MASTER´S THESIS

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Students' understanding of multiplication

A mixed method study done in grade 6

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The work with this master thesis is now over and also our five-year master's course. It has been an educational and content-rich journey. The work with this thesis has required a collaboration with each other that can be characterized by mutual trust, respect, and responsibility. We have learned lot from this journey, especially how important it is to support each other throughout the whole process, especially when the days are long and tough. We want to thank each other for the support, hard work, willpower and caring for each other during this journey, and that we always had the goal to make each other better. There are also other people who have helps us along the way to make this process possible to carry out and thus deserves a ‘thank you’.

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Now it is a new chapter for us as we are finished with our master’s degree and will go out in the working world to start contributing to children’s education. We are very excited.

Bodø, May 2023

Ingvild Reinfjord Ruth-Amalie Breckan

Abstract

Mastery of multiplication includes many components. These components are, in particular, subject knowledge of the operation multiplication, procedural knowledge to determine the value of products, the ability to apply it in contexts, and, last but not least, the ability to retrieve the basic facts of multiplication from memory. The aim of this study is to gain new insight and expand the knowledge of the concept multiplication among students in 6th grade; what do they understand, and what is missing. The research questions for this master thesis are as follow:

1. How far do students in grade 6 have a conceptual understanding of the operation multiplication?
2. How far can students in grade 6 apply their conceptual understanding of the operation multiplication to describe and solve problems in their everyday life?
3. How far can students in grade 6 compute the value of every product using different strategies