\colon O_f \to \varphi_f(O_f))_{f\in C^\infty(M,N)}$ is an atlas, turning $C^\infty(M,N)$ into a smooth locally convex and locally metrisable manifold. The manifold structure is independent of the choice of the local addition.*

c) *If L is another locally convex and locally metrisable manifold, then a map $f\colon L \times M \to N$ is smooth if and only if $f^\wedge\colon L \to C^\infty(M,N)$ is smooth. In other words,*

$$C^\infty(L \times M, N) \to C^\infty(L, C^\infty(M,N)), \quad f \mapsto f^\wedge$$

is a bijection (which is even natural).

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Part II.

Hopf algebra character groups as Lie groups

ANNALES

DE

L'INSTITUT FOURIER

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Character groups of Hopf algebras as infinite-dimensional Lie groups

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CHARACTER GROUPS OF HOPF ALGEBRAS AS INFINITE-DIMENSIONAL LIE GROUPS

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Abstract. — In this article character groups of Hopf algebras are studied from the perspective of infinite-dimensional Lie theory. For a graded and connected Hopf algebra we obtain an infinite-dimensional Lie group structure on the character group with values in a locally convex algebra. This structure turns the character group into a Baker–Campbell–Hausdorff–Lie group which is regular in the sense of Milnor. Furthermore, we show that certain subgroups associated to Hopf ideals become closed Lie subgroups of the character group.

If the Hopf algebra is not graded, its character group will in general not be a Lie group. However, we show that for any Hopf algebra the character group with values in a weakly complete algebra is a pro-Lie group in the sense of Hofmann and Morris.

Résumé. — Dans cet article, nous étudions les groupes de caractères des algèbres de Hopf du point de vue de la théorie de Lie de dimension infinie. Pour une algèbre de Hopf connexe et graduée, nous munissons le groupe de caractères d'une structure de groupe de Lie de dimension infinie, à valeurs dans une algèbre localement convexe. Cette structure permet de voir le groupe de caractères comme un groupe de Lie de Baker–Campbell–Hausdorff, qui est régulier au sens de Milnor. De plus, nous montrons que certains sous-groupes associés aux idéaux de Hopf sont alors des sous-groupes de Lie du groupe de caractères.

Si l'algèbre de Hopf n'est pas graduée, son groupe de caractères ne sera pas un groupe de Lie, en général. Cependant, nous montrons que pour une algèbre de Hopf quelconque, le groupe de caractères à valeurs dans une algèbre faiblement complète est un groupe pro-Lie au sens de Hofmann et Morris.

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Introduction and statement of results

Hopf algebras and their character groups appear in a variety of mathematical and physical contexts. To name just a few, they arise in noncommutative geometry, renormalisation of quantum field theory (see [7, 10]) and numerical analysis (cf. [4]). We recommend [5] as a convenient introduction to Hopf algebras and their historical development.

In their seminal work [8, 9] Connes and Kreimer associate to the group of characters of a Hopf algebra of Feynman graphs a Lie algebra. It turns out that this (infinite-dimensional) Lie algebra is an important tool to analyse the structure of the character group. In fact, the character group is then called "infinite-dimensional Lie group" meaning that it is a projective limit of finite-dimensional Lie groups with an associated infinite-dimensional Lie algebra. These structures enable the treatment of certain differential equations on the group which are crucial to understand the renormalisation procedure. Moreover, on a purely algebraic level it is always possible to construct a Lie algebra associated to the character group of a Hopf algebra. Thus it seems natural to ask whether the differential equations and the Lie algebras are connected to some kind of Lie group structure on the character group. Indeed, in [3] the character group of the Hopf algebra of rooted trees was turned into an infinite-dimensional Lie group. Its Lie algebra is closely related to the Lie algebra constructed in [7] for the character group.

These observations hint at a general theme which we explore in the present paper. Our aim is to study character groups (with values in a commutative locally convex algebra) of a Hopf algebra from the perspective of infinite-dimensional Lie theory. We base our investigation on a concept of C^r-maps between locally convex spaces known as Keller's C^r_c-theory(1) [21] (see [12, 28, 30] for streamlined expositions and Appendix A for a quick reference). In the framework of this theory, we treat infinite-dimensional Lie group structures for character groups of Hopf algebras. If the Hopf algebra is graded and connected, it turns out that the character group can be made into an infinite-dimensional Lie group. We then investigate Lie theoretic properties of these groups and some of their subgroups. In particular, the Lie algebra associated to the Lie group structure on the group of characters turns out to be the Lie algebra of infinitesimal characters.

(1) Although Keller's C^r_c-theory is in general inequivalent to the "convenient setting" of calculus [22], in the important case of Fréchet spaces both theories coincide (e.g. Example 4.7).

The character group of an arbitrary Hopf algebra can in general not be turned into an infinite-dimensional Lie group and we provide an explicit example for this behaviour. However, it turns out that the character group of an arbitrary Hopf algebra (with values in a finite-dimensional algebra) is always a topological group with strong structural properties, i.e. it is always the projective limit of finite-dimensional Lie groups. Groups with these properties — so called pro-Lie groups — are accessible to Lie theoretic methods (cf. [18]) albeit they may not admit a differential structure.

We now go into some more detail and explain the main results of the present paper. Let us recall first the definition of the character group of a Hopf algebra $(\mathcal{H}, m_{\mathcal{H}}, 1_{\mathcal{H}}, \Delta_{\mathcal{H}}, \varepsilon_{\mathcal{H}}, S_{\mathcal{H}})$ over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Fix a commutative locally convex algebra B. Then the character group $G(\mathcal{H}, B)$ of $\mathcal{H}$ with values in B is defined as the set of all unital algebra characters

$$G(\mathcal{H}, B) := \left\{ \phi \in \operatorname{Hom}_{\mathbb{K}}(\mathcal{H}, B) \middle| \begin{array}{c} \phi(ab) = \phi(a)\phi(b), \forall a, b \in \mathcal{H} \\ \text{and } \phi(1_{\mathcal{H}}) = 1_B \end{array} \right\},$$

together with the convolution product $\phi \star \psi := m_B \circ (\phi \otimes \psi) \circ \Delta_{\mathcal{H}}$.

Closely related to this group is the Lie algebra of infinitesimal characters

$$\mathfrak{g}(\mathcal{H}, B) := \{\phi \in \operatorname{Hom}_{\mathbb{K}}(\mathcal{H}, B) \mid \phi(ab) = \varepsilon_{\mathcal{H}}(a)\phi(b) + \varepsilon_{\mathcal{H}}(b)\phi(a)\}$$

with the commutator Lie bracket $[\phi, \psi] := \phi \star \psi - \psi \star \phi$.

It is well known that for a certain type of Hopf algebra (e.g. graded and connected) the exponential series induces a bijective map $\mathfrak{g}(\mathcal{H}, B) \to G(\mathcal{H}, B)$. In this setting, the ambient algebra $(\operatorname{Hom}_{\mathbb{K}}(\mathcal{H}, B), \star)$ becomes a locally convex algebra with respect to the topology of pointwise convergence and we obtain the following result.

Theorem A. — *Let $\mathcal{H}$ be a graded and connected Hopf algebra and B a commutative locally convex algebra, then the group $G(\mathcal{H}, B)$ of B-valued characters of $\mathcal{H}$ is a ($\mathbb{K}$-analytic) Lie group.*

The Lie algebra of $G(H, B)$ is the Lie algebra $\mathfrak{g}(\mathcal{H}, B)$ of infinitesimal characters.

Note that this Lie group structure recovers the Lie group structure on the character group of the Hopf algebra of rooted trees which has been constructed in [3]. For further information we refer to Example 4.7.

We then investigate the Lie theoretic properties of the character group of a graded connected Hopf algebra. To understand these results first recall the notion of regularity for Lie groups.

Let G be a Lie group modelled on a locally convex space, with identity element $\mathbf{1}$, and $r \in \mathbb{N}_0 \cup \{\infty\}$. We use the tangent map of the left translation

$\lambda_g\colon G \to G$, $x \mapsto gx$ by $g \in G$ to define $g.v := T_{\mathbf{1}}\lambda_g(v) \in T_g G$ for $v \in T_{\mathbf{1}}(G) =: \mathbf{L}(G)$. Following [15], G is called C^r*-semiregular* if for each C^r-curve $\gamma\colon [0,1] \to \mathbf{L}(G)$ the initial value problem

$$\begin{cases} \eta'(t) = \eta(t).\gamma(t) \\ \eta(0) \;= \mathbf{1} \end{cases}$$

has a (necessarily unique) C^{r+1}-solution $\mathrm{Evol}(\gamma) := \eta\colon [0,1] \to G$. If further the map

$$\mathrm{evol}\colon C^r([0,1], \mathbf{L}(G)) \to G,\ \gamma \mapsto \mathrm{Evol}(\gamma)(1)$$

is smooth, we call G C^k-regular. If G is C^r-regular and $r \leqslant s$, then G is also C^s-regular. A C^∞-regular Lie group G is called *regular (in the sense of Milnor)* — a property first defined in [28]. Every finite-dimensional Lie group is C^0-regular (cf. [30]). Several important results in infinite-dimensional Lie theory are only available for regular Lie groups (see [15, 28, 30], cf. also [22] and the references therein).

Concerning the Lie theoretic properties of the character groups our results subsume the following theorem.

Theorem B. — *Let $\mathcal{H}$ be a graded and connected Hopf algebra and B be a commutative locally convex algebra.*

(a) *Then $G(H, B)$ is a Baker–Campbell–Hausdorff–Lie group which is exponential, i.e. the Lie group exponential map is a global $\mathbb{K}$-analytic diffeomorphism.*
(b) *If B is sequentially complete then $G(\mathcal{H}, B)$ is a C^0-regular Lie group.*

To illustrate the importance of regularity, let us digress and consider the special case of the $\mathbb{C}$-valued character group of the Hopf algebra of Feynman graphs. In the Connes–Kreimer theory of renormalisation of quantum field theories, a crucial step is to integrate paths in the Lie algebra associated to this group. These integrals, called “time-ordered exponentials” (cf. [10, Definition 1.50]), encode information on the renormalisation procedure. By [10, Chapter 1, 7.2] the “main advantage” of the time-ordered differentials is that they are the solutions of certain differential equation on the character group (see [10, Proposition 1.51 (3)]). In view of the Lie group structure induced by Theorem A on the character group, these differential equations turn out to be the differential equations of regularity of the Lie group. Hence, in our language loc. cit. proves that the $\mathbb{C}$-valued characters of the Hopf algebra of Feynman graphs are regular in the sense of Milnor.

Returning to the results contained in the present paper, we turn to a class of closed subgroups of character groups which are closed Lie subgroups. For a Hopf ideal $\mathcal{J}$ of a Hopf algebra $\mathcal{H}$, consider the annihilator

$$\operatorname{Ann}(\mathcal{J}, B) := \{\phi \in \operatorname{Hom}_{\mathbb{K}}(\mathcal{H}, B) \mid \phi(a) = 0_B,\ \forall a \in \mathcal{J}\}.$$

Then $\operatorname{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ becomes a subgroup and we obtain the following result.

Theorem C. — *Let $\mathcal{H}$ be a connected and graded Hopf algebra and B be a commutative locally convex algebra.*

(a) *Then $\operatorname{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is a closed Lie subgroup of $G(\mathcal{H}, B)$ whose Lie algebra is $\operatorname{Ann}(\mathcal{J}, B) \cap \mathfrak{g}(\mathcal{H}, B)$.*

(b) *There is a canonical isomorphism of (topological) groups*

$$\operatorname{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B) \cong G(\mathcal{H}/\mathcal{J}, B),$$

where $\mathcal{H}/\mathcal{J}$ is the quotient Hopf algebra. If $\mathcal{H}/\mathcal{J}$ is a connected and graded Hopf algebra (e.g. $\mathcal{J}$ is a homogeneous ideal) then this map is an isomorphism of Lie groups.

Note that in general $\mathcal{H}/\mathcal{J}$ will not be graded and connected. In these cases the isomorphism $\operatorname{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B) \cong G(\mathcal{H}/\mathcal{J}, B)$ extends the construction of Lie group structures for character groups to Hopf algebras which are quotients of connected and graded Hopf algebras. However, this does not entail that the character groups of all Hopf algebras are infinite-dimensional Lie groups. In general, the character group will only be a topological group with respect to the topology of pointwise convergence. We refer to Example 4.11 for an explicit counterexample of a character group which can not be turned into an infinite-dimensional Lie group.

Finally, we consider a class of character groups of (non-graded) Hopf algebras whose topological group structure is accessible to Lie theoretic methods. The class of characters we consider are character groups with values in an algebra which is "weakly complete", i.e. the algebra is as a topological vector space isomorphic to $\mathbb{K}^I$ for some index set I. All finite-dimensional algebras are weakly complete, we refer to the Diagram B.1 and Appendix C for more information. Then we obtain the following result:

Theorem D. — *Let $\mathcal{H}$ be an arbitrary Hopf algebra and B be a commutative weakly complete algebra. Then the following holds*

(a) *the topological group $G(\mathcal{H}, B)$ is a projective limit of finite-dimensional Lie groups (*a pro-Lie group *in the sense of* [18]*).*

A pro-Lie group is associated to a Lie algebra which we identify for $G(\mathcal{H}, B)$:

(b) *the pro-Lie algebra* $\mathcal{L}(G(\mathcal{H}, B))$ *of the pro-Lie group* $G(\mathcal{H}, B)$ *is the Lie algebra of infinitesimal characters* $\mathfrak{g}(\mathcal{H}, B)$.

The pro-Lie group structure of the character group is exploited in the theory of renormalisation (cf. [10]). We refer to Remark 5.8 for concrete examples of computations in loc. cit. which use the pro-Lie structure.

Note that pro-Lie groups are in general only topological groups without a differentiable structure attached to them. However, these groups admit a Lie theory which has been developed in the extensive monograph [18]. The results on the pro-Lie structure are somewhat complementary to the infinite-dimensional Lie group structure. If the Hopf algebra $\mathcal{H}$ is graded and connected and B is a commutative weakly complete algebra, then the pro-Lie group structure of $G(\mathcal{H}, B)$ is compatible with the infinite-dimensional Lie group structure of $G(\mathcal{H}, B)$ obtained in Theorem A.

1. Linear maps on (connected) coalgebras

In this preliminary section we collect first some basic results and notations used throughout the paper (also cf. Appendices A–C). Most of the results are not new, however, we state them together with a proof for the reader's convenience.

Notation 1.1. — We write $\mathbb{N} := \{1, 2, 3, \ldots\}$, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. In this article (with the exception of Appendix C), $\mathbb{K}$ denotes either the field $\mathbb{R}$ of real or the field $\mathbb{C}$ of complex numbers.

Notation 1.2 (Terminology).

(a) By the term *(co-)algebra*, we always mean an (co-)associative unital $\mathbb{K}$-(co-)algebra.

(b) The *unit group* or *group of units* of an algebra $\mathcal{A}$ is the group of its invertible elements and is denoted by $\mathcal{A}^\times$.

(c) A *locally convex space* is a locally convex Hausdorff topological vector space and a *weakly complete space* is a locally convex space which is topologically isomorphic to $\mathbb{K}^I$ for an index set I (see Definition C.1).

(d) A *locally convex algebra (weakly complete) algebra* is a topological algebra whose underlying topological space is locally convex (weakly complete). Cf. also Lemma 5.3.

(e) Finally, a *continuous inverse algebra* (*CIA*) is a locally convex algebra with an open unit group and a continuous inversion.

If we want to emphasize that an algebraic structure, such as a vector space or an algebra, carries no topology, we call it an *abstract vector space* or *abstract algebra*, etc.

Throughout this section, let $\mathcal{C} = (\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ denote an abstract coalgebra and let B denote a locally convex topological algebra, e.g. a Banach algebra.

Definition 1.3 (Algebra of linear maps on a coalgebra). — *We consider the locally convex space*

$$A := \mathrm{Hom}_{\mathbb{K}}(\mathcal{C}, B)$$

of all $\mathbb{K}$*-linear maps from* $\mathcal{C}$ *into* B*. We will give this space the topology of pointwise convergence, i.e. we embed* A *into the product space* $B^{\mathcal{C}}$ *with the product topology.*

The space A *becomes a unital algebra with respect to the* convolution product *(cf.* [33, Section IV]*)*

$$\star \colon A \times A \to A,\ (h, g) \mapsto m_B \circ (h \otimes g) \circ \Delta_{\mathcal{C}}.$$

Here $m_B \colon B \otimes B \to B$ *is the algebra multiplication. The unit with respect to* $\star$ *is the map* $1_A = u_B \circ \varepsilon_{\mathcal{C}}$ *where we defined* $u_B : \mathbb{K} \to B$, $z \mapsto z1_B$.

One of the most interesting choices for the target algebra B of $A = \mathrm{Hom}_{\mathbb{K}}(\mathcal{C}, B)$ is $B = \mathbb{K}$: In this case, A is the algebraic dual of the abstract vector space $\mathcal{C}$ with the weak*-topology.

Lemma 1.4. — *Consider the algebra* $A = \mathrm{Hom}_{\mathbb{K}}(\mathcal{C}, B)$ *with the convolution* $\star$ *as above. The map* $\star \colon A \times A \to A$ *is continuous, whence* $(A, \star)$ *is a locally convex algebra,*

Proof. — Since the range space $A = \mathrm{Hom}_{\mathbb{K}}(\mathcal{C}, B)$ carries the topology of pointwise convergence, we fix an element $c \in \mathcal{C}$. We write $\Delta_{\mathcal{C}}(c) \in \mathcal{C} \otimes \mathcal{C}$ in Sweedler's sigma notation (see [20, Notation 1.6] or [33, Section 1.2]) as a finite sum

$$\Delta_{\mathcal{C}}(c) = \sum_{(c)} c_1 \otimes c_2.$$

Then the convolution product $\phi \star \psi$ evaluated at point c is of the form:

$$\begin{aligned}(\phi \star \psi)(c) = m_B \circ (\phi \otimes \psi) \circ \Delta_{\mathcal{C}}\,(c) &= m_B \circ (\phi \otimes \psi) \left(\sum_{(c)} c_1 \otimes c_2\right) \\ &= \sum_{(c)} m_B\,(\phi(c_1) \otimes \psi(c_2)) = \sum_{(c)} \phi(c_1) \cdot \psi(c_2).\end{aligned}$$

This expression is continuous in (ϕ, ψ) since point evaluations are continuous as well as multiplication in the locally convex algebra B. □

Remark 1.5. — Note that the multiplication of the locally convex algebra B is assumed to be a continuous bilinear map $B \times B \to B$. However, we did not need to put a topology on the space $B \otimes B$ nor did we say anything about the continuity of the linear map $m_B \colon B \otimes B \to B$.

LEMMA 1.6 (Properties of the space A). — *Let $A = \mathrm{Hom}_{\mathbb{K}}(\mathcal{C}, B)$ as above.*

(a) *As a locally convex space (without algebra structure), the space A is isomorphic to B^I, where the cardinality of the index set I is equal to the dimension of $\mathcal{C}$.*
(b) *If the vector space $\mathcal{C}$ is of countable dimension and B is a Fréchet space, A is a Fréchet space as well.*
(c) *The locally convex algebra A is (Mackey/sequentially) complete if and only if the algebra B is (Mackey/sequentially) complete.*

Proof.

(a) A linear map is uniquely determined by its valued on a basis $(c_i)_{i \in I}$ of $\mathcal{C}$.
(b) As a locally convex space $A \cong B^I$. Since I is countable and B is a Fréchet space, A is a countable product of Fréchet spaces, whence a Fréchet space.
(c) By part (a), B is a closed vector subspace of A. So every completeness property of A is inherited by B. On the other hand, products of Mackey complete (sequentially complete, complete) spaces are again of this type. □

The terms *abstract gradings* and *dense gradings* used in the next lemma are defined in Definition B.1 and B.2, respectively.

LEMMA 1.7. — *Let $\mathcal{C}$ be an abstract coalgebra, let B be a locally convex algebra, and set $A = \mathrm{Hom}_{\mathbb{K}}(\mathcal{C}, B)$ as above.*

(a) *If $\mathcal{C}$ admits an (abstract) grading $\mathcal{C} = \bigoplus_{n=0}^{\infty} \mathcal{C}_n$, the bijection*

$$A = \mathrm{Hom}_{\mathbb{K}}\left(\bigoplus_{n=0}^{\infty} \mathcal{C}_n, B\right) \to \prod_{n=0}^{\infty} \mathrm{Hom}_{\mathbb{K}}(\mathcal{C}_n, B),\ \phi \mapsto (\phi|_{\mathcal{C}_n})_{n \in \mathbb{N}_0} \tag{1.1}$$

turns A into a densely graded algebra with respect to the grading $(\mathrm{Hom}_{\mathbb{K}}(\mathcal{C}_n, B))_{n \in \mathbb{N}_0}$.
(b) *If in addition $\mathcal{C}$ is connected and B is a CIA, then A is a CIA as well.*

Proof.

(a) It is clear that the map (1.1) is an isomorphism of topological vector spaces. Via this dualisation, the axioms of the graded coalgebra (see Definition B.1(c)) directly translate to the axioms of densely graded locally convex algebra (see Definition B.2(b)).

(b) By Lemma B.7, we know that the densely graded locally convex algebra A is a CIA, if $A_0 = \mathrm{Hom}_{\mathbb{K}}(\mathcal{C}_0, B)$ is a CIA. Since we assume that $\mathcal{C}$ is connected, this means that $\mathcal{C} \cong \mathbb{K}$ and hence $A_0 \cong B$. The assertion follows. □

From Lemma 1.7 and Lemma A.6 one easily deduce the following proposition.

Proposition 1.8 ($A^\times$ is a Lie group). — *Let $\mathcal{C}$ be an abstract graded connected coalgebra and let B be a Mackey complete CIA. Then the unit group $A^\times$ of the densely graded algebra $A = (\mathrm{Hom}_{\mathbb{K}}(\mathcal{C}, B), \star)$ is a BCH–Lie group. The Lie algebra of the group $A^\times$ is $(A, [\cdot,\cdot])$, where $[\cdot,\cdot]$ denotes the usual commutator bracket.*

Furthermore, the Lie group exponential function of $A^\times$ is given by the exponential series, whence it restricts to the exponential function constructed in Lemma B.5.

Theorem 1.9 (Regularity of $A^\times$). — *Let $\mathcal{C}$ be an abstract graded connected coalgebra and let B be a Mackey complete CIA. As above, we set $A := (\mathrm{Hom}_{\mathbb{K}}(\mathcal{C}, B), \star)$ and assume that B is commutative or locally m-convex (i.e. the topology is generated by a system of submultiplicative seminorms).*

(a) *The Lie group $A^\times$ is C^1-regular.*

(b) *If in addition, the space B is sequentially complete, then $A^\times$ is C^0-regular.*

In both cases the associated evolution map is even $\mathbb{K}$-analytic.

Proof. — Since B is a commutative CIA or locally m-convex, the algebra B has the (GN)-property by Lemma A.8. The algebra $A := \mathrm{Hom}_{\mathbb{K}}(\mathcal{C}, B)$ is densely graded with $A_0 \cong B$ by Lemma 1.7. We claim that since $A_0 = B$ has the (GN)-property, the same holds for A (the details are checked in Lemma 1.10 below).

By Lemma 1.6, we know that A and B share the same completeness properties, i.e. the algebra A is Mackey complete if B is so and the same hold for sequential completeness. In conclusion, the assertion follows directly from Lemma A.9. □

Lemma 1.10. — *Let A be a densely graded algebra and denote the dense grading by $(A_N)_{N\in\mathbb{N}_0}$. Then A has the (GN)-property (see Definition A.7) if and only if the subalgebra A_0 has the (GN)-property.*

Proof. — Let us first see that the condition is necessary. Pick a continuous seminorm p_0 on A_0. Then $P := p_0 \circ \pi_0$ is a continuous seminorm on A (where $\pi_0 \colon A \to A_0$ is the canonical projection). Following Definition A.7 there is a continuous seminorm Q on A and a number $M > 0$ such that the following condition holds

$$\forall n \in \mathbb{N} \text{ and each } (\overline{a}_1, \ldots, \overline{a}_n) \in A^n \text{ with } Q(\overline{a}_i) \leqslant 1 \text{ for } 1 \leqslant i \leqslant n, \quad \text{we have } P(\overline{a}_1 \cdots \overline{a}_n) \leqslant M^n. \tag{1.2}$$

Now we set $q_0 := Q|_{A_0}$ and observe that q_0 is a continuous seminorm as the inclusion $A_0 \to A$ is continuous and linear. A trivial computation now shows that p_0, q_0 and M satisfy (1.2). We conclude that A_0 has the (GN)-property.

For the converse assume that A_0 has the (GN)-property and fix a continuous seminorm P on A. The topology on A is the product topology, i.e. it is generated by the canonical projections $\pi_N \colon A \to A_N$. Hence we may assume that P is of the form

$$P = \max_{0\leqslant N\leqslant L} (p_N \circ \pi_N)$$

where each $p_N \colon A_N \to [0, \infty[$ is a continuous seminorm on A_N. The number $L \in \mathbb{N}_0$ is finite and remains fixed for the rest of the proof.

The key idea is here that P depends only on a finite number of spaces in the grading. Now the multiplication increases the degree of elements except for factors of degree 0. However, these contributions can be controlled by the (GN)-property in A_0.

We will now construct a continuous seminorm Q on A and a number $M > 0$ such that (1.2) holds.

Construction of the seminorm Q. Let $w = (w_1, w_2, \ldots, w_r) \in \mathbb{N}^r$ (where $r \in \mathbb{N}$) be a multi-index and denote by $|w|$ the sum of the entries of the multi-index w. Define for $r \leqslant L$ and $w \in \mathbb{N}^r$ with $|w| \leqslant L$ a continuous r-linear map

$$f_w \colon A_{w_1} \times A_{w_2} \times \cdots \times A_{w_r} \to A_{|w|}, \, (b_1, \ldots, b_r) \mapsto b_1 \cdots b_r,$$

Since $L < \infty$ is fixed and the w_i are strictly positive for $1 \leqslant i \leqslant r$, there are only finitely many maps f_w of this type. This allows us to define for each $1 \leqslant k \leqslant L$ a seminorm q_k on A_k with the following property: For all

$r \leqslant L$ and $w \in \mathbb{N}^r$ with $|w| \leqslant L$ we obtain an estimate

$$p_{|w|}\left(f_w(b_1,\dots,b_r)\right) \leqslant q_{w_1}(b_1)\cdot q_{w_2}(b_2)\cdots q_{w_r}(b_r). \tag{1.3}$$

Consider for each $N \leqslant L$ the continuous trilinear map

$$g_N\colon A_0 \times A_N \times A_0 \to A_N,\ (c,d,e) \mapsto c\cdot d\cdot e.$$

As there are only finitely many of these maps, we can define a seminorm q_0 on A_0 and a seminorm $q_N^{\sim}$ on A_N for each $1 \leqslant N \leqslant L$ such that

$$q_N\left(g_N(c,d,e)\right) \leqslant q_0(c)q_N^{\sim}(d)q_0(e) \text{ holds for } 1 \leqslant N \leqslant L. \tag{1.4}$$

Enlarging the seminorm q_0, we may assume that $q_0 \geqslant p_0$.

Now we use the fact that the subalgebra A_0 has the (GN)-property. Hence, there is a continuous seminorm $q_0^{\sim}$ on A_0 and a number $M_0 \geqslant 1$ such that for $n \in \mathbb{N}$ and elements $c_i \in A_0,\ 1 \leqslant i \leqslant n$ with $q_0^{\sim}(c_i) \leqslant 1$ we have

$$q_0(c_1\cdots c_n) \leqslant M_0^n. \tag{1.5}$$

Finally, we define the seminorm Q via

$$Q := \max_{0\leqslant k\leqslant L}\left(q_k^{\sim}\circ \pi_k\right).$$

Clearly Q is a continuous seminorm on A. Moreover, we set

$$M := M_0^{2(L+1)}\cdot(L+1).$$

The seminorms P, Q and the constant M satisfy (1.2)**.** Let $n \in \mathbb{N}$ and $(\overline{a}_1,\dots,\overline{a}_n) \in A^n$ with $Q(\overline{a}_j) \leqslant 1$ be given. It remains to show that $P(\overline{a}_1\cdots\overline{a}_n) \leqslant M^n$. Each element $\overline{a}_j$ can be written as a converging series

$$\overline{a}_j = \sum_{k=0}^{\infty} a_j^{(k)} \text{ with } a_j^{(k)} \in A_k.$$

Plugging this representation into P, we obtain the estimate

$$\begin{aligned}P\left(\overline{a}_1\cdots\overline{a}_n\right) &= P\left(\sum_{N=0}^{\infty}\sum_{\substack{\alpha\in\mathbb{N}_0^n\\|\alpha|=N}} a_1^{(\alpha_1)}\cdots a_n^{(\alpha_n)}\right)\\ &= \max_{0\leqslant N\leqslant L} p_N\left(\sum_{\substack{\alpha\in\mathbb{N}_0^n\\|\alpha|=N}} a_1^{(\alpha_1)}\cdots a_n^{(\alpha_n)}\right) \leqslant \max_{0\leqslant N\leqslant L}\sum_{\substack{\alpha\in\mathbb{N}_0^n\\|\alpha|=N}} p_N\left(a_1^{(\alpha_1)}\cdots a_n^{(\alpha_n)}\right).\end{aligned}$$

For fixed $0 \leqslant N \leqslant L$ the number of summands in this sum is bounded from above by $(N+1)^n \leqslant (L+1)^n$ since for the entries of α there are at most $N+1$ choices. We claim that each summand can be estimated as

$$p_N\left(a_1^{(\alpha_1)} \cdots a_n^{(\alpha_n)}\right) \leqslant \left(M_0^{2(L+1)}\right)^n. \tag{1.6}$$

If this is true then one easily deduces that $P(\overline{a}_1 \cdots \overline{a}_n) \leqslant (L+1)^n \cdot (M_0^{2(L+1)})^n = M^n$ and the assertion follows.

Hence we have to prove that (1.6) holds. To this end, fix $0 \leqslant N \leqslant L$ and $\alpha \in \mathbb{N}_0^n$.

Case $N = 0$. — Then $\alpha = (0, \ldots, 0)$ and we have

$$p_0\left(a_1^{(0)} \cdots a_n^{(0)}\right) \leqslant q_0\left(a_1^{(0)} \cdots a_n^{(0)}\right) \overset{(1.5)}{\leqslant} M_0^n \leqslant \left(M_0^{2(L+1)}\right)^n.$$

Case $N \geqslant 1$. — The product $a_1^{(\alpha_1)} \cdots a_n^{(\alpha_n)}$ may contain elements from the subalgebra A_0 and elements from the subspaces A_k with $k \geqslant 1$. Combining each element contained in A_0 with elements to the left or the right, we rewrite the product as

$$a_1^{(\alpha_1)} \cdots a_n^{(\alpha_n)} = b_1 \cdots b_r$$

for some $r \leqslant \min\{n, L\}$. Deleting all zeroes from α, we obtain a multi-index $w \in \mathbb{N}^r$. Now by construction $b_k \in A_{w_k}$ is a product $b_k = c_k \cdot d_k \cdot e_k$, where each $d_k \in A_{w_k}$ is one of the a_j and c_k and e_k are finite products of A_0-factors in the original product.

Since each c_k is a product of at most n elements of A_0, all of which have $q_0^{\sim}$-norm at most 1, we may apply (1.5) to obtain the estimate:

$$q_0(c_k) \leqslant M_0^n.$$

For the same reason, we have the corresponding estimate $q_0(e_k) \leqslant M_0^n$.

Combining these results, we derive

$$\begin{aligned} p_N\left(a_1^{(\alpha_1)} \cdots a_n^{(\alpha_n)}\right) &= p_N(b_1 \cdots b_r) \overset{(1.3)}{\leqslant} \prod_{k=1}^r q_{w_k}(b_k) = \prod_{k=1}^r q_{w_k}(c_k d_k e_k) \\ &\overset{(1.4)}{\leqslant} \prod_{k=1}^r \underbrace{q_0(c_k)}_{\leqslant M_0^n} \cdot \underbrace{q_{w_k}^{\sim}(d_k)}_{\leqslant 1} \cdot \underbrace{q_0(e_k)}_{\leqslant M_0^n} \leqslant \prod_{k=1}^r (M_0^2)^n = (M_0^{2r})^n \\ &\leqslant \left(M_0^{2(L+1)}\right)^n \end{aligned}$$

□

2. Characters on graded connected Hopf algebras

In this section we construct Lie group structures on character groups of (graded and connected) Hopf algebras.

Notation 2.1. — From now on $\mathcal{H} = (\mathcal{H}, m_{\mathcal{H}}, u_{\mathcal{H}}, \Delta_{\mathcal{H}}, \varepsilon_{\mathcal{H}}, S_{\mathcal{H}})$ will denote a fixed Hopf algebra and we let B be a fixed *commutative* locally convex algebra. Using only the coalgebra structure of $\mathcal{H}$, we obtain the locally convex algebra

$$A := (\mathrm{Hom}_{\mathbb{K}}(\mathcal{H}, B), \star) \quad \text{(see Lemma 1.4).}$$

Note that our framework generalises the special case $B = \mathbb{K}$ which is also an interesting case. For example, the Hopf algebra of rooted trees (see Example 4.6) is a connected, graded Hopf algebra and its group of $\mathbb{K}$-valued characters turns out to be the Butcher group from numerical analysis (cf. Example 4.7).

We will now consider groups of characters of Hopf algebras:

Definition 2.2. — *A linear map $\phi\colon \mathcal{H} \to B$ is called (B-valued)* character *if it is a homomorphism of unital algebras, i.e.*

$$\phi(ab) = \phi(a)\phi(b) \text{ for all } a, b \in \mathcal{H} \text{ and } \phi(1_{\mathcal{H}}) = 1_B. \tag{2.1}$$

Another way of saying this is that ϕ is a character, if

$$\phi \circ m_{\mathcal{H}} = m_B \circ (\phi \otimes \phi) \text{ and } \phi(1_{\mathcal{H}}) = 1_B. \tag{2.2}$$

The set of characters is denoted by $G(\mathcal{H}, B)$.

Lemma 2.3. — *The set of characters $G(\mathcal{H}, B)$ is a closed subgroup of $(A^{\times}, \star)$. With the induced topology, $G(\mathcal{H}, B)$ is a topological group. Inversion in this group is given by the map $\phi \mapsto \phi \circ S_{\mathcal{H}}$ and the unit element is $1_A := u_B \circ \varepsilon_{\mathcal{H}}\colon \mathcal{H} \to B$, $x \mapsto \varepsilon_{\mathcal{H}}(x)1_B$.*

Proof. — The fact that the characters of a Hopf algebra form a group with respect to the convolution product is well-known, see for example [26, Proposition II.4.1 3)]. Note that in loc. cit. this is only stated for a connected graded Hopf algebra although the proof does not use the grading at all.

The closedness of $G(\mathcal{H}, B)$ follows directly from Definition 2.2 and the fact that we use the topology of pointwise convergence on A. Continuity of the convolution product was shown in Lemma 1.4. Inversion is continuous as the precomposition with the antipode is obviously continuous with respect to pointwise convergence. □

Our goal in this section is to turn the group of characters into a Lie group. Hence, we need a modelling space for this group. This leads to the following definition:

Definition 2.4. — *A linear map $\phi \in \mathrm{Hom}_{\mathbb{K}}(\mathcal{H}, B)$ is called an* infinitesimal character *if*

$$\phi \circ m_{\mathcal{H}} = m_B \circ (\phi \otimes \varepsilon_{\mathcal{H}} + \varepsilon_{\mathcal{H}} \otimes \phi), \tag{2.3}$$

which means for $a, b \in \mathcal{H}$ that $\phi(ab) = \phi(a)\varepsilon_{\mathcal{H}}(b) + \varepsilon_{\mathcal{H}}(a)\phi(b)$.

We denote by $\mathfrak{g}(\mathcal{H}, B)$ the set of all infinitesimal characters.

Lemma 2.5. — *The infinitesimal characters $\mathfrak{g}(\mathcal{H}, B)$ form a closed Lie subalgebra of $(A, [\cdot,\cdot])$, where $[\cdot,\cdot]$ is the commutator bracket of $(A, \star)$.*

Proof. — As for $G(\mathcal{H}, B)$, the closedness follows directly from the definition. The fact that the infinitesimal characters form a Lie subalgebra is well-known, see for example [26, Proposition II.4.2]. □

Remark 2.6. — From now on we assume for the rest of this section that the Hopf algebra $\mathcal{H}$ is *graded and connected* (see Definition B.1). Thus by Lemma 1.7 the locally convex algebra $A = \mathrm{Hom}_{\mathbb{K}}(\mathcal{H}, B)$ is densely graded.

Every infinitesimal character $\phi \in \mathfrak{g}(\mathcal{H}, B)$ maps $1_{\mathcal{H}}$ to $0_{\mathbb{K}} \cdot 1_B$ since $\phi(1_{\mathcal{H}} \cdot 1_{\mathcal{H}}) = \phi(1_{\mathcal{H}})\varepsilon_{\mathcal{H}}(1_{\mathcal{H}}) + \varepsilon_{\mathcal{H}}(1_{\mathcal{H}})\phi(1_{\mathcal{H}}) = 2\phi(1_{\mathcal{H}})$. Now $\mathcal{H} = \bigoplus_{n=0} \mathcal{H}_n$ is assumed to be connected and we have $\mathcal{H}_0 = \mathbb{K}1_{\mathcal{H}}$, whence $\phi|_{\mathcal{H}_0} = 0$. Translating this to the densely graded algebra A, we observe $\mathfrak{g}(\mathcal{H}, B) \subseteq \mathcal{I}_A$ (the ideal of all elements which vanish on $\mathcal{H}_0$, cf. Definition B.2). Similarly, a character maps $1_{\mathcal{H}}$ to 1_B by definition. Hence, $G(\mathcal{H}, B) \subseteq 1_A + \mathcal{I}_A$.

Theorem 2.7. — *Let $\mathcal{H}$ be an abstract graded connected Hopf algebra $\mathcal{H}$. For any commutative locally convex algebra B, the group $G(\mathcal{H}, B)$ of B-valued characters of $\mathcal{H}$ is a ($\mathbb{K}-$analytic) Lie group.*

Furthermore, we observe the following properties

(i) *The Lie algebra $\mathbf{L}(G(H, B))$ of $G(H, B)$ is the Lie algebra $\mathfrak{g}(\mathcal{H}, B)$ of infinitesimal characters with the commutator bracket $[\phi, \psi] = \phi \star \psi - \psi \star \phi$.*

(ii) *$G(\mathcal{H}, B)$ is a BCH–Lie group which is exponential, i.e. the exponential map is a global $\mathbb{K}$-analytic diffeomorphism and is given by the exponential series.*

(iii) *The model space of $G(\mathcal{H}, B)$ is a Fréchet space whenever $\mathcal{H}$ is of countable dimension (e.g. a Hopf algebra of finite type) and B is a Fréchet space (e.g. B is finite-dimensional or a Banach algebra).*

In the special case that B is a weakly complete algebra, the modelling space $\mathfrak{g}(\mathcal{H}, B)$ is weakly complete as well.

Proof. — The locally convex algebra $A = \mathrm{Hom}_{\mathbb{K}}(\mathcal{H}, B)$ is densely graded. Hence, by Lemma B.5, the exponential series converges on the closed vector subspace $\mathcal{I}_A$ and defines a $C^\omega_{\mathbb{K}}$-diffeomorphism:

$$\exp_A \colon \mathcal{I}_A \to 1_A + \mathcal{I}_A,\ \phi \mapsto \exp[\phi].$$

This implies that the closed vector subspace $\mathfrak{g}(\mathcal{H}, B)$ is mapped to a closed analytic submanifold of $1_A + \mathcal{I}_A \subseteq A$. By Lemma B.10, we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{I}_A \arrow[r, "\exp_A"] & 1_A + \mathcal{I}_A \\ \mathfrak{g}(\mathcal{H},B) \arrow[u, "\subseteq"] \arrow[r, "\mathrm{Exp}:=\exp_A|^{G(\mathcal{H},B)}_{\mathfrak{g}(\mathcal{H},B)}"'] & G(\mathcal{H},B) \arrow[u, "\subseteq"'] \end{tikzcd} \tag{2.4}$$

which shows that the group $G(\mathcal{H}, B)$ is a closed analytic submanifold of $1_A + \mathcal{I}_A \subseteq A$.

The group multiplication is a restriction of the continuous bilinear map $\star\colon A \times A \to A$ to the analytic submanifold $G(\mathcal{H}, B) \times G(\mathcal{H}, B)$ and since the range space $G(\mathcal{H}, B) \subseteq A$ is closed this ensures that the restriction is analytic as well. Inversion in $G(\mathcal{H}, B)$ is composition with the antipode map $G(\mathcal{H}, B) \to G(\mathcal{H}, B)$, $\phi \mapsto \phi \circ S$ (see Lemma 2.3). This is a restriction of the continuous linear (and hence analytic) map $A \to A$, $\phi \mapsto \phi \circ S$ to closed submanifolds in the domain and range and hence inversion is analytic as well. This shows that $G(\mathcal{H}, B)$ is an analytic Lie group.

(i) The map $\mathrm{Exp}\colon \mathfrak{g}(\mathcal{H}, B) \to G(\mathcal{H}, B)$, $\phi \mapsto \exp_A(\phi)$ is an analytic diffeomorphism. Hence, we take the tangent map at point 0 in the diagram (2.4) to obtain:

$$\begin{tikzcd} \mathcal{I}_A \arrow[r, "T_0 \exp_A"] & \mathcal{I}_A \\ \mathfrak{g}(\mathcal{H},B) \arrow[u, "\subseteq"] \arrow[r, "T_0\mathrm{Exp}"'] & T_{1_A}(G(\mathcal{H},B)) \arrow[u, "\subseteq"'] \end{tikzcd}$$

By Lemma B.6(b) we know that $T_0 \exp_A = \mathrm{id}_A$ which implies $T_0\mathrm{Exp} = \mathrm{id}_{\mathfrak{g}(\mathcal{H},B)}$. Hence as locally convex spaces $\mathbf{L}(G(\mathcal{H}, B)) = T_{1_A}(G(\mathcal{H}, B)) = \mathfrak{g}(\mathcal{H}, B)$. It remains to show that the Lie bracket on $T_{1_A}(G(\mathcal{H}, B))$ induced by the Lie group structure is the commutator bracket on $\mathfrak{g}(\mathcal{H}, B)$. This is what we will show in the following:

For a fixed $\phi \in G(\mathcal{H}, B)$, the inner automorphism

$$c_\phi \colon G(\mathcal{H}, B) \to G(\mathcal{H}, B),\ \psi \mapsto \phi \star \psi \star (\phi \circ S)$$

is a restriction of continuous linear map on the ambient space A. Taking the derivative of c_ϕ we obtain the usual formula for the adjoint action

$$\mathbf{L}(c_\phi) = \mathrm{Ad}(\phi) \colon \mathfrak{g}(\mathcal{H}, B) \to \mathfrak{g}(\mathcal{H}, B),\ \psi \mapsto \phi \star \psi \star (\phi \circ S).$$

For fixed $\psi \in \mathfrak{g}(\mathcal{H}, B)$, this formula can be considered as a restriction of a continuous polynomial in ϕ. We then define ad $:= T_{1_A}\mathrm{Ad}(\cdot).\psi$ (cf. [30, Example II.3.9]) which yields the Lie bracket of ϕ and ψ in $\mathfrak{g}(\mathcal{H}, B)$:

$$[\phi, \psi] = \mathrm{ad}(\phi).\psi = \phi \star \psi + \psi \star (\phi \circ S). \tag{2.5}$$

Recall that the antipode is an anti-coalgebra morphism by [25, Proposition 1.3.1]. Thus the map $A \to A$, $f \mapsto f \circ S$ is an anti-algebra morphism which is continuous with respect to the topology of pointwise convergence. We conclude for an infinitesimal character ϕ that $\exp_A(\phi \circ S) = \exp_A(\phi) \circ S = (\exp_A(\phi))^{-1}$, i.e. $\exp_A(\phi \circ S)$ is the inverse of $\exp_A(\phi)$ with respect to $\star$. As $\exp_A$ restricts to a bijection from $\mathfrak{g}(\mathcal{H}, B)$ to $G(\mathcal{H}, B)$ we derive from Lemma B.6(a) that $\phi \circ S = -\phi$. This implies together with (2.5) that $[\phi, \psi] = \phi \star \psi - \psi \star \phi$.

(ii) We already know that the exponential series defines an analytic diffeomorphism $\mathrm{Exp} \colon \mathfrak{g}(\mathcal{H}, B) \to G(\mathcal{H}, B)$ between Lie algebra and Lie group. It only remains to show that Exp is the exponential function of the Lie group. To this end let $\phi \in \mathfrak{g}(\mathcal{H}, B)$ be given. The analytic curve

$$\gamma_\phi \colon (\mathbb{R}, +) \to G(\mathcal{H}, B),\ t \mapsto \mathrm{Exp}(t\phi)$$

is a group homomorphism by Lemma B.6(a) and we have $\gamma_\phi'(0) = \phi$ by Lemma B.6(b). By definition of a Lie group exponential function (see [30, Definition II.5.1]) we have $\exp(\phi) = \gamma_\phi(1) = \mathrm{Exp}(\phi)$.

(iii) Assume that the underlying vector space of the algebra $\mathcal{H}$ is of countable dimension. The Lie algebra $\mathfrak{g}(\mathcal{H}, B)$ is a closed vector subspace of A, the latter is Fréchet by Lemma 1.6(b). Hence, $\mathfrak{g}(\mathcal{H}, B)$ is Fréchet as well.

If B is weakly complete then $\mathfrak{g}(\mathcal{H}, \mathbb{K})$ is a closed vector subspace of a weakly complete space by Lemma 1.6(a). Since closed subspaces of weakly complete spaces are again weakly complete (e.g. [18, Theorem A2.11]), the assertion follows. □

Remark 2.8.

(a) Character groups of (graded and connected)
Hopf algebras with values in locally convex algebras arise naturally in application in physics. Namely, in renormalisation of quantum field theories one considers characters of the Hopf algebra of Feynman graphs with values in algebras of polynomials or the algebra of germs of meromorphic functions (see e.g. [26, 10]).

(b) In Theorem 2.7 we did not need to assume that the Hopf algebra is of finite type, i.e. that the steps of the grading are all finite-dimensional. Moreover, the Lie group structure does not depend on the grading on the Hopf algebra $\mathcal{H}$, i.e. if $\mathcal{H}$ admits two connected gradings, then the Lie group structures induced by Theorem 2.7 coincide. Note that the Hopf algebra of Feynman graphs admits two connected gradings which are natural and of interest in physics (cf. [10, Proposition 1.30] and [31]).

Remark 2.9. — Consider the special case of a $\mathbb{K}$-Hopf algebra $\mathcal{H}$ which is graded, connected and commutative. Recall that an *affine group scheme* is a covariant representable functor from the category $\mathbf{CommAlg}_{\mathbb{K}}$ of commutative algebras over $\mathbb{K}$ to the category of groups. The functor

$$G(\mathcal{H}, \cdot)\colon \mathbf{CommAlg}_{\mathbb{K}} \longrightarrow \mathbf{Groups},$$

which sends $\mathbb{K}$-algebras to their associated character groups and an algebra morphism $f\colon A \to B$ to $G(\mathcal{H}, f)\colon G(\mathcal{H}, A) \to G(\mathcal{H}, B)$, $\phi \mapsto f \circ \phi$, is an affine group scheme (cf. e.g. [34, 1.4]). In this case, [29, Proposition 4.13] implies that the group $G(\mathcal{H}, B)$ is a projective limit (in **Groups**) of linear affine groups (see also [34]). We will encounter a similar phenomenon (albeit in the case of an arbitrary Hopf algebra) in Section 5.

Assume further that the Hopf algebra $\mathcal{H}$ is of finite type. Then the Milnor–Moore Theorem [29, Theorem 5.18] asserts that $\mathcal{H}$ can be recovered as the graded dual of the universal enveloping algebra of a dense Lie subalgebra of $\mathfrak{g}(\mathcal{H}, \mathbb{K})$. Hence the affine group scheme is completely determined by the Lie algebra of infinitesimal characters (cf. [10, p. 76–77] for a full account).(2) However, we remark that our results do not rely on the Milnor–Moore Theorem or techniques for affine group schemes.

(2) In general, the Lie algebra encountered in the Milnor–Moore Theorem is not the full Lie algebra of infinitesimal characters (a fact which is somewhat opaque in [10]). This is due to the fact that one needs to take the restricted dual, i.e. the direct sum of the duals of the finite-dimensional steps of the grading. The Lie algebra obtained from the restricted dual is a dense subalgebra of the infinitesimal characters which are contained in the full dual, cf. Lemma 1.7.

We now turn to the question whether the Lie group constructed in Theorem 2.7 is a *regular* Lie group. To derive the regularity condition, we need to restrict our choice of target algebras to *Mackey complete* algebras. Let us note first the following result.

Lemma 2.10. — *Let $\mathcal{H}$ be a graded and connected Hopf algebra and B be a commutative locally convex algebra. Then $G(\mathcal{H}, B)$ and $\mathfrak{g}(\mathcal{H}, B)$ are contained in a closed subalgebra of $A = (\mathrm{Hom}_{\mathbb{K}}(\mathcal{H}, B), \star)$ which is a densely graded continuous inverse algebra.*

Proof. — Consider the closed subalgebra $\Lambda := (\mathbb{K}1_{A_0}) \times \prod_{n\in\mathbb{N}} A_n$. By Lemma B.8 this algebra is densely graded (with respect to the grading induced by A) and a continuous inverse algebra. Note that $\mathcal{I}_A = \ker \pi_0$ and $\pi_0(1_A + \mathcal{I}_A) = 1_A$ imply that $\mathcal{I}_A$ and $1_A + \mathcal{I}_A$ are contained in Λ. Thus Remark 2.6 shows that Λ contains $G(\mathcal{H}, B)$ and $\mathfrak{g}(\mathcal{H}, B)$. ☐

Theorem 2.11. — *Let $\mathcal{H}$ be a graded and connected Hopf algebra and B be a Mackey complete locally convex algebra.*

(a) *The Lie group $G(\mathcal{H}, B)$ is C^1-regular.*
(b) *If in addition, B is sequentially complete, then $G(\mathcal{H}, B)$ is even C^0-regular.*

In both cases, the associated evolution map is even a $\mathbb{K}$-analytic map.

Proof. — We have seen in Lemma 2.10 that $G(\mathcal{H}, B)$ and $\mathfrak{g}(\mathcal{H}, B)$ are contained in a closed subalgebra Λ of $A = (\mathrm{Hom}_{\mathbb{K}}(\mathcal{H}, B), \star)$. By construction Λ is a densely graded CIA. Note that Λ inherits all completeness properties from B since it is closed in A and this space inherits its completeness properties from B by Lemma 1.6. Combine Lemma 2.3 and Lemma B.7(a) to see that $G(\mathcal{H}, B)$ is a closed subgroup of the group of units of the CIA Λ. Theorem 2.7 and Lemma A.6 show that $\Lambda^\times$ and $G(\mathcal{H}, B)$ are BCH–Lie groups and thus [30, Theorem IV.3.3] entails that $G(\mathcal{H}, B)$ is a closed Lie subgroup of $\Lambda^\times$.

Since $\Lambda_0 \cong \mathbb{K}$ (cf. Lemma B.8) is a commutative CIA, we deduce from Lemma A.8 that Λ_0 has the (GN)-property. Hence Lemma 1.10 shows that also Λ has the (GN)-property. We deduce from Lemma A.9 that the unit group $\Lambda^\times$ is a C^1-regular Lie group which is even C^0-regular if B is sequentially complete. For the rest of this proof, fix $k \in \{0, 1\}$ and let B be sequentially complete if $k = 0$.

Our first goal is to show that $G(\mathcal{H}, B)$ is C^k-semiregular, i.e. that every C^k-curve into the Lie algebra $\mathfrak{g}(\mathcal{H}, B)$ admits a C^{k+1}-evolution in the character group (cf. [15]). To this end, let us recall the explicit regularity condition for the unit group.

Step 1: The initial value problem for C^k-regularity in the group $\Lambda^\times$. The Lie group $\Lambda^\times$ is open in the CIA Λ. Take the canonical identification $T\Lambda^\times \cong \Lambda^\times \times \Lambda$. Recall that the group operation of $\Lambda^\times$ is the restriction of the bilinear convolution $\star\colon \Lambda \times \Lambda \to \Lambda$. Consider for $\theta \in \Lambda^\times$ the left translation $\lambda_\theta(h) := \theta \star h, h \in \Lambda^\times$. Then the identification of the tangent spaces yields $T_{1_\Lambda}\lambda_\theta(X) = \theta \star X$ for all $X \in T_{1_\Lambda}\Lambda^\times = \Lambda$. Summing up, the initial value problem associated to C^k-regularity of $\Lambda^\times$ becomes

$$\begin{cases} \eta'(t) = \eta(t).\gamma(t) = T_{1_\Lambda}\lambda_{\eta(t)}(\gamma(t)) = \eta(t) \star \gamma(t) & t \in [0,1], \\ \eta(0) = 1_\Lambda \end{cases} \tag{2.6}$$

where $\gamma \in C^k([0,1],\Lambda)$.

Fix $\gamma \in C^k([0,1],\mathfrak{g}(\mathcal{H},B))$. Now $\mathfrak{g}(\mathcal{H},B) \subseteq \Lambda$ holds and $\Lambda^\times$ is C^k-regular. Thus γ admits a C^{k+1}-evolution η in $\Lambda^\times$, i.e. $\eta\colon [0,1] \to \Lambda^\times$ is of class C^{k+1} and solves (2.6) with respect to γ. We will now show that η takes its values in $G(\mathcal{H},B)$.

Step 2: An auxiliary map to see that $G(\mathcal{H},B)$ is C^k-semiregular. Consider

$$F\colon [0,1] \times \mathcal{H} \times \mathcal{H} \to B,\ (t,x,y) \mapsto \eta(t)(xy) - \eta(t)(x)\eta(t)(y).$$

If F vanishes identically, the evolution η is multiplicative for each fixed t. Note that as η is a C^{k+1}-curve and Λ carries the topology of pointwise convergence, for each $(x,y) \in \mathcal{H} \times \mathcal{H}$ the map $F_{x,y} := F(\cdot,x,y)\colon [0,1] \to B$ is a C^{k+1}-map. Furthermore, $F(0,\cdot,\cdot) \equiv 0$ as $\eta(0) = 1_\Lambda \in G(\mathcal{H},B)$. Thus for each pair $(x,y) \in \mathcal{H} \times \mathcal{H}$ the fundamental theorem of calculus yields

$$F(t,x,y) = F_{x,y}(t) = \int_0^t \frac{\partial}{\partial t} F_{x,y}(t)\mathrm{d}t. \tag{2.7}$$

In the following formulae, abbreviate $\eta_t := \eta(t)$ and $\gamma_t := \gamma(t)$ to shorten the notation. To evaluate the expression (2.7) we compute the derivative of $F_{x,y}$ as

$$\begin{aligned} \frac{\partial}{\partial t} F_{x,y}(t) &= \frac{\partial}{\partial t}\eta_t(xy) - \left(\frac{\partial}{\partial t}\eta_t(x)\right)\eta_t(y) - \left(\frac{\partial}{\partial t}\eta_t(y)\right)\eta_t(x) \\ &\overset{(2.6)}{=} [\eta_t \star \gamma_t](xy) - [\eta_t \star \gamma_t](x)\eta_t(y) - [\eta_t \star \gamma_t](y)\eta_t(x). \end{aligned} \tag{2.8}$$

We use Sweedler's sigma notation to write $\Delta_\mathcal{H}(x) = \sum_{(x)} x_1 \otimes x_2$ and $\Delta_\mathcal{H}(y) = \sum_{(y)} y_1 \otimes y_2$. As $\Delta_\mathcal{H}$ is an algebra homomorphism, the convolution in (2.8) can then be rewritten as

$$\begin{aligned}\frac{\partial}{\partial t}F_{x,y}(t) = \sum_{(x)(y)} \eta_t(x_1y_1)\gamma_t(x_2y_2) - \sum_{(x)} \eta_t(x_1)\gamma_t(x_2)\eta_t(y) \\ - \sum_{(y)} \eta_t(y_1)\gamma_t(y_2)\eta_t(x).\end{aligned} \tag{2.9}$$

Recall that the curve γ takes its values in the infinitesimal characters, whence we have the identity $\gamma_t(ab) = \varepsilon(a)\gamma_t(b) + \varepsilon(b)\gamma_t(a)$. Plugging this into the first summand in (2.9) and using that η_t is linear for all t we obtain the identity

$$\begin{aligned}&\sum_{(x)}\sum_{(y)}\eta_t(x_1y_1)\gamma_t(x_2y_2) \\ &\quad= \sum_{(x)}\sum_{(y)}(\eta_t(\varepsilon(x_2)x_1y_1)\gamma_t(y_2) + \eta_t(x_1(\varepsilon(y_2)y_1))\gamma_t(x_2)) \\ &\quad= \sum_{(y)} \eta_t(xy_1)\gamma_t(y_2) + \sum_{(x)} \eta_t(x_1y)\gamma_t(x_2).\end{aligned} \tag{2.10}$$

As B is commutative inserting (2.10) into (2.9) yields

$$\begin{aligned}\frac{\partial}{\partial t}F_{x,y}(t) = \sum_{(y)}(\eta_t(xy_1) - \eta_t(x)\eta_t(y_1))\gamma_t(y_2) \\ + \sum_{(x)}(\eta_t(x_1y) - \eta_t(x_1)\eta_t(y))\gamma_t(x_2).\end{aligned} \tag{2.11}$$

Since η_t is linear for each fixed t it suffices to check that η_t is multiplicative for all pairs of elements in a set spanning the vector space $\mathcal{H}$. As $\mathcal{H}$ is graded, the homogeneous elements span the vector space $\mathcal{H}$. We will now use the auxiliary mapping F and its partial derivative to prove that the evolution η_t is multiplicative on all homogeneous elements in $\mathcal{H}$ (whence on all elements in $\mathcal{H}$).

Step 3: The evolution $\eta(t)$ is multiplicative on $\mathcal{H}_0$ and maps $1_\mathcal{H}$ to 1_B. The Hopf algebra $\mathcal{H}$ is graded and connected, i.e. $\mathcal{H} = \bigoplus_{n\in\mathbb{N}_0}\mathcal{H}_n$ and $\mathcal{H}_0 = \mathbb{K}1_\mathcal{H}$. By construction this entails $\Delta_\mathcal{H}(\mathcal{H}_0) \subseteq \mathcal{H}_0 \otimes \mathcal{H}_0$ and the infinitesimal character γ vanishes on $\mathcal{H}_0$. Thus for $x, y \in \mathcal{H}_0$ we have $\frac{\partial}{\partial t}F_{x,y}(t) = 0$ for all $t \in [0,1]$ by (2.11). We conclude from (2.7) for $x, y \in \mathcal{H}_0$ the formula

$$\eta(t)(xy) - \eta(t)(x)\eta(t)(y) = F(t,x,y) = 0 \quad \forall t \in [0,1].$$

Hence $\eta(t)(xy) = \eta(t)(x)\eta(t)(y)$ for all elements in degree 0. Specialising to $x = 1_\mathcal{H} = y$ we see that $\eta(t)(1_\mathcal{H})$ is an idempotent in the CIA B.

Furthermore, since $\eta(t) \in \Lambda^\times$, we have $1_B = 1_\Lambda(1_\mathcal{H}) = \eta(t) \star (\eta(t))^{-1}(1_\mathcal{H})$, whence $\eta(t)(1_\mathcal{H}) \in B^\times$ for all $t \in [0,1]$. As $B^\times$ is a group it contains only one idempotent, i.e. $\eta(t)(1_\mathcal{H}) = 1_B$.

Step 4: The evolution $\eta(t)$ is multiplicative for all homogeneous elements. As $\mathcal{H}$ is connected, η_t is linear and $\eta_t(1_\mathcal{H}) = 1_B$ holds, we see that (2.11) vanishes if either x or y are contained in $\mathcal{H}_0$. We conclude from (2.7) that

$$\eta_t(xy) = \eta_t(x)\eta_t(y) \ \forall t \in [0,1] \text{ if } x \text{ or } y \text{ are contained in degree } 0$$

Denote for a homogeneous element $x \in \mathcal{H}$ by $\deg x$ its degree with respect to the grading. To prove that η_t is multiplicative for elements of higher degree and $t \in [0,1]$ we proceed by induction on the sum of the degrees of x and y. Having established multiplicativity of η_t if at least one element is in $\mathcal{H}_0$, we have already dealt with the cases $\deg x + \deg y \in \{0,1\}$.

Induction step for $\deg x + \deg y \geqslant 2$. — We assume that for homogeneous elements a, b with $\deg a + \deg b \leqslant \deg x + \deg y - 1$ the formula $\eta_t(ab) = \eta_t(a)\eta_t(b)$ holds.

Since $\mathcal{H}$ is connected, for each $z \in \mathcal{H}_n$ with $n \geqslant 1$ the coproduct can be written as

$$\Delta_\mathcal{H}(z) = z \otimes 1_\mathcal{H} + 1_\mathcal{H} \otimes z + \tilde{\Delta}(z)$$

where $\tilde{\Delta}(z) = \sum_{\widetilde{(z)}} \tilde{z}_1 \otimes \tilde{z}_2 \in \varepsilon_\mathcal{H}^{-1}(0) \otimes \varepsilon_\mathcal{H}^{-1}(0)$ is the reduced coproduct. Note that by construction the elements $\tilde{z}_1, \tilde{z}_2$ are homogeneous of degree strictly larger than 0. Let us plug this formula for the coproduct into (2.11). We compute for the first sum in (2.11):

$$\begin{aligned}
&\sum_{(y)} (\eta_t(xy_1) - \eta_t(x)\eta_t(y_1))\gamma_t(y_2) \\
&= (\eta_t(xy) - \eta_t(x)\eta_t(y)) \underbrace{\gamma_t(1_\mathcal{H})}_{=0} + \underbrace{(\eta_t(x1_\mathcal{H}) - \eta_t(x)\eta_t(1_\mathcal{H}))}_{=0} \gamma_t(y) \\
&\qquad + \sum_{\widetilde{(y)}} (\eta_t(x\tilde{y}_1) - \eta_t(x)\eta_t(\tilde{y}_1))\gamma_t(\tilde{y}_2) \\
&= \sum_{\widetilde{(y)}} \underbrace{(\eta_t(x\tilde{y}_1) - \eta_t(x)\eta_t(\tilde{y}_1))}_{C_{x,\tilde{y}_1} :=} \gamma_t(\tilde{y}_2)
\end{aligned} \tag{2.12}$$

By construction we have $\deg \tilde{y}_1, \deg \tilde{y}_2 \in [1, \deg y - 1]$. Then $\deg \tilde{y}_1 < \deg y$ implies $\deg x + \deg \tilde{y}_1 < \deg x + \deg y$ and thus $C_{x,\tilde{y}_1}$ vanishes by the induction assumption.

As the two sums in (2.11) are symmetric, interchanging the roles of x and y together with an analogous argument as above shows that also the second sum vanishes. Hence, arguing as in Step 3, we see that $\eta_t(xy) = \eta_t(x)\eta_t(y)$ holds for all homogeneous elements $x, y \in \mathcal{H}$.

In conclusion, the evolution $\eta\colon [0,1] \to \Lambda^\times$ of γ takes its values in the closed subgroup $G(\mathcal{H}, B)$ and thus $G(\mathcal{H}, B)$ is C^k-semiregular.

Step 5: $G(\mathcal{H}, B)$ is C^k-regular. Let $\iota\colon \mathfrak{g}(\mathcal{H}, B) \to \Lambda$ be the canonical inclusion mapping. Consider $\iota_*\colon C^k([0,1], \mathfrak{g}(\mathcal{H}, B)) \to C^k([0,1], \Lambda)$, $c \mapsto \iota \circ c$. As ι is continuous and linear the map ι_* is continuous and linear by [16, Lemma 1.2], whence smooth and even $\mathbb{K}$-analytic.

Let $\mathrm{evol}_\Lambda\colon C^k([0,1], \Lambda) \to \Lambda^\times$ be the (smooth) evolution map of the C^k-regular Lie group $\Lambda^\times$. Then the map

$$\mathrm{evol}\colon C^k([0,1], \mathfrak{g}(\mathcal{H}, B)) \to \Lambda^\times, \mathrm{evol}_{\Lambda^\times} \circ \iota_*$$

is $\mathbb{K}$-analytic by Theorem 1.9 and maps a C^k-curve in the Lie algebra of $G(\mathcal{H}, B)$ to its time 1 evolution. As the closed subgroup $G(\mathcal{H}, B)$ is C^k-semiregular by Step 4, evol factors through a $\mathbb{K}$-analytic map

$$\mathrm{evol}_{G(\mathcal{H},B)}\colon C^k([0,1], \mathfrak{g}(\mathcal{H}, B)) \to G(\mathcal{H}, B).$$

Summing up, $G(\mathcal{H}, B)$ is C^k-regular and the evolution map is $\mathbb{K}$-analytic. □

3. Subgroups associated to Hopf ideals

So far, we were only able to turn the character group of a graded connected Hopf algebra $\mathcal{H}$ into a Lie group. In this section, we will show that the character group of a quotient $\mathcal{H}/\mathcal{J}$ can be regarded as a closed Lie subgroup of the character group of $\mathcal{H}$ and thus carries a Lie group structure as well. See Remark 4.10 for examples of Hopf ideals and quotients arising in renormalisation of quantum field theories. It should be noted that this does not imply that the character group of *every* Hopf algebra can be endowed with a Lie group structure (see Example 4.11).

Definition 3.1 (Hopf ideal). — *Let $\mathcal{H}$ be a Hopf algebra. We say $\mathcal{J} \subseteq \mathcal{H}$ is a* Hopf ideal *if the subset $\mathcal{J}$ is*

(a) *a two-sided (algebra) ideal,*
(b) *a coideal, i.e. $\varepsilon(\mathcal{J}) = 0$ and $\Delta(\mathcal{J}) \subseteq \mathcal{J} \otimes \mathcal{H} + \mathcal{H} \otimes \mathcal{J}$ and*
(c) *stable under the antipode, i.e. $S(\mathcal{J}) \subseteq \mathcal{J}$.*

Let $\mathcal{H}$ be a graded Hopf algebra. Then we call $\mathcal{J}$ homogeneous *if for all $c \in \mathcal{J}$ with $c = \sum_{i=1}^n c_i$ and each c_i homogeneous we have $c_i \in \mathcal{J}$ for $1 \leqslant i \leqslant n$.*

Definition 3.2 (Quotient Hopf algebra and the annihilator of an ideal). *Let $\mathcal{H}$ be a Hopf algebra and let $\mathcal{J} \subseteq \mathcal{H}$ be a Hopf ideal.*

(a) *The quotient vector space $\mathcal{H}/\mathcal{J}$ carries a natural Hopf algebra structure (see* [33, Theorem 4.3.1.]*). This structure turns the canonical quotient map $q\colon \mathcal{H} \to \mathcal{H}/\mathcal{J}$ into a morphism of Hopf algebras.*

(b) *Let B be a locally convex algebra. Then the algebra $\mathrm{Hom}_{\mathbb{K}}(\mathcal{H}/\mathcal{J}, B)$ is canonically isomorphic to the* annihilator of *$\mathcal{J}$:*

$$\mathrm{Ann}(\mathcal{J}, B) = \{\phi \in \mathrm{Hom}_{\mathbb{K}}(\mathcal{H}, B) \mid \phi(\mathcal{J}) = 0_B\}$$

which is a closed unital subalgebra of $\mathrm{Hom}_{\mathbb{K}}(\mathcal{H}, B)$.

If $\mathcal{H}$ is graded and the ideal $\mathcal{J}$ is homogeneous then the grading of $\mathcal{H}$ induces a natural grading on the quotient $\mathcal{H}/\mathcal{J}$. However, like the example of the universal enveloping algebra as a quotient of the tensor algebra (see Examples 4.1 and 4.2) shows, there are interesting ideals which occur naturally but are not homogeneous.

Lemma 3.3. — *Let $\mathcal{J}$ be a Hopf ideal of the Hopf algebra $\mathcal{H}$ with quotient mapping $q\colon \mathcal{H} \to \mathcal{H}/\mathcal{J}$. Let B be a commutative locally convex algebra. Then $\mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is a closed subgroup of the topological group $G(\mathcal{H}, B)$. Furthermore it is isomorphic as a topological group to $G(\mathcal{H}/\mathcal{J}, B)$ via the following isomorphism:*

$$q_*\colon G(\mathcal{H}/\mathcal{J}, B) \to \mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B),\ \phi \mapsto \phi \circ q.$$

Proof. — We first prove that $\mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is a closed subgroup. It is stable under the group product and contains the unit because $\mathrm{Ann}(\mathcal{J}, B)$ is a unital subalgebra. To see that it is stable under inversion recall from Lemma 2.3 that inversion in $G(\mathcal{H}, B)$ is given by precomposition with the antipode. Hence for $\phi \in \mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ we find $\phi^{-1}(\mathcal{J}) = \phi \circ S(\mathcal{J}) \subseteq \phi(\mathcal{J}) = 0$. Finally, $\mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is closed as a subset of $G(\mathcal{H}, B)$ because $\mathrm{Ann}(\mathcal{J}, B)$ is closed in A, and $G(\mathcal{H}, B)$ carries the subset topology.

The map $q_*\colon G(\mathcal{H}/\mathcal{J}, B) \to \mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is clearly an isomorphism of groups. Continuity of q_* and $(q_*)^{-1}$ follows from the fact that we use pointwise convergence on all spaces. □

Theorem 3.4. — *Let $\mathcal{H}$ be a graded connected Hopf algebra and $\mathcal{J} \subseteq \mathcal{H}$ be a (not necessarily homogeneous) Hopf ideal. Furthermore, we fix a commutative locally convex algebra B. Then*

(*i*) *$G(\mathcal{H}/\mathcal{J}, B) \cong \mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B) \subseteq G(\mathcal{H}, B)$ is a closed Lie subgroup, and even an exponential BCH–Lie group.*

(*ii*) $\mathrm{Ann}(\mathcal{J}, B) \cap \mathfrak{g}(\mathcal{H}, B) \subseteq \mathfrak{g}(\mathcal{H}, B)$ *is a closed Lie subalgebra, and a BCH–Lie algebra.*

(*iii*) *The map* exp *restricts to a global* $\mathbb{K}$*-analytic diffeomorphism* $\mathrm{Ann}(\mathcal{J}, B) \cap \mathfrak{g}(\mathcal{H}, B) \to \mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$.

(*iv*) *If* $\mathcal{J}$ *is homogeneous then the Lie group structure on* $\mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ *agrees with the one already obtained on* $G(\mathcal{H}/\mathcal{J}, B)$.

Proof.

(ii) $\mathrm{Ann}(\mathcal{J}, B) \cap \mathfrak{g}(\mathcal{H}, B)$ is a Lie subalgebra because $[\phi, \psi] = \phi \star \psi - \psi \star \phi$ and $\mathrm{Ann}(\mathcal{J}, B)$ is stable under convolution. It is closed because $\mathrm{Ann}(\mathcal{J}, B)$ is closed in A. As a closed Lie subalgebra of a BCH–Lie algebra, $\mathrm{Ann}(\mathcal{J}, B) \cap \mathfrak{g}(\mathcal{H}, B)$ is also a BCH–Lie algebra.

(iii) From Theorem 2.7 we deduce that it suffices to prove that exp restricts to a bijection $\mathrm{Ann}(\mathcal{J}, B) \cap \mathfrak{g}(\mathcal{H}, B) \to \mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$.

Recall from Theorem 2.7 that the Lie group exponential map of $G(\mathcal{H}, \mathcal{J})$ is a global diffeomorphism which is given on $\mathfrak{g}(\mathcal{H}, B) \subseteq \mathcal{I}_A$ by a convergent power series. Hence exp maps elements in a closed unital subalgebra into the subalgebra, i.e. $\exp(\phi) \in \mathrm{Ann}(\mathcal{J}, B)$ for each $\phi \in \mathrm{Ann}(\mathcal{J}, B) \cap \mathcal{I}_A$ and thus

$$\exp(\mathrm{Ann}(\mathcal{J}, B) \cap \mathfrak{g}(\mathcal{H}, B)) \subseteq \mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B).$$

The logarithm log on $G(\mathcal{H}, B) \subseteq (1_A + \mathcal{I}_A)$ is also given by a power series which converges on $\mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ and by the same argument, we obtain:

$$\log(\mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)) \subseteq \mathrm{Ann}(\mathcal{J}, B) \cap \mathfrak{g}(\mathcal{H}, B).$$

In conclusion, exp restricts to a bijection $\mathrm{Ann}(\mathcal{J}, B) \cap \mathfrak{g}(\mathcal{H}, B) \to \mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ as desired.

(i) It now follows from (ii), (iii) and [30, Theorem IV.3.3] that the group $\mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is a closed Lie subgroup, and an exponential BCH–Lie group.

(iv) We have already seen that $G(\mathcal{H}/\mathcal{J}, B)$ and $\mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ are isomorphic as topological groups and that $G(\mathcal{H}/\mathcal{J}, B)$ and also $\mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ are Baker–Campbell–Hausdorff–Lie groups. The Automatic Smoothness Theorem [30, Theorem IV.1.18] implies that we have in fact ($\mathbb{K}$-analytic) Lie group isomorphisms. □

Note that when $\mathcal{J}$ is homogeneous and B is Mackey complete, it follows from Theorem 3.4(iv) that the Lie subgroup $\mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is again a regular Lie group. Namely, with Theorem 2.11 we obtain a regularity result:

Corollary 3.5. — *Let $\mathcal{H}$ be a graded connected Hopf algebra and B be a commutative and Mackey complete locally convex algebra. Furthermore, let $\mathcal{J} \subseteq \mathcal{H}$ be a homogeneous Hopf ideal, then the Lie subgroup $\operatorname{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is a C^1-regular Lie group. If B is in addition sequentially complete, then $\operatorname{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is C^0-regular.*

Note that we have not established the regularity condition of $\operatorname{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ for non-homogeneous $\mathcal{J}$. Hence, we pose the following problem:

Problem. — *Let $\mathcal{H}$ be a graded and connected Hopf algebra and B be a commutative and Mackey complete locally convex algebra. Is the Lie group $\operatorname{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ C^k-regular (with $k \in \mathbb{N}_0 \cup \{\infty\}$) if $\mathcal{J}$ is a non-homogeneous Hopf Ideal? It suffices to prove that $\operatorname{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is a semiregular Lie subgroup of $G(\mathcal{H}, B)$. However, the idea used to prove Theorem 2.11 seems to carry over only to homogeneous ideals.*

In the special case that B is a weakly complete algebra (e.g. a finite-dimensional algebra) we deduce from Remark 5.8(c) the following corollary.

Corollary 3.6. — *Let $\mathcal{H}$ be a connected graded Hopf algebra and B be a commutative and weakly complete locally convex algebra. Furthermore, let $\mathcal{J} \subseteq \mathcal{H}$ be a Hopf ideal. Then the Lie subgroup $\operatorname{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, B)$ is regular.*

4. (Counter-)examples for Lie groups arising as Hopf algebra characters

In this section we give several examples for Lie groups arising from the construction in the last section. In the literature many examples for graded and connected Hopf algebras are studied (we refer the reader to [5] and the references and examples therein). In particular, the so called combinatorial Hopf algebras provide a main class of examples for graded and connected Hopf algebras (see [24] for an overview). A prime example for a combinatorial Hopf algebra is the famous Connes–Kreimer Hopf algebra of rooted trees. Its character group corresponds to the Butcher group from numerical analysis and we discuss it as our main example below. Furthermore, we discuss several (counter-)examples to statements in Theorem 2.7 for characters of Hopf algebras which are *not* graded.

Tensor algebras and universal enveloping algebras

Example 4.1 (Tensor algebra). — Consider an abstract vector space $\mathcal{V}$. Then the tensor algebra

$$T(\mathcal{V}) := \bigoplus_{n=0}^{\infty} \mathcal{V}^{\otimes n} \text{ with } \mathcal{V}^{\otimes n} := \underbrace{\mathcal{V} \otimes \cdots \otimes \mathcal{V}}_{n}$$

has a natural structure of a graded connected Hopf algebra which we denote by $(T(\mathcal{V}), \otimes, u, \Delta, \varepsilon, S)$ where

$$\Delta(v) = 1 \otimes v + v \otimes 1 \quad \text{and} \quad S(v) = -v \quad \text{for} \quad v \in \mathcal{V}.$$

By Theorem 2.7 the character group of $T(\mathcal{V})$ is a BCH–Lie group. This group can be described explicitly.

Every linear functional on $\mathcal{V}$ has a unique extension to a character of the Hopf algebra $T(\mathcal{V})$, yielding a bijection to the algebraic dual $\mathcal{V}^*$:

$$\Phi\colon G(T(\mathcal{V}), \mathbb{K}) \to \mathcal{V}^*,\ \phi \mapsto \phi|_{\mathcal{V}}.$$

We claim that Φ is a group isomorphism (where we view $\mathcal{V}^*$ as a group with respect to its additive structure). Let $v \in \mathcal{V}$ and $\phi, \psi \in G(T(\mathcal{V}), \mathbb{K})$ be given, then

$$\begin{aligned}(\phi \star \psi)(v) &= m_{\mathbb{K}} \circ (\phi \otimes \psi)(\Delta(v)) = m_{\mathbb{K}}(\phi \otimes \psi)(v \otimes 1 + 1 \otimes v) \\ &= \phi(v)\psi(1) + \phi(1)\psi(v) = \phi(v) + \psi(v).\end{aligned}$$

Thus, the group $(G(T(\mathcal{V}), \mathbb{K}), \star)$ is isomorphic to the additive group $(\mathcal{V}^*, +)$. As $\mathcal{V}^*$ is endowed with the weak*-topology, it is easy to check that Φ and Φ^{-1} are both continuous, hence Φ is also an isomorphism of topological groups. Since both Lie groups are known to be BCH–Lie groups, the the Automatic Smoothness Theorem [30, Theorem IV.1.18.] guarantees that Φ is also an analytic diffeomorphism, hence an isomorphism in the category of analytic Lie groups.

Example 4.2 (Universal enveloping algebra). — The *universal enveloping algebra* $\mathcal{U}(\mathfrak{g})$ of an abstract non-graded Lie algebra $\mathfrak{g}$ can be constructed as a quotient of the connected graded Hopf algebra $T(\mathfrak{g})$ and hence, its character group is a Lie group by Theorem 3.4. Note that we cannot use Theorem 2.7 directly since in general $\mathcal{U}(\mathfrak{g})$ does not possess a natural connected grading (the grading of the tensor algebra induces only a filtration on $\mathcal{U}(\mathfrak{g})$, see [20, Theorem V.2.5]). If $\mathfrak{g}$ is abelian, the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ coincides with the symmetric algebra $S(\mathfrak{g})$ (cf. [20, V.2 Example 1]). It is possible to give an explicit description of the group $G(\mathcal{U}(\mathfrak{g}), \mathbb{K})$:

Every character of $\phi \in G(\mathcal{U}(\mathfrak{g}), \mathbb{K})$ corresponds to a Lie algebra homomorphism $\phi|_{\mathfrak{g}} \colon \mathfrak{g} \to \mathbb{K}$ which in turn factors naturally through a linear map $\phi^{\sim} \colon \mathfrak{g}/(\mathfrak{g}') \to \mathbb{K}$, yielding a bijection

$$\Phi \colon G(\mathcal{U}(\mathfrak{g}), \mathbb{K}) \to \big((\mathfrak{g}/\mathfrak{g}')^*, +\big),\ \phi \mapsto \big(\phi^{\sim} \colon v + \mathfrak{g}' \mapsto \phi(v)\big).$$

Like in the case of the tensor algebra and the symmetric algebra, one easily verifies that this is an isomorphism of topological groups and since both Lie groups $G(\mathcal{U}(\mathfrak{g}), \mathbb{K})$ and $\big((\mathfrak{g}/\mathfrak{g}')^*, +\big)$ are BCH–Lie groups, we use again the Automatic Smoothness Theorem [30, Theorem IV.1.18.] to see that they are also isomorphic as analytic Lie groups.

In particular, this shows that the character group of $\mathcal{U}(\mathfrak{g})$ only sees the abelian part of $\mathfrak{g}$ and is therefore not very useful for studying the Lie algebra $\mathfrak{g}$.

Remark 4.3 (Universal enveloping algebra of a graded Lie algebra). — We remark that there is a notion of a *graded Lie algebra* which differs from the usual notion of a Lie algebra. Here the Lie bracket of the graded Lie algebra $\mathfrak{g} = \bigoplus_{p \in \mathbb{N}_0} \mathfrak{g}_p$ satisfies the Koszul sign convention, i.e. $[x, y] = (-1)^{pq+1}[y, x]$ for $x \in \mathfrak{g}_p$ and $y \in \mathfrak{g}_q$. Such graded Lie algebras also admit a universal enveloping algebra which inherits a grading from the Lie algebra grading. We refer to [29, 5.] for definitions and more details. (Note that the gradings encountered so far can be seen as even gradings with respect to the Koszul sign convention, cf. [10, Remark 1.24].) Then, the universal enveloping algebra becomes a graded and connected Hopf algebra.

Characters of the Hopf algebra of rooted trees

We examine the Hopf algebra of rooted trees which arises naturally in numerical analysis, renormalisation of quantum field theories and noncommutative geometry (see [4] for a survey). To construct the Hopf algebra, recall some notation first.

Notation 4.4.

(1) A *rooted tree* is a connected *finite* graph without cycles with a distinguished node called the *root*. We identify rooted trees if they are graph isomorphic via a root preserving isomorphism.

Let $\mathcal{T}$ be *the set of all rooted trees* and write $\mathcal{T}_0 := \mathcal{T} \cup \{\emptyset\}$ where $\emptyset$ denotes the empty tree. The *order* $|\tau|$ of a tree $\tau \in \mathcal{T}_0$ is its number of vertices.

(2) An *ordered subtree*[3] of $\tau \in \mathcal{T}_0$ is a subset s of all vertices of τ which satisfies
   (i) s is connected by edges of the tree τ,
   (ii) if s is non-empty, it contains the root of τ.

   The set of all ordered subtrees of τ is denoted by $\mathrm{OST}(\tau)$. Associated to an ordered subtree $s \in \mathrm{OST}(\tau)$ are the following objects:
   - A forest (collection of rooted trees) denoted as $\tau \setminus s$. The forest $\tau \setminus s$ is obtained by removing the subtree s together with its adjacent edges from τ. We denote by $\#(\tau \setminus s)$ the number of trees in the forest $\tau \setminus s$.
   - s_τ, the rooted tree given by vertices of s with root and edges induced by that of the tree τ.

Notation 4.5. — A *partition* p of a tree $\tau \in \mathcal{T}_0$ is a subset of edges of the tree. We denote by $\mathcal{P}(\tau)$ the set of all partitions of τ (including the empty partition). Associated to a partition $p \in \mathcal{P}(\tau)$ are the following objects

- A forest $\tau \setminus p$ which is defined as the forest that remains when the edges of p are removed from the tree τ. Write $\#(\tau \setminus p)$ for the number of trees in $\tau \setminus p$.
- The *skeleton* p_τ, is the tree obtained by contracting each tree of $\tau \setminus p$ to a single vertex and by re-establishing the edges of p.

Example 4.6 (The Connes–Kreimer Hopf algebra of rooted trees [7]*).* — Consider the algebra $\mathcal{H}_{CK}^{\mathbb{K}} := \mathbb{K}[\mathcal{T}]$ of polynomials which is generated by the trees in $\mathcal{T}$. We denote the structure maps of this algebra by m (multiplication) and u (unit). Indeed $\mathcal{H}_{CK}^{\mathbb{K}}$ becomes a bialgebra with the coproduct

$$\Delta\colon \mathcal{H}_{CK}^{\mathbb{K}} \to \mathcal{H}_{CK}^{\mathbb{K}} \otimes \mathcal{H}_{CK}^{\mathbb{K}},\ \tau \mapsto \sum_{s \in \mathrm{OST}(\tau)} (\tau \setminus s) \otimes s_\tau$$

and the counit $\varepsilon\colon \mathcal{H}_{CK}^{\mathbb{K}} \to \mathbb{K}$ defined via $\varepsilon(1_{\mathcal{H}_{CK}^{\mathbb{K}}}) = 1$ and $\varepsilon(\tau) = 0$ for all $\tau \in \mathcal{T}$. Furthermore, one can define an antipode S via

$$S\colon \mathcal{H}_{CK}^{\mathbb{K}} \to \mathcal{H}_{CK}^{\mathbb{K}},\ \tau \mapsto \sum_{p \in \mathcal{P}(\tau)} (-1)^{|p_\tau|} (\tau \setminus p)$$

such that $\mathcal{H}_{CK}^{\mathbb{K}} = (\mathcal{H}_{CK}^{\mathbb{K}}, m, u, \Delta, \varepsilon, S)$ is a $\mathbb{K}$-Hopf algebra (see [6, 5.1] for more details and references).

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[3] The term "ordered" refers to that the subtree remembers from which part of the tree it was cut.

Furthermore, the Hopf algebra $\mathcal{H}_{CK}^{\mathbb{K}}$ is graded as a Hopf algebra by the *number of nodes grading*: For each $n \in \mathbb{N}_0$ we define the nth degree via

$$\text{For } \tau_i \in \mathcal{T}, 1 \leqslant i \leqslant k, k \in \mathbb{N}_0 \quad \tau_1 \cdot \tau_2 \cdot \ldots \cdot \tau_k \in (\mathcal{H}_{CK}^{\mathbb{K}})_n \iff \sum_{r=1}^{k} |\tau_k| = n$$

Clearly, $\mathcal{H}_{CK}^{\mathbb{K}}$ is connected with respect to the number of nodes grading and we identify $(\mathcal{H}_{CK}^{\mathbb{K}})_0$ with $\mathbb{K}\emptyset$. Thus we can apply Theorem 2.7 for every commutative CIA B to see that the B valued characters $G(\mathcal{H}_{CK}^{\mathbb{K}}, B)$ form an exponential BCH–Lie group.

It turns out that the $\mathbb{K}$-valued characters of the Connes–Kreimer Hopf algebra $\mathcal{H}_{CK}^{\mathbb{K}}$ can be identified with the Butcher group from numerical analysis.

Example 4.7 (The Butcher Group). — Let us recall the definition of the ($\mathbb{K}$-)Butcher group. As a set the ($\mathbb{K}$-)Butcher group is the set of tree maps

$$G_{\mathrm{TM}}^{\mathbb{K}} := \{a \colon \mathcal{T}_0 \to \mathbb{K} \mid a(\emptyset) = 1\}$$

together with the group operation

$$a \cdot b(\tau) := \sum_{s \in \mathrm{OST}(\tau)} b(s_\tau) a(\tau \setminus s) \quad \text{with} \quad a(\tau \setminus s) := \prod_{\theta \in \tau \setminus s} a(\theta).$$

In [3] we have constructed a Lie group structure for the ($\mathbb{K}$-)Butcher group as follows: Identify $G_{\mathrm{TM}}^{\mathbb{K}}$ with the closed affine subspace $e + \mathbb{K}^{\mathcal{T}} = \{a \in \mathcal{T}_0 \mid a(\emptyset) = 1\}$ of $\mathbb{K}^{\mathcal{T}_0}$ with the topology of pointwise convergence. Then the subspace topology turns $G_{\mathrm{TM}}^{\mathbb{K}}$ into a BCH–Lie group modelled on the Fréchet space $\mathbb{K}^{\mathcal{T}}$.

Note that the group operation of the Butcher group is closely related to the coproduct of the Hopf algebra of rooted trees. Indeed the obvious morphism

$$\Phi \colon G(\mathcal{H}_{CK}^{\mathbb{K}}, \mathbb{K}) \to G_{\mathrm{TM}}^{\mathbb{K}}, \, \varphi \mapsto (\tau \mapsto \varphi(\tau))$$

is an isomorphism of (abstract) groups (see also [6, Eq. 38]). Moreover, it turns out that Φ is an isomorphism of Lie groups if we endow these groups with the Lie group structures discussed in Example 4.6 and Example 4.7.

LEMMA 4.8. — *The group isomorphism $\Phi \colon G(\mathcal{H}_{CK}^{\mathbb{K}}, \mathbb{K}) \to G_{TM}^{\mathbb{K}}$ is an isomorphism of $\mathbb{K}$-analytic Lie groups.*

Proof. — We already know that Φ is an isomorphism of abstract groups whose inverse is given by $\Phi^{-1}(a) = \varphi_a$ where φ_a is the algebra homomorphism defined via

$$\varphi_a(1_{\mathcal{H}_{CK}^{\mathbb{K}}}) = 1 \text{ and } \varphi_a(\tau) = a(\tau)$$

Recall from Lemma A.6 that the Lie groups $A^\times := \mathrm{Hom}_K(\mathcal{H}_{CK}^{\mathbb{K}}, \mathbb{K})^\times$ and $G(\mathcal{H}_{CK}^{\mathbb{K}}, \mathbb{K}) \subseteq A^\times$ carry the subspace topology with respect to the topology of pointwise convergence on the space $\mathrm{Hom}_K(\mathcal{H}_{CK}^{\mathbb{K}}, \mathbb{K})$. Furthermore, the topology on $G_{\mathrm{TM}}^{\mathbb{K}}$ is subspace topology with respect to the topology of pointwise convergence on $\mathbb{K}^{\mathcal{T}_0}$. Hence a straight forward computation shows that Φ and Φ^{-1} are continuous, i.e. they are isomorphisms of topological groups. Since both $G(\mathcal{H}_{CK}^{\mathbb{K}}, \mathbb{K})$ and $G_{\mathrm{TM}}^{\mathbb{K}}$ are BCH–Lie groups, the Automatic Smoothness Theorem [30, Theorem IV.1.18] asserts that Φ and Φ^{-1} are smooth (even real analytic). Thus Φ is an isomorphism of ($\mathbb{K}$-analytic) Lie groups. □

So far we have seen that our Theorem 2.7 generalises the construction of the Lie group structure of the Butcher group from [3]. In loc. cit. we have also endowed the subgroup of symplectic tree maps with a Lie group structure. This can be seen as a special case of the construction given in Theorem 3.4 as the next example shows. Thus the results from [3] are completely subsumed in the more general framework developed in the present paper.

Example 4.9 (The subgroup of symplectic tree maps). — In [3, Theorem 5.8], it was shown that the subgroup of symplectic tree maps $S_{\mathrm{TM}}^{\mathbb{K}} \subseteq G_{\mathrm{TM}}^{\mathbb{K}}$ is a closed Lie subgroup of the Butcher group and that the subgroup is itself an exponential Baker–Campbell–Hausdorff Lie group.

The symplectic tree maps are defined as those $a \in G_{\mathrm{TM}}^{\mathbb{K}}$ such that

$$a(\tau \circ \upsilon) + a(\upsilon \circ \tau) = a(\tau)a(\upsilon) \text{ for all } \tau, \upsilon \in \mathcal{T},$$

where $\tau \circ \upsilon$ denotes the rooted tree obtained by connecting τ and υ with an edge between the roots of τ and υ, and where the root of τ is the root of $\tau \circ \upsilon$ (4).

To cast [3, Theorem 5.8] in the context of Theorem 3.4, let $\mathcal{J} \subseteq \mathcal{H}_{CK}$ be the algebra ideal generated by the elements $\{\tau \circ \upsilon + \upsilon \circ \tau - \tau\upsilon\}_{\tau,\upsilon \in \mathcal{T}}$. Note that by definition of the Butcher product we have $|\tau \circ \upsilon| = |\upsilon \circ \tau| = |\tau\upsilon|$. Hence the generating elements of $\mathcal{J}$ are homogeneous elements with respect to the number of nodes grading (see Example 4.6). Consequently $\mathcal{J}$ is a homogeneous (algebra) ideal. It is possible to show that $\mathcal{J}$ is also a co-ideal and stable under the antipode.

If $a \in S_{\mathrm{TM}}^{\mathbb{K}}$, then $\varphi_a = \Phi^{-1}(a) \in \mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, \mathbb{K})$, since φ_a is an algebra morphism and zero on the generators of $\mathcal{J}$. The inverse implication also holds. Therefore, the restriction of Φ is a bijection between

(4) This is known as the *Butcher product* and should not be confused with the product in the Butcher group (cf. [3, Remark 5.1]).

$S^{\mathbb{K}}_{\mathrm{TM}}$ and $\mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, \mathbb{K})$. By Theorem 3.4, $\mathrm{Ann}(\mathcal{J}, B) \cap G(\mathcal{H}, \mathbb{K}) \subseteq G(\mathcal{H}, \mathbb{K})$ is a closed Lie subgroup and an exponential BCH–Lie group. Using Lemma 4.8, we can show that this structure is isomorphic to the Lie group structure $S^{\mathbb{K}}_{\mathrm{TM}} \subseteq G^{\mathbb{K}}_{\mathrm{TM}}$ constructed in [3, Theorem 5.8].

In addition to the Lie group structure on $S^{\mathbb{K}}_{\mathrm{TM}}$ already constructed in [3, Theorem 5.8] we derive from Corollary 3.5 that the Lie group $S^{\mathbb{K}}_{\mathrm{TM}}$ is C^0-regular.

Finally, let us mention certain character groups connected to Hopf ideals arising in the renormalisation of quantum field theories. Note that these Hopf ideals are not contained in the Hopf algebra of rooted trees, but instead in the larger Hopf algebra of Feynman graphs. The definition of these ideals and the ambient Hopf algebra is rather involved, whence we refer to the references given in the next remark for details.

Remark 4.10. — The Connes–Kreimer theory of renormalisation allows one to formulate renormalisation of quantum field theories in the language of (characters of) the Hopf algebra of Feynman graphs (cf. e.g. [7]). In [31] this idea is used to study the combinatorics of the renormalisation of gauge theories. Namely, loc. cit. proves that certain identities from physics, the so called "Ward–Takahashi" and "Slavnov–Taylor" identities, generate Hopf ideals in the Hopf algebra of Feynman graphs. Then in [32] character groups related to the quotient Hopf algebras associated to these Hopf ideals are studied.

Characters of Hopf algebras without connected grading

For the rest of this section let us investigate the case of a Hopf algebra $\mathcal{H}$ without a connected grading. It will turn out that the results achieved for graded connected Hopf algebras (and quotients thereof) do not hold for Hopf algebras without grading. It should be noted however, that for scalar valued characters, we can show that they still form a so called pro-Lie group (see Theorem 5.6)

Let $\mathcal{H}$ be a Hopf algebra without a grading then the dual space $A := \mathcal{H}^* = \mathrm{Hom}_{\mathbb{K}}(\mathcal{H}, \mathbb{K})$ is still a locally convex algebra (see Lemma 1.4). However, in general, neither its unit group will be an open subset, nor will the group of characters $G(\mathcal{H}, \mathbb{K})$ be a Lie group modelled on a locally convex space. We give two examples for this behaviour.

Example 4.11. — Let Γ be an abstract group. Then the group algebra $\mathbb{K}\Gamma$ carries the structure of a cocommutative Hopf algebra by [20, III.3

Example 2]. The algebraic dual $A := (\mathbb{K}\Gamma)^*$ is isomorphic to the direct product algebra $\mathbb{K}^\Gamma$ consisting of all functions on the group with pointwise multiplication. Its unit group $(\mathbb{K} \setminus \{0\})^\Gamma$ is a topological group (as a direct product of the topological group $\mathbb{K} \setminus \{0\}$ with itself). However, in general, it will not be open:

(a) Let Γ be an infinite group, then the unit group $A^\times = (\mathbb{K}^\times)^\Gamma$ is not open in $\mathbb{K}^\Gamma$. Hence, A is not a CIA and in particular, the unit group $A^\times$ does not inherit a Lie group structure from Lemma A.6 or Proposition 1.8, respectively.

Furthermore, the universal property of the group algebra $\mathbb{K}\Gamma$ (see [20, III.2 Example 2]) implies that a linear map $\phi\colon \mathbb{K}\Gamma \to \mathbb{K}$ is a character if and only if the map $\phi|_\Gamma\colon \Gamma \to \mathbb{K}^\times$ is a group homomorphism.

Thus, the group $G(\mathbb{K}\Gamma, \mathbb{K})$ is (as a topological group) isomorphic to the group of group homomorphisms from Γ to $\mathbb{K}^\times$ with the topology of pointwise convergence.

(b) Let $\Gamma = (\mathbb{Z}^{(I)}, +)$ be a free abelian of countable infinite rank. Then it is easy to see that $G(\mathbb{K}\Gamma, \mathbb{K})$ is topologically isomorphic to the infinite product $(\mathbb{K}^\times)^I$. This topological group is not locally contractible, hence it can not be (locally) homeomorphic to a topological vector space and thus cannot carry a locally convex manifold structure. In particular, Theorem 2.7 does not generalise to the character group of $\mathbb{K}\mathbb{Z}^{(I)}$.

If a non-graded Hopf algebra $\mathcal{H}$ is finite-dimensional, then $A = \mathcal{H}^*$ is a finite-dimensional algebra and hence it is automatically a CIA. The group of characters will then be a (finite-dimensional) Lie group. However, by [34, 2.2] the characters are linearly independent whence this Lie group will always be finite. Hence there cannot be a bijection between the character group and the Lie algebra of infinitesimal characters (which in this case will be 0-dimensional). This shows that even when $G(\mathcal{H}, \mathbb{K})$ is a Lie group it may fail to be exponential. We consider a concrete example of this behaviour:

Example 4.12. — Take a *finite* non trivial group Γ and consider the finite-dimensional algebra $\mathcal{H} := \mathbb{K}^\Gamma$ of functions on the group with values in $\mathbb{K}$ together with the pointwise operations. There is a suitable coalgebra structure and antipode (see [25, Example 1.5.2]) which turns algebra $\mathbb{K}^\Gamma$ into a Hopf algebra. Furthermore, we can identify its dual with the group algebra $\mathbb{K}\Gamma$ of Γ (as [20, III. Example 3] shows).

With this identification we can identify $G(\mathcal{H}, \mathbb{K})$ with the group Γ (with the discrete topology). Obviously, there is no bijection between the group of characters and the $\mathbb{K}$-Lie algebra of infinitesimal characters (which in this case is trivial).

5. Character groups as pro-Lie groups

In this section, we show that the group of characters of an abstract Hopf algebra (graded or not) can always be considered as a projective limit of finite-dimensional Lie groups, i.e. the group of characters is a *pro-Lie group*. The range space B has to the ground field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ or a commutative weakly complete algebra (see Lemma 5.3).

The category of pro-Lie groups admits a very powerful structure theory which is similar to the theory of finite-dimensional Lie groups (see [18]). This should provide Lie theoretic tools to work with character groups, even in the examples where the methods of locally convex Lie groups do not apply (like Examples 4.11 and 4.12). It should be noted, however, that the concept of a pro-Lie group is of purely topological nature and involves no differential calculus. See [17] for an article dedicated to the problem of determining which pro-Lie groups do admit a locally convex differential structure and which do not.

Definition 5.1 (pro-Lie group). — *A topological group G is called* pro-Lie group *if one of the following equivalent conditions holds:*

(a) *G is isomorphic (as a topological group) to a closed subgroup of a product of finite-dimensional (real) Lie groups.*

(b) *G is the projective limit of a directed system of finite-dimensional (real) Lie groups (taken in the category of topological groups)*

(c) *G is complete and each identity neighbourhood contains a closed normal subgroup N such that G/N is a finite-dimensional (real) Lie group.*

The fact that these conditions are equivalent is surprisingly complicated to show and can be found in [14] or in [18, Theorem 3.39]. The class of pro-Lie groups contains all compact groups (see e.g. [19, Corollary 2.29]) and all connected locally compact groups (Yamabe's Theorem, see [35]). However, this does not imply that all pro-Lie groups are locally compact. In fact, the pro-Lie groups constructed in this paper will almost never be locally compact.

In absence of a differential structure we cannot define a Lie algebra as a tangent space. However, it is still possible to define a Lie functor.

Definition 5.2 (The pro-Lie algebra of a pro-Lie group). — *Let G be a pro-Lie group. Consider the space $\mathcal{L}(G)$ of all continuous G-valued one-parameter subgroups, endowed with the compact-open topology.*

(a) *The space $\mathcal{L}(G)$ is the projective limit of finite-dimensional Lie algebras and hence, carries a natural structure of a locally convex topological Lie algebra over $\mathbb{R}$ (see* [18, Definition 2.11]*). As a topological vector space, $\mathcal{L}(G)$ is weakly complete, i.e. isomorphic to $\mathbb{R}^I$ for an index set I (see also Definition C.1).*

(b) *Assigning the pro-Lie algebra to a pro-Lie group yields a functor: Assign to a morphism of pro-Lie groups, i.e. a continuous group homomorphism $\phi\colon G \to H$, a morphism of topological real Lie algebras $\mathcal{L}(\phi)\colon \mathcal{L}(G) \to \mathcal{L}(H)$, $\gamma \mapsto \gamma\circ\phi$. We thus obtain the so called pro-Lie functor.*

For more information on pro-Lie groups, pro-Lie algebras and the pro-Lie functor, see [18, Chapter 3].

Many pro-Lie groups arise as groups of invertible elements of topological algebras:

Lemma 5.3 (Fundamental lemma of weakly complete algebras). — *For a topological $\mathbb{K}$-algebra A, the following are equivalent:*

(a) *The underlying topological vector space A is (forgetting the multiplicative structure) weakly complete, i.e. isomorphic to $\mathbb{K}^I$ for an index set I.*

(b) *There is an abstract $\mathbb{K}$-coalgebra $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$ such that A is isomorphic to* $(\mathrm{Hom}(\mathcal{C}, \mathbb{K}), \star)$.

(c) *The topological algebra A is the projective limit of a directed system of* finite-dimensional *$\mathbb{K}$-algebras (taken in the category of topological $\mathbb{K}$-algebras)*

A topological algebra with these properties is called weakly complete algebra.

Proof.

(a)⇒(b) The category of weakly complete topological vector spaces over $\mathbb{K}$ and the category of abstract $\mathbb{K}$-vector spaces are dual. This implies that $A = \mathbb{K}^I$ is the algebraic dual space of the vector space $\mathcal{C} := \mathbb{K}^{(I)}$ of finite supported functions. The continuous multiplication $\mu_A\colon A \times A \to A$ dualises to an abstract comultiplication $\Delta_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$. (see [27, Theorem 4.4] or Appendix C for details of this duality.)

(b)⇒(c) This is a direct consequence of the *Fundamental Theorem of Coalgebras* ([27, Theorem 4.12]) stating that $\mathcal{C}$ is the directed union of finite-dimensional coalgebras. Dualising this, yields a projective limit of topological algebras.

(c)⇒(a) The projective limit of finite-dimensional $\mathbb{K}$-vector spaces is always topologically isomorphic to $\mathbb{K}^I$. (see Appendix C) □

Proposition 5.4 (The group of units of a weakly complete algebra is a pro-Lie group). — *Let A be a weakly complete $\mathbb{K}$-algebra as in Lemma 5.3. Then the group of units $A^\times$ is a pro-Lie group. Its pro-Lie algebra $\mathcal{L}(A)$ is (as a* real *Lie algebra) canonically isomorphic to* $(A, [\cdot,\cdot])$ *via the isomorphism*

$$A \to \mathcal{L}(A^\times),\ x \mapsto \gamma_x : (t \mapsto \exp(tx)),$$

where $\exp\colon A \to A^\times$ *denotes the usual exponential series which converges on A.*

Proof. — Let $A = \lim_{\leftarrow} A_\alpha$ with finite-dimensional $\mathbb{K}$-algebras A_α (by Lemma 5.3). Then the unit group is given by

$$A^\times = \lim_{\leftarrow} A_\alpha^\times$$

in the category of topological groups. Each group $A_\alpha^\times$ is a finite-dimensional (linear) real Lie group. Hence, $A^\times$ is a pro-Lie group and in particular, inversion is continuous, which is not obvious for unit groups of topological algebras.

The exponential series converges on each algebra A_α and hence on the projective limit A. The correspondence between continuous one-parameter subgroups $\gamma \in \mathcal{L}(A^\times)$ and elements in A holds in each A_α and hence it holds on A. □

Remark 5.5. — As Example 4.11 shows, the group of units $A^\times$ of a weakly complete algebra need not be an open subset of A, nor will the exponential series be local homeomorphism around 0.

Theorem 5.6 (The character group of a Hopf algebra is a pro-Lie group). — *Let $\mathcal{H}$ be an abstract Hopf algebra and B be a commutative weakly complete $\mathbb{K}$-algebra (e.g. $B := \mathbb{K}$ or $B = \mathbb{K}[[X]]$). Then the group of B-valued characters $G(\mathcal{H}, B)$ endowed with the topology of pointwise convergence is pro-Lie group.*

Its pro-Lie algebra is isomorphic to the locally convex Lie algebra $\mathfrak{g}(\mathcal{H}, B)$ of infinitesimal characters via the canonical isomorphism

$$\mathfrak{g}(\mathcal{H}, B) \to \mathcal{L}(G(\mathcal{H}, B)),\ \phi \mapsto (t \mapsto \exp(t\phi)),$$

Remark 5.7. — The pro-Lie algebra $\mathcal{L}(G)$ of a pro-Lie group G is a priori only a *real* Lie algebra[5] However, since we already know that $\mathfrak{g}(\mathcal{H}, B)$ is a complex Lie algebra if $\mathbb{K} = \mathbb{C}$ (Lemma 2.5), we may use the isomorphism given in the theorem above to endow the real Lie algebra structure with a complex one.

Proof of Theorem 5.6. — The space $A := \operatorname{Hom}_{\mathbb{K}}(\mathcal{H}, B)$ is a topological algebra by Lemma 1.4. The underlying topological vector space is isomorphic to B^I by Lemma 1.6. Thus, A is a weakly complete algebra since B is weakly complete.

By Proposition 5.4, we may conclude that $A^\times$ is a pro-Lie group. The group $G(\mathcal{H}, B)$ is a closed subgroup of this pro-Lie group by Lemma 2.3. From part (a) of Definition 5.1 it follows that closed subgroups of pro-Lie groups are pro-Lie groups. Hence, $G(\mathcal{H}, B)$ is a pro-Lie group.

It remains to show that the pro-Lie algebra $\mathcal{L}(G(\mathcal{H}, B))$ is isomorphic to $\mathfrak{g}(\mathcal{H}, B)$. Since $G(\mathcal{H}, B)$ is a closed subgroup of $A^\times$, every continuous 1-parameter-subgroup γ of $G(\mathcal{H}, B)$ is also a 1-parameter subgroup of $A^\times$ and (by Proposition 5.4) of the form

$$\gamma_\phi \colon \mathbb{R} \to A^\times,\ t \mapsto \exp(t\phi)$$

for a unique element $\phi \in A$. It remains to show the following equivalence:

$$(\forall t \in \mathbb{R}\colon\ \exp(t\phi) \in G(\mathcal{H}, B)) \iff \phi \in \mathfrak{g}(\mathcal{H}, B).$$

At the end of the proof of Lemma B.10, there is a chain of equivalences. While the equivalence of the first line with the second uses the bijectivity of the exponential function which does not hold in the pro-Lie setting, the equivalence of the second line with all following lines hold by Remark B.11 also in this setting. Substituting $t\phi$ for ϕ, we obtain the following equivalence:

$$\begin{aligned}&\forall t \in \mathbb{R}\colon\ \exp(t\phi) \in G(\mathcal{H}, B)\\ &\qquad\qquad \iff \exp_{A_\otimes}(t\phi \circ m_{\mathcal{H}}) = \exp_{A_\otimes}(t(\phi \diamond 1_A + 1_A \diamond \phi)).\end{aligned}$$

Here the exponential function $\exp_{A_\otimes}$ is taken in $A_\otimes := \operatorname{Hom}_{\mathbb{K}}(\mathcal{H} \otimes \mathcal{H}, B)$.

This shows that the 1-parameter subgroups $\gamma_{\phi \circ m_{\mathcal{H}}}$ and $\gamma_{\phi \diamond 1_A + 1_A \diamond \phi}$ agree and by Proposition 5.4 applied to $A_\otimes$, we obtain that

$$\phi \circ m_{\mathcal{H}} = \phi \diamond 1_A + 1_A \diamond \phi$$

[5] This is due to the fact that the finite-dimensional *real* Lie groups form a full subcategory of the category of topological groups while the finite-dimensional *complex* Lie groups do not. In fact, there are infinitely many non-isomorphic complex Lie group structures (elliptic curves) on the torus $(\mathbb{R}/\mathbb{Z})^2$, inducing the same real Lie group structure.

which is equivalent to ϕ being an infinitesimal character. This finishes the proof. □

Remark 5.8.

(1) It is remarkable that Theorem 5.6 holds without any assumption on the given abstract Hopf algebra (in particular, we do not assume that it is graded or connected.)

(2) For a weakly complete commutative algebra B (e.g. $B = \mathbb{K}[[X]]$ or B finite-dimensional) and a graded and connected Hopf algebra $\mathcal{H}$ the results of Theorem 2.7 and Theorem 5.6 apply both to $G(\mathcal{H}, B)$.

In this case, the infinite-dimensional Lie group $G(\mathcal{H}, B)$ inherits additional structural properties as a projective limit of finite-dimensional Lie groups. In particular, the regularity of $G(\mathcal{H}, B)$ (cf. Theorem 2.11) then follows from [17].

For example, these observations apply to the Connes–Kreimer Hopf algebra $\mathcal{H}^{\mathbb{K}}_{CK}$ and the Butcher group $G(\mathcal{H}^{\mathbb{K}}_{CK}, \mathbb{K})$ (see Example 4.7). In fact, the structure as a pro-Lie group (implicitely) enabled some of the computations made in [3] to treat the Lie theoretic properties of the Butcher group.

Furthermore, in the Connes–Kreimer theory of renormalisation of quantum field theories, the pro-Lie group structure of certain character groups has been exploited. See e.g. [10, Proofs of Proposition 1.52 and Lemma 1.54] for explicit examples of results relying on this structure.

Appendix A. Locally convex differential calculus and manifolds

See [12, 21] for references on differential calculus in locally convex spaces.

Definition A.1. — *Let $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, $r \in \mathbb{N} \cup \{\infty\}$ and E, F locally convex $\mathbb{K}$-vector spaces and $U \subseteq E$ open. Moreover we let $f : U \to F$ be a map. If it exists, we define for $(x, h) \in U \times E$ the directional derivative*

$$df(x, h) := D_h f(x) := \lim_{t \to 0} t^{-1}(f(x + th) - f(x)) \quad (\text{where } t \in \mathbb{K}^{\times})$$

We say that f is $C^r_{\mathbb{K}}$ if the iterated directional derivatives

$$d^{(k)} f(x, y_1, \ldots, y_k) := (D_{y_k} D_{y_{k-1}} \cdots D_{y_1} f)(x)$$

exist for all $k \in \mathbb{N}_0$ such that $k \leqslant r$, $x \in U$ and $y_1, \ldots, y_k \in E$ and define continuous maps $d^{(k)}f \colon U \times E^k \to F$. If it is clear which $\mathbb{K}$ is meant, we simply write C^r for $C^r_{\mathbb{K}}$. If f is $C^\infty_{\mathbb{K}}$ we say that f is smooth.[(6)]

Definition A.2. — *Let E, F be real locally convex spaces and $f \colon U \to F$ defined on an open subset U. We call f* real analytic *(or $C^\omega_{\mathbb{R}}$) if f extends to a $C^\infty_{\mathbb{C}}$-map $\tilde{f} \colon \tilde{U} \to F_{\mathbb{C}}$ on an open neighbourhood $\tilde{U}$ of U in the complexification $E_{\mathbb{C}}$.*

For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ the composition of $C^r_{\mathbb{K}}$-maps (if possible) is again a $C^r_{\mathbb{K}}$-map (cf. [12, Propositions 2.7 and 2.9]).

Definition A.3. — *Fix a Hausdorff topological space M and a locally convex space E over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. An ($E$-)manifold chart (U_κ, κ) on M is an open set $U_\kappa \subseteq M$ together with a homeomorphism $\kappa \colon U_\kappa \to V_\kappa \subseteq E$ onto an open subset of E. Two such charts are called C^r-compatible for $r \in \mathbb{N}_0 \cup \{\infty, \omega\}$ if the change of charts map $\nu^{-1} \circ \kappa \colon \kappa(U_\kappa \cap U_\nu) \to \nu(U_\kappa \cap U_\nu)$ is a C^r-diffeomorphism. A $C^r_{\mathbb{K}}$-atlas of M is a family of pairwise C^r-compatible manifold charts, whose domains cover M. Two such C^r-atlases are equivalent if their union is again a C^r-atlas.*

A locally convex C^r-manifold *M modelled on E is a Hausdorff space M with an equivalence class of C^r-atlases of (E-)manifold charts.*

Direct products of locally convex manifolds, tangent spaces and tangent bundles as well as C^r-maps of manifolds may be defined as in the finite-dimensional setting (cf. [30]).

Definition A.4. — *A $\mathbb{K}$-analytic* Lie group *is a group G equipped with a $C^\omega_{\mathbb{K}}$-manifold structure modelled on a locally convex space, such that the group operations are $\mathbb{K}$-analytic. For a Lie group G we denote by $\mathbf{L}(G)$ the associated Lie algebra.*

Definition A.5 (Baker–Campbell–Hausdorff (BCH-)Lie groups and Lie algebras).

(a) *A Lie algebra $\mathfrak{g}$ is called* Baker–Campbell–Hausdorff–Lie algebra *(BCH–Lie algebra) if there exists an open 0-neighbourhood $U \subseteq \mathfrak{g}$ such that for $x, y \in U$ the BCH-series $\sum_{n=1}^\infty H_n(x,y)$ converges and defines an analytic function $U \times U \to \mathfrak{g}$. (The H_n are defined as $H_1(x,y) = x + y$, $H_2(x,y) = \frac{1}{2}[x,y]$ and for $n \geqslant 3$ by sums of iterated brackets, see* [30, Definition IV.1.5.].*)*

(6) A map f is of class $C^\infty_{\mathbb{C}}$ if and only if it is *complex analytic* i.e., if f is continuous and locally given by a series of continuous homogeneous polynomials (cf. [1, Proposition 7.7]). We then also say that f is of class $C^\omega_{\mathbb{C}}$.

(b) *A locally convex Lie group G is called* BCH–Lie group *if it satisfies one of the following equivalent conditions (cf.* [30, Theorem IV.1.8]*)*
   (i) *G is a $\mathbb{K}$-analytic Lie group whose Lie group exponential function is $\mathbb{K}$-analytic and a local diffeomorphism in* 0.
   (ii) *The exponential map of G is a local diffeomorphism in* 0 *and* $\mathbf{L}(G)$ *is a BCH–Lie algebra.*

Lemma A.6 (Unit groups of CIAs are Lie groups [11, Theorem 5.6]). — *Let A be a Mackey complete CIA. Then the group of units $A^\times$ is a $C^\omega_\mathbb{K}$-Lie group with the manifold structure endowed from the locally convex space A. The Lie algebra of the group $A^\times$ is $(A, [\cdot,\cdot])$, where $[\cdot,\cdot]$ is the commutator bracket.*

Moreover, the group $A^\times$ is a Baker–Campbell–Hausdorff–Lie group, i.e. the exponential map is a local $C^\omega_\mathbb{K}$-diffeomorphism around 0. *This exponential map is given by the exponential series and its inverse is locally given by the logarithm series.*

To establish regularity of unit groups of CIAs in [16] a sufficient criterion, called property "$(*)$" in ibid., was introduced. We recall this now:

Definition A.7 ((GN)-property). — *A locally convex algebra A satisfies the* (GN)-property, *if for every continuous seminorm p on A, there exists a continuous seminorm q and $M \geqslant 0$ such that for all $n \in \mathbb{N}$, we have*

$$\left\|\mu_A^{(n)}\right\|_{p,q} := \sup\{p(\mu_A^{(n)}(x_1,\ldots,x_n)) \mid q(x_i) \leqslant 1,\ 1 \leqslant i \leqslant n\} \leqslant M^n.$$

Here, $\mu_A^{(n)} \colon A \times \cdots \times A \to A$, $(a_1,\ldots,a_n) \mapsto a_1 \cdots a_n$.

Lemma A.8 ([16]). — *A locally convex algebra which is either a commutative continuous inverse algebra or locally m-convex has the (GN)-property.*

Lemma A.9 ([16, Proposition 3.4]). — *Let A be a CIA with the (GN)-property.*

(a) *If A is Mackey complete, then the Lie group $A^\times$ is C^1-regular.*
(b) *If A is sequentially complete, then $A^\times$ is C^0-regular.*

In both cases, the associated evolution map is even $\mathbb{K}$-analytic.

Appendix B. Graded algebra and characters

In this section we recall basic tools from abstract algebra. All results and definitions given in this appendix are well known (see for example [20, 25, 29, 33]. However, for the reader's convenience we recall some details of the construction and proofs. We assume that the reader is familiar with the definition of algebras, coalgebras and Hopf algebras.

Definition B.1 (Abstract gradings).

(a) *Let $\mathcal{V}$ be an abstract $\mathbb{K}$-vector space. A family of vector subspaces $(\mathcal{V}_n)_{n\in\mathbb{N}_0}$ is called (abstract)* $\mathbb{N}_0$-grading *(or just* grading*) of $\mathcal{V}$, if the canonical linear addition map*

$$\Sigma\colon \bigoplus_{n\in\mathbb{N}_0} \mathcal{V}_n \to E,\ (v_n)_{n\in\mathbb{N}_0} \mapsto \sum_{n\in\mathbb{N}_0} v_n$$

is an isomorphism of $\mathbb{K}$-vector spaces, i.e. is bijective.

(b) *By a* graded algebra*, we mean an abstract $\mathbb{K}$-algebra $\mathcal{A}$, together with an abstract grading $(\mathcal{A}_n)_{n\in\mathbb{N}_0}$ of the underlying vector space $\mathcal{A}$ such that*

$$1_{\mathcal{A}} \in \mathcal{A}_0 \text{ and } \mathcal{A}_n \cdot \mathcal{A}_m \subseteq \mathcal{A}_{n+m} \text{ for all } n, m \in \mathbb{N}_0.$$

This implies in particular that $\mathcal{A}_0$ is a unital subalgebra and that the projection $\pi_0\colon \mathcal{A} \to \mathcal{A}_0$ onto $\mathcal{A}_0$ is an algebra homomorphism.

(c) *A* graded coalgebra *is an abstract coalgebra $(\mathcal{C}, \Delta_{\mathcal{C}}, \varepsilon_{\mathcal{C}})$, together with an abstract grading $(\mathcal{C}_n)_{n\in\mathbb{N}_0}$ of the underlying vector space $\mathcal{C}$ such that for all $n \in \mathbb{N}_0$*

$$\Delta_{\mathcal{C}}(\mathcal{C}_n) \subseteq \bigoplus_{i+j=n} \mathcal{C}_i \otimes \mathcal{C}_j \text{ and } \bigoplus_{n\geqslant 1} \mathcal{C}_n \subseteq \ker(\varepsilon_{\mathcal{C}}).$$

A graded coalgebra is called connected *if $\mathcal{C}_0$ is one dimensional.*

(d) *An abstract Hopf algebra $(\mathcal{H}, m_{\mathcal{H}}, u_{\mathcal{H}}, \Delta_{\mathcal{H}}, \varepsilon_{\mathcal{H}})$ is called* graded Hopf algebra *if there is an abstract grading $(\mathcal{H}_n)_{n\in\mathbb{N}_0}$ of the underlying vector space $\mathcal{H}$ which is an algebra grading and a coalgebra grading at the same time. We call a graded Hopf algebra* connected *if $\mathcal{H}_0$ is one dimensional. An element $a \in \mathcal{A}_n$ (with $n \in \mathbb{N}_0$) is called* homogeneous (of degree n).

Definition B.2 (Dense Gradings).

(a) *Let E be a locally convex space. A family of vector subspaces $(E_n)_{n\in\mathbb{N}_0}$ is called a* dense grading *of E, if the canonical linear summation map*

$$\Sigma\colon \bigoplus_{n\in\mathbb{N}_0} E_n \to E,\ (x_n)_{n\in\mathbb{N}_0} \mapsto \sum_{n\in\mathbb{N}_0} x_n$$

extends (uniquely) to an isomorphism of locally convex spaces

$$\overline{\Sigma}\colon \prod_{n\in\mathbb{N}_0} E_n \to E.$$

Notice that Each E_n is closed in E and E is a Fréchet space if and only if each E_n is a Fréchet space.

(b) *By* a densely graded locally convex algebra, *we mean a locally convex algebra A, together with a dense grading of the underlying locally convex space A such that*

$$A_n \cdot A_m \subseteq A_{n+m} \text{ for all } n, m \in \mathbb{N}_0 \text{ and } 1_A \in A_0.$$

This implies that A_0 is a closed unital subalgebra and that the projection $\pi_0\colon A \to A_0$ is a continuous algebra homomorphism.

Denote the kernel of π_0 by $\mathcal{I}_A := \ker(\pi_0) = \overline{\bigoplus_{n\geqslant 1} A_n}$. The kernel $\mathcal{I}_A$ is a closed ideal. Each element in A has a unique decomposition $a = a_0 + b$ with $a_0 \in A_0$ and $b \in \mathcal{I}_A$.

For the reader's convenience we summarise important examples of topological algebras and some of their properties discussed in this appendix in the following chart. Here the arrows indicate that a given property is stronger than another (normal arrow) or that an example possesses the property (dashed arrow), respectively.

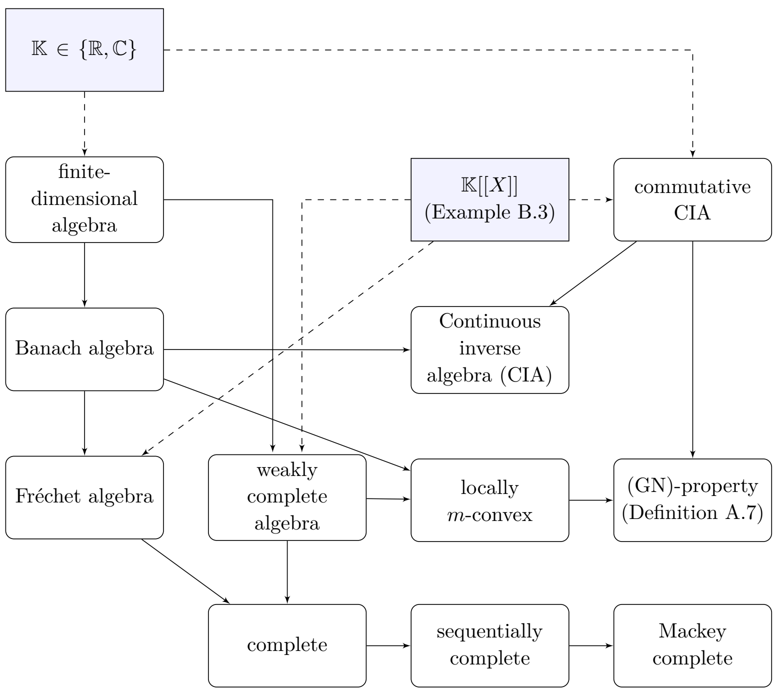

Figure B.1. Important properties and examples of locally convex algebras.

Note that *densely graded algebras* (see Definition B.2(b)) are not included in Diagram B.1 as every locally convex algebra A admits the trivial grading $A_0 = A$ and $A_n = 0$ for $n \geqslant 1$.

Example B.3 (Formal power series). — Let $\mathbb{K}[[X]]$ be the algebra of formal power series in one variable. We give this algebra the topology of pointwise convergence of coefficients, i.e. the initial topology with respect to the coordinate maps:

$$\kappa_n \colon \mathbb{K}[[X]] \to \mathbb{K}, \sum_{k=0}^{\infty} c_k X^k \mapsto c_n.$$

As a topological vector space, the algebra $\mathbb{K}[[X]]$ is isomorphic to the Fréchet space $\mathbb{K}^{\mathbb{N}_0} = \prod_{k=0}^{\infty} \mathbb{K}$ We see that $\mathbb{K}[[X]]$ is a densely graded algebra with respect to the grading $(\mathbb{K}X^n = \{cX^n \mid c \in \mathbb{K}\})_{n \in \mathbb{N}_0}$. Note that $\mathbb{K}[[X]]$ is a CIA by Lemma B.7(b).

In the following, we will identify a densely graded locally convex algebra A with the product space $\prod_{n=0}^{\infty} A_n$ such that each element $a \in A$ is a tuple $(a_n)_{n \in \mathbb{N}_0}$.

Lemma B.4 (Functional calculus for densely graded algebras). — *Let $A = \prod_{n=0}^{\infty} A_n$ be a densely graded locally convex algebra with $\mathcal{I}_A = \ker \pi_0$. Then there exists a unique continuous map*

$$\mathbb{K}[[X]] \times \mathcal{I}_A \to A,\ (f, a) \mapsto f[a]$$

such that for all $a \in \mathcal{I}_A$, we have $X[a] = a$ and the map

$$\mathbb{K}[[X]] \to A,\ f \mapsto f[a]$$

is a morphism of unital algebras. If $f = \sum_{k=0}^{\infty} c_k X^k$ and $a = (a_n)_{n \in \mathbb{N}_0}$ with $a_0 = 0$ are given, the following explicit formula holds:

$$f[a] = \left(\sum_{k=0}^{n} c_k \sum_{\substack{\alpha \in \mathbb{N}^k \\ |\alpha| = n}} a_{\alpha_1} \cdot \ldots \cdot a_{\alpha_k} \right)_{n \in \mathbb{N}_0}. \tag{B.1}$$

Furthermore, the map $\mathbb{K}[[X]] \times \mathcal{I}_A \to A$, $(f, a) \mapsto f[a]$ is a $C^{\omega}_{\mathbb{K}}$-map (cf. Appendix A).

Proof. — First of all, the explicit formula is well-defined and continuous on $\mathbb{K}[[X]] \times \mathcal{I}_A$ since every component is a continuous polynomial in finitely many evaluations of the spaces $\mathbb{K}[[X]]$ and $\mathcal{I}_A$. As A is densely graded and thus isomorphic to the locally convex product of the spaces $A_n, n \in \mathbb{N}_0$, this implies that the map is continuous. In fact, this already implies that the map is $C^\omega_{\mathbb{K}}$ for $\mathbb{K} = \mathbb{C}$. For $\mathbb{K} = \mathbb{R}$ one has to be a little bit more careful since there exist maps into products which are not $C^\omega_{\mathbb{R}}$ although every component is $C^\omega_{\mathbb{R}}$ (cf. [13, Example 3.1]). However, if each component is a continuous polynomial, the real case follows from the complex case as real polynomials complexify to complex polynomials by [2, Theorem 3].

Let $a \in \mathcal{I}_A$ be a fixed element. It remains to show that $\mathbb{K}[[X]] \to A$, $f \mapsto f[a]$ is an algebra homomorphism. By construction it is clear that $f \mapsto f[a]$ is linear and maps X^0 to 1_A and X^1 to a. Since f is continuous and linear, it suffices to establish the multiplicativity for series of the form X^N, i.e. it suffices to prove that

$$(X^N[a]) \cdot (X^M[a]) = X^{N+M}[a]$$

which follows from the easily verified fact that $X^N[a] = a^N$.

To establish uniqueness of the map obtained, we remark the following: A continuous map on $\mathbb{K}[[X]]$ is determined by its values on the dense space of polynomials $\mathbb{K}[X]$, and an algebra homomorphism on $\mathbb{K}[X]$ is determined by its value on the generator X. Here this value has to be a. □

Lemma B.5 (Exponential and logarithm). — *Consider the formal series*

$$\exp(X) := \sum_{k=0}^{\infty} \frac{X^k}{k!} \quad \textit{and} \quad \log(1+X) := \sum_{k=1}^{\infty} (-1)^{k+1} \frac{X^k}{k}.$$

Let A be a densely graded locally convex algebra. The exponential function restricted to the closed vector subspace $\mathcal{I}_A$

$$\exp_A \colon \mathcal{I}_A \to 1_A + \mathcal{I}_A,\ a \mapsto \exp[a]$$

is a $C^\omega_{\mathbb{K}}$-diffeomorphism with inverse

$$\log_A \colon 1_A + \mathcal{I}_A \to \mathcal{I}_A,\ (1_A + a) \mapsto \log(1+X)[a].$$

Proof. — The maps are $C^\omega_{\mathbb{K}}$ by Lemma B.4. As formal power series $E(X) := \exp(X) - 1$ and $L(X) := \log(X+1)$ are inverses with respect to composition of power series. Note that $\mathbb{K}[[X]]$ is a densely graded locally convex algebra with $E(X), L(X) \in \mathcal{I}_{\mathbb{K}[[X]]}$. Apply functional calculus (Lemma B.4) to $\mathbb{K}[[X]]$ and $E(X), L(X)$. This yields for $a \in \mathcal{I}_A$ the identity

$$E[L[a]] = (E \circ L)[a] = X[a] = a.$$

Similarly $L[E[a]] = a$, whence $\exp_A$ and $\log_A$ are mutually inverse. □

Lemma B.6. — *Let $A = \prod_{n\in\mathbb{N}_0} A_n$ be a densely graded locally convex algebra with exponential map $\exp_A$. Then the following assertions hold:*

(a) *For $a, b \in \mathcal{I}_A$ with $ab = ba$ we have $\exp_A(a+b) = \exp_A(a)\exp_A(b)$.*
(b) *The derivative of $\exp_A$ at 0 is $T_0 \exp_A = \mathrm{id}_{\mathcal{I}_A}$.*

Proof. — By construction of $\exp_A$ we derive from Lemma B.4 for $x \in \mathcal{I}_A$ the formula $\exp_A(x) = \lim_{N\to\infty} \sum_{k=0}^N \frac{x^k}{k!}$. The algebra A is densely graded and we have for every $n \in \mathbb{N}_0$ a continuous linear projection $\pi_n \colon A \to A_n$. By definition we have for $x \in \mathcal{I}_A$ that $\pi_0(x) = 0$. Hence, the definition of a densely graded algebra implies for $x, y \in \mathcal{I}_A$ that $\pi_j(x^k y^l) = 0$ if $k + l > j$.

(a) As a and b commute we can compute as follows:

$$\exp_A(a)\exp_A(b) = \lim_{N\to\infty} \left(\sum_{k+l\leqslant N} \frac{a^k b^l}{k!l!} + \underbrace{\sum_{\substack{l+k>N,\\ l,k\leqslant N}} \frac{a^k b^l}{k!l!}}_{=:S_N} \right) \tag{B.2}$$

$$= \lim_{N\to\infty} \left(\sum_{n=0}^N \sum_{k+l=n} \frac{a^k b^l}{k!l!} + S_N \right) = \lim_{N\to\infty} \left(\sum_{n=0}^N \frac{(a+b)^n}{n!} + S_N \right)$$

The first summand in the lower row converges to $\exp_A(a+b)$.

Now the definition of S_N shows that $\pi_j(S_N) = 0$ if $N \geqslant j$. Apply the continuous map π_j to both sides of (B.2) for $j \in \mathbb{N}_0$ to derive

$$\pi_j(\exp_A(a)\exp_A(b)) = \lim_{N\to\infty} \left(\pi_j \left(\sum_{n=0}^N \frac{(a+b)^n}{n!} \right) + \pi_j(S_N) \right)$$

On the right hand side the second term vanishes if $N > j$. Thus in passing to the limit we obtain $\pi_j(\exp_A(a)\exp_A(b)) = \pi_j(\exp_A(a+b))$ for all $j \in \mathbb{N}_0$.

(b) The image of $\exp_A$ is the affine subspace $1_A + \mathcal{I}_A$ whose tangent space (as a submanifold of A) is $\mathcal{I}_A$. We can thus identify $\exp_A$ with $F \colon \mathcal{I}_A \to A$, $a \mapsto \exp[a]$ to compute $T_0 \exp_A$ as $dF(0;\cdot)$. The projections $\pi_n, n \in \mathbb{N}_0$ are continuous linear, whence it suffices to compute $d\pi_n \circ F(0;\cdot) = \pi_n dF(0;\cdot)$ for all $n \in \mathbb{N}_0$. Now for $n \in \mathbb{N}_0$ and $a \in \mathcal{I}_A$ compute the derivative in 0:

$$\begin{aligned}\pi_n dF(0;a) &= d\pi_n \circ F(0;a) = \lim_{t\to 0} t^{-1}(\pi_n \circ F(ta) - \pi_n \circ F(0)) \\ &\overset{\text{(B.1)}}{=} \lim_{t\to 0} \sum_{k=1}^n \frac{1}{k!} \sum_{\substack{\alpha\in\mathbb{N}^k\\ |\alpha|=n}} t^{k-1} a_{\alpha_1} \cdot \ldots \cdot a_{\alpha_k} = a_n = \pi_n \circ \mathrm{id}_{\mathcal{I}_A}(a) \;\square\end{aligned}$$

Lemma B.7 (Unit groups of densely graded algebras). — *Let $A = \prod_{n=0}^\infty A_n$ be a densely graded locally convex algebra.*

(a) *An element $a \in A$ with decomposition $a = a_0 + b$ is invertible in A if and only a_0 is invertible in A_0.*

(b) *The algebra A is a CIA if and only if A_0 is a CIA. In particular, if $A_0 = \mathbb{K}$, then A is a CIA.*

Proof.

(a) The map $\pi_0 \colon A \to A_0$ is an algebra homomorphism. This implies that invertible elements $a \in A$ are mapped to invertible elements $a_0 \in A_0$. For the converse, take an element $a \in A$ with decomposition $a = a_0 + b$ with $b \in \mathcal{I}_A$ and a_0 is invertible in A_0. Then we may multiply by a_0^{-1} from the left and obtain the equality
$$a_0^{-1}a = 1 + a_0^{-1}b.$$
This shows that a is invertible if we are able to prove that $1 + a_0^{-1}b$ is invertible. Apply the formal power series
$$(1 - X)^{-1} = \sum_{k=0}^{\infty} X^k$$
to the element $-a_0^{-1}b \in \mathcal{I}_A$ and obtain the inverse of $1 + a_0^{-1}b$.

(b) We have seen in part (a) that the units in the algebra A satisfy
$$A^\times = \pi_0^{-1}(A_0^\times)$$
and hence one of the unit groups is open if and only if the other one is open. It remains to establish that continuity of inversion in $A_0^\times$ implies continuity of inversion in $A^\times$. In part (a) we have seen that inversion of $a = a_0 + b$ in A is given by
$$a^{-1} = \left((1 - X)^{-1}\right)\left[-a_0^{-1}b\right] \cdot a_0^{-1}$$
So, the continuity of inversion in A follows from the continuity of inversion in A_0 and the continuity of the functional calculus (Lemma B.4). □

Lemma B.8. — *Let $A = \prod_{n \in \mathbb{N}_0} A_n$ be a densely graded locally convex algebra over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. Then $\Lambda := \mathbb{K}1_{A_0} \times \prod_{n \in \mathbb{N}} A_n \subseteq A$ is a closed subalgebra of A. Furthermore, Λ is densely graded with respect to the grading induced by $(A_n)_{n \in \mathbb{N}_0}$ and Λ is a CIA.*

Proof. — Clearly Λ is a subalgebra of A and the subspace topology turns this subalgebra into a locally convex algebra over $\mathbb{K}$. By definition Λ is the product of the (closed) subspaces $(\Lambda_n)_{n \in \mathbb{N}_0}$. Hence Λ is a closed subalgebra of A with dense grading $(\Lambda_n)_{n \in \mathbb{N}_0}$. Finally we have the isomorphism of locally convex algebras $\Lambda_0 = \mathbb{K}1_{A_0} \cong \mathbb{K}$. Hence Λ_0 is a CIA and thus Λ is a CIA by Lemma B.7(b). □

Auxiliary results concerning characters of Hopf algebras

Fix for this section a $\mathbb{K}$-Hopf algebra $\mathcal{H} = (\mathcal{H}, m_{\mathcal{H}}, u_{\mathcal{H}}, \Delta_{\mathcal{H}}, \varepsilon_{\mathcal{H}}, S_{\mathcal{H}})$ and a commutative locally convex algebra B. Furthermore, we assume that $\mathcal{H}$ is graded and connected, i.e. $\mathcal{H} = \bigoplus_{n\in\mathbb{N}_0} \mathcal{H}$ and $\mathcal{H}_0 \cong \mathbb{K}$. The aim of this section is to prove that the exponential map $\exp_A$ of $A := \mathrm{Hom}_{\mathbb{K}}(\mathcal{H}, B)$ restricts to a bijection from the infinitesimal characters to the characters.

Lemma B.9 (Cocomposition with Hopf multiplication). — *Let $\mathcal{H} \otimes \mathcal{H}$ be the tensor Hopf algebra (cf.* [25, p. 8]*). With respect to the topology of pointwise convergence and the convolution product, the algebras*

$$A := \mathrm{Hom}_{\mathbb{K}}(\mathcal{H}, B) \qquad A_\otimes := \mathrm{Hom}_{\mathbb{K}}(\mathcal{H} \otimes \mathcal{H}, B)$$

become locally convex algebras (see Lemma 1.4). This structure turns

$$\cdot \circ m_{\mathcal{H}} \colon \mathrm{Hom}_{\mathbb{K}}(\mathcal{H}, B) \to \mathrm{Hom}_{\mathbb{K}}(\mathcal{H} \otimes \mathcal{H}, B),\ \phi \mapsto \phi \circ m_{\mathcal{H}}.$$

into a continuous algebra homomorphism.

Proof. — From the usual identities for the structure maps of Hopf algebras (cf. [25, p. 7 Fig. 1.3]) it is easy to see that $\cdot \circ m_{\mathcal{H}}$ is an algebra homomorphism. Clearly $\cdot \circ m_{\mathcal{H}}$ is continuous with respect to the topologies of pointwise convergence. □

Lemma B.10. — *The analytic diffeomorphism $\exp_A \colon \mathcal{I}_A \to 1+\mathcal{I}_A$ maps the set of infinitesimal characters $\mathfrak{g}(\mathcal{H}, B)$ bijectively onto the set of characters $G(\mathcal{H}, B)$.*(7)

Proof. — We regard the tensor product $\mathcal{H}\otimes\mathcal{H}$ as a graded and connected Hopf algebra with respect to the tensor grading, i.e.

$$\mathcal{H} \otimes \mathcal{H} = \bigoplus_{n\in\mathbb{N}_0} (\mathcal{H} \otimes \mathcal{H})_n \text{ with } (\mathcal{H} \otimes \mathcal{H})_n = \bigoplus_{i+j=n} \mathcal{H}_i \otimes \mathcal{H}_j \text{ for all } n \in \mathbb{N}_0.$$

The set of linear maps $A_\otimes := \mathrm{Hom}_{\mathbb{K}}(H \otimes H, B)$ together with the convolution product $\star_{A_\otimes}$ forms a densely graded locally convex algebra. Let $m_B \colon B \otimes B \to B$, $b_1 \otimes b_2 \mapsto b_1 \cdot b_2$ be the multiplication in B. Define a bilinear map

$$\beta \colon A \times A \to A_\otimes,\ (\phi, \psi) \mapsto \phi \diamond \psi := m_B \circ (\phi \otimes \psi).$$

(7) It is hard to find a complete proof in the literature, whence we chose to include a proof for the reader's convenience.

We prove now that β is continuous. To this end consider a fixed element $c = \sum_{k=1}^n c_{1,k} \otimes c_{2,k} \in \mathcal{H} \otimes \mathcal{H}$. We have to prove that $\phi \diamond \psi(c)$ depends continuously on ϕ and ψ:

$$\begin{aligned}(\phi \diamond \psi)(c) &= m \circ (\phi \otimes \psi)(c) = m \circ (\phi \otimes \psi) \left(\sum_{k=1}^n c_{1,k} \otimes c_{2,k}\right) \\ &= \sum_{k=1}^n m\left(\phi(c_{1,k}) \otimes \psi(c_{2,k})\right) = \sum_{k=1}^n \phi(c_{1,k}) \cdot \psi(c_{2,k})\end{aligned}$$

This expression is continuous in (ϕ, ψ) since point evaluations are continuous as well as multiplication in the locally convex algebra B. The convolution in A can be written as $\star_A = \beta \circ \Delta$. We obtain

$$(\phi_1 \diamond \psi_1) \star_{A_\otimes} (\phi_2 \diamond \psi_2) = (\phi_1 \star_A \phi_2) \diamond (\psi_1 \star_A \psi_2). \tag{B.3}$$

Recall, that $1_A := u_B \circ \varepsilon_{\mathcal{H}}$ is the neutral element of the algebra A. From equation (B.3), it follows at once, that the continuous linear maps

$$\begin{aligned} &\beta(\cdot, 1_A)\colon A \to A_\otimes,\ \phi \mapsto \phi \diamond 1_A \\ \text{and} \quad &\beta(1_A, \cdot)\colon A \to A_\otimes,\ \phi \mapsto 1_A \diamond \phi \end{aligned} \tag{B.4}$$

are continuous algebra homomorphisms. We will now exploit $\diamond$ to prove that the bijection $\exp_A\colon \mathcal{I}_A \to 1_A + \mathcal{I}_A$ (see Lemma B.5) maps the set $\mathfrak{g}(H,B)$ onto $G(H,B)$. Let $\phi \in \mathcal{I}_A$ be given and recall:

(a) The Hopf algebra product $m_{\mathcal{H}}$ maps $\mathcal{H}_0 \otimes \mathcal{H}_0$ into $\mathcal{H}_0$. Now $\mathcal{H}_0 \otimes \mathcal{H}_0 = (\mathcal{H} \otimes \mathcal{H})_0$ (tensor grading) entails for $\phi \in \mathcal{I}_A$ that $\phi \circ m_{\mathcal{H}} \in \mathcal{I}_{A_\otimes}$.

(b) From (B.3) we derive that

$$(\phi \diamond 1_A) \star_{A_\otimes} (1_A \diamond \phi) = \phi \diamond \phi = (1_A \diamond \phi) \star_{A_\otimes} (\phi \diamond 1_A).$$

If $\phi \in A$ is an infinitesimal character then $\phi \circ m_{\mathcal{H}} = \phi \diamond 1_A + 1_A \diamond \phi$.

Combining (a) and (b) we see that Lemma B.6 is applicable and as a consequence

$$\exp_{A_\otimes}(\phi \diamond 1_A + 1_A \diamond \phi) = \exp_{A_\otimes}(\phi \diamond 1_A) \star_{A_\otimes} \exp_{A_\otimes}(1_A \diamond \phi). \tag{B.5}$$

Note that it suffices to check multiplicativity of $\exp_A(\phi)$ as $\exp_A(\phi)(1_{\mathcal{H}}) = 1_B$ is automatically satisfied. To prove the assertion we establish the following equivalences:

$$\begin{aligned}
\phi \in \mathfrak{g}(\mathcal{H},B) &\overset{\text{Def}}{\iff} \phi \circ m_{\mathcal{H}} = \phi \diamond 1_A + 1_A \diamond \phi \\
&\overset{\text{(a)}}{\iff} \exp_{A_\otimes}(\phi \circ m_{\mathcal{H}}) = \exp_{A_\otimes}(\phi \diamond 1_A + 1_A \diamond \phi) \\
&\overset{\text{(B.5)}}{\iff} \exp_{A_\otimes}(\phi \circ m_{\mathcal{H}}) = \exp_{A_\otimes}(\phi \diamond 1_A) \star_{A_\otimes} \exp_{A_\otimes}(1_A \diamond \phi) \\
&\overset{\text{(B.4)}}{\iff} \exp_{A_\otimes}(\phi \circ m_{\mathcal{H}}) = \big(\exp_A(\phi) \diamond 1_A\big) \star_{A_\otimes} \big(1_A \diamond \exp_A(\phi)\big) \\
&\overset{\text{(B.3)}}{\iff} \exp_{A_\otimes}(\phi \circ m_{\mathcal{H}}) = \big(\exp_A(\phi) \star_A 1_A\big) \diamond \big(1_A \star_A \exp_A(\phi)\big) \\
&\iff \exp_{A_\otimes}(\phi \circ m_{\mathcal{H}}) = \exp_A(\phi) \diamond \exp_A(\phi) \\
&\overset{\text{B.9}}{\iff} \exp_A(\phi) \circ m_{\mathcal{H}} = \exp_A(\phi) \diamond \exp_A(\phi) \\
&\overset{\text{Def}}{\iff} \exp_A(\phi) \in G(\mathcal{H}, B)
\end{aligned}$$

□

Remark B.11. — The chain of equivalences in the proof of Lemma B.10 uses the dense grading of $A = \mathrm{Hom}_{\mathbb{K}}(\mathcal{H}, B)$ twice to show that the first to third lines are equivalent. However, for arbitrary $\mathcal{H}$ and weakly complete B the second and third line are still equivalent: In this case $A_\otimes$ is weakly complete and Lemma 5.3(c) allows us to embed $A_\otimes$ into $P := \prod_{i \in I} A_i$ (product of Banach algebras in the category of topological algebras). This implies that the formula $\exp_{A_\otimes}(a + b) = \exp_{A_\otimes}(a) \exp_{A_\otimes}(b)$ used in (B.5) still holds as the power series defining $\exp_A$ converges on P and satisfies the formula (which is component-wise true in every Banach algebra).

Appendix C. Weakly complete vector spaces and duality

The purpose of this section is to exhibit the duality between the category of abstract vector spaces and the category of weakly complete topological vector spaces. Although none of this is needed for the results of this paper, many ideas of this paper appear to be more natural in this wider setting.

Throughout this section, let $\mathbb{K}$ be a fixed Hausdorff topological field. Although, in this paper, we are only interested in the cases $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$, the statements in this appendix hold for an arbitrary Hausdorff field of any characteristic, including the discrete ones.[8] We start with a definition.

Definition C.1. — *A topological vector space E over the topological field $\mathbb{K}$ is called* weakly complete topological vector space *(or* weakly complete space *for short) if one of the following equivalent conditions is satisfied:*

(a) *There exists a set I such that E is topologically isomorphic to $\mathbb{K}^I$.*
(b) *There exists an abstract $\mathbb{K}$-vector space $\mathcal{V}$ such that E is topologically isomorphic to $\mathcal{V}^* := \mathrm{Hom}_{\mathbb{K}}(\mathcal{V}, \mathbb{K})$ with the weak*-topology*
(c) *The space E is the projective limit of its finite-dimensional Hausdorff quotients, with each n-dimensional quotient being topologically isomorphic to $\mathbb{K}^n$*

For the case $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, these conditions are also equivalent to the following conditions:

(d) *The space E is locally convex and is complete with respect to the weak topology.*
(e) *The space E is locally convex, it carries its weak topology and is complete with this topology.*

The proof that (a) $\iff$ (b) $\iff$ (c) can be found in [18, Appendix 2]). The characterisations (d) and (e) justify the name *weakly complete*.

Remark C.2. — Part (b) of the preceding definition tells us that the *algebraic dual* $\mathcal{V}^*$ of an abstract vector space $\mathcal{V}$ becomes a weakly complete topological vector space with respect to the weak*-topology, i.e. the topology of pointwise convergence.

Conversely, given a weakly complete vector space E, we can consider the *topological dual* E' of all *continuous* linear functionals. Although there are many vector space topologies on this topological dual, we will always take E' as an abstract vector space.

One of the main problems when working in infinite-dimensional linear (and multilinear) algebra is that a vector space $\mathcal{V}$ is no longer isomorphic to its bidual $(\mathcal{V}^*)^*$. The main purpose of this section is to convince the reader that the reason for this bad behaviour of the bidual is due to the fact that the wrong definition of a bidual is used (at least for infinite-dimensional spaces).

[8] In functional analysis, usually only $\mathbb{R}$ and $\mathbb{C}$ with their usual field topologies are considered, where in algebra usually an arbitrary field with the discrete topology is considered. Our setup includes both cases (and many more, e.g. the p-adic numbers, etc.).

If we start with an abstract vector space $\mathcal{V}$, then its dual is a weakly complete space $\mathcal{V}^*$ and consequently, one should *not* take the algebraic dual $(\mathcal{V}^*)^*$ but the topological dual $(\mathcal{V}^*)'$ which is the natural choice. For a finite-dimensional space the construction coincides with the usual definition of the bidual. In the general case however, the so obtained bidual is now canonically isomorphic to the original space as the following proposition shows:

Proposition C.3 (Duality and Reflexivity). — *Let E be a weakly complete space and let $\mathcal{V}$ be an abstract vector space. There are natural isomorphisms*

$$\begin{aligned} \eta_E\colon E &\longrightarrow (E')^* \\ x &\longmapsto (\eta_E(x) := \phi_x\colon E' \to \mathbb{K},\ \lambda \mapsto \lambda(x)) \end{aligned}$$

and

$$\begin{aligned} \eta_{\mathcal{V}}\colon \mathcal{V} &\longrightarrow (\mathcal{V}^*)' \\ v &\longmapsto (\eta_{\mathcal{V}}(v) := \lambda_v\colon \mathcal{V}^* \to \mathbb{K},\ \phi \mapsto \phi(v)). \end{aligned}$$

Proof (Sketch). — Let E be a weakly complete vector space. We may assume that $E = \mathbb{K}^I$ for a set I. Then each projection map $\pi_i\colon \mathbb{K}^I \to \mathbb{K}$ on the i-th component is an element in E'. It is easy to see that $(\pi_i)_{i\in I}$ is in fact a basis of the abstract vector space E'. This means that the algebraic dual of E' is topologically isomorphic to $\mathbb{K}^I$. Using this identification, one can check that the map η_E is the identity.

Similarly, let $\mathcal{V}$ be an abstract vector space. By Zorn's Lemma, pick a basis $(b_i)_{i\in I}$. Then the dual space $\mathcal{V}^*$ is isomorphic to $\mathbb{K}^I$. And therefore, the dual of that one $(\mathcal{V}^*)'$ has a basis $(\pi_i)_{i\in I}$. Under this identification, the linear map $\eta_{\mathcal{V}}$ is the identity. □

Definition C.4 (The weakly complete tensor product). — *One way to understand Proposition C.3 is that every element x in a weakly complete space E can be identified with a linear functional $\phi_x = \eta_E(x) \in (E')^*$ on the abstract vector space E'. This enables us to define a tensor product of two elements $x \in E$ and $y \in F$ as the tensor product of the corresponding linear functionals*

$$\begin{aligned} x \otimes y := \phi_x \otimes \phi_y\colon E' \otimes F' &\longrightarrow \mathbb{K} \\ \lambda \otimes \mu &\longmapsto \phi_x(\lambda)\cdot\phi_y(\mu) = \lambda(x)\cdot\mu(y). \end{aligned}$$

This element $x \otimes y$ is now a linear functional on the abstract vector space $E' \otimes F'$ which motivates the definition:

$$E \mathbin{\widetilde{\otimes}} F := (E' \otimes F')^*.$$

If the spaces E and F are of the form $E = \mathbb{K}^I$ and $F = \mathbb{K}^J$, it is easy to verify that the space $\mathbb{K}^I \mathbin{\widetilde{\otimes}} \mathbb{K}^J = (E' \otimes F')^*$ is canonically isomorphic to $\mathbb{K}^{I\times J}$. This could have been taken as the definition of the weakly complete tensor product in the first place. However, the definition we chose has the advantage that is independent of the choice of coordinates, i.e. the specific isomorphisms $E \cong \mathbb{K}^I$ and $F \cong \mathbb{K}^J$, respectively.

Proposition C.5 (The universal property of the weakly complete tensor product). — *Let E, F, H be weakly complete spaces and let $\beta\colon E\times F \to H$ be a continuous bilinear map. Then there exists a unique continuous linear map $\beta^\sim\colon E \mathbin{\widetilde{\otimes}} F \to H$ such that the following diagram commutes:*

$$\begin{tikzcd} E\times F \arrow[r, "\beta"] \arrow[d, "\otimes"'] & H \\ E \mathbin{\widetilde{\otimes}} F \arrow[ur, "\beta^\sim"'] & \end{tikzcd}$$

For the case $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$, this universal property also holds for arbitrary complete locally convex spaces H, showing that this weakly complete tensor product is just a special case of the usual projective tensor product *for locally convex vector spaces.*

Definition C.6 (The monoidal categories $\mathbf{WCVS}_{\mathbb{K}}$ and $\mathbf{VS}_{\mathbb{K}}$).

(i) Denote the category of weakly complete spaces and continuous linear maps by $\mathbf{WCVS}_{\mathbb{K}}$. Together with the weakly complete tensor product and the ground field $\mathbb{K}$ as unit object, we obtain a monoidal category $(\mathbf{WCVS}_{\mathbb{K}}, \mathbin{\widetilde{\otimes}}, \mathbb{K})$.

(ii) Denote the monoidal category of abstract vector spaces, abstract linear maps, the usual abstract tensor product and the ground field as unit object by $(\mathbf{VS}_{\mathbb{K}}, \otimes, \mathbb{K})$.

The two categories $\mathbf{WCVS}_{\mathbb{K}}$ and $\mathbf{VS}_{\mathbb{K}}$ are dual to each other. The dualities are given by the contravariant monoidal functors *algebraic dual*

$$\begin{aligned} (\cdot)^*\colon \mathbf{VS}_{\mathbb{K}} &\longrightarrow \mathbf{WCVS}_{\mathbb{K}} \\ \mathcal{V} &\longmapsto \mathcal{V}^* \\ (\Phi\colon \mathcal{V} \to \mathcal{W}) &\longmapsto (\Phi^*\colon \mathcal{W}^* \to \mathcal{V}^*,\, \phi \mapsto \phi\circ\Phi) \end{aligned}$$

and *topological dual*

$$\begin{aligned} (\cdot)'\colon \mathbf{WCVS}_{\mathbb{K}} &\longrightarrow \mathbf{VS}_{\mathbb{K}} \\ E &\longmapsto E' \\ (T\colon E \to F) &\longmapsto (T'\colon F' \to E',\, \lambda \mapsto \lambda\circ T) \end{aligned}$$

(cf. Proposition C.3). The duality interchanges direct sums in the abstract category with direct products in the weakly complete category, hence graded vector spaces (Definition B.1) are assigned to densely graded vector spaces (Definition B.2). For more information about this duality, we refer to [27, p. 679] and to [18, Appendix 2].

We can naturally define weakly complete algebras, weakly complete coalgebras, weakly complete bialgebras and weakly complete Hopf algebras using the weakly complete tensor product in $(\mathbf{WCVS}_{\mathbb{K}}, \widetilde{\otimes}, \mathbb{K})$. By duality, we get the correspondence:

Abstract world $(\mathbf{VS}_{\mathbb{K}}, \otimes, \mathbb{K})$	**Weakly complete world** $(\mathbf{WCVS}_{\mathbb{K}}, \widetilde{\otimes}, \mathbb{K})$
abstract vector space linear map graded vector space	weakly complete vector space continuous linear map densely graded weakly complete vector space
abstract coalgebra	weakly complete algebra
abstract algebra	weakly complete coalgebra
abstract bialgebra	weakly complete bialgebra
abstract Hopf algebra characters infinitesimal characters	weakly complete Hopf algebra group like elements primitive elements

Remark C.7. — Note that while a weakly complete algebra is an algebra with additional structure (namely a topology), a weakly complete coalgebra is in general *not* a coalgebra. This is due to the fact that the weakly complete comultiplication $\Delta\colon C \to C \widetilde{\otimes} C$ takes values in the completion $C \widetilde{\otimes} C$, while for a coalgebra it would be necessary that it takes its values in $C \otimes C$ and the canonical inclusion map $C \otimes C \mapsto C \widetilde{\otimes} C$ goes into the wrong direction (see also [27, p. 680]). In particular, a Hopf algebra in the weakly complete category is not a Hopf algebra in general.

Using the duality, we may translate theorems from the abstract category to the weakly complete category, for example the Fundamental Lemma of weakly complete algebras (Lemma 5.3) follows directly from the the fundamental theorem of coalgebras, stating that every abstract coalgebra is the direct union of its finite-dimensional subcoalgebras. It should be mentioned that one of the first proofs of the fundamental theorem of coalgebras by

Larson [23, Prop. 2.5] used this duality and worked in the framework of topological algebras to show the result about abstract coalgebras.

Let $\mathcal{H}$ be an abstract Hopf algebra and $H := \mathcal{H}^*$ the corresponding weakly complete Hopf algebra. Then the characters of $\mathcal{H}$ are exactly the group like elements in H, while the infinitesimal characters of $\mathcal{H}$ are exactly the primitive elements of H. This allows us to rephrase the scalar valued case of Theorem 2.7.

Theorem C.8 (Group like elements in a weakly complete Hopf algebra). *Let H be a densely graded weakly complete Hopf algebra over $\mathbb{R}$ or $\mathbb{C}$ with $H_0 = \mathbb{K}$. Then the group like elements of H form a closed Lie subgroup of the open unit group $H^\times$. The Lie algebra of this group is the weakly complete Lie algebra of primitive elements.*

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Part III.

Applications of the interplay between Lie theory and Riemannian geometry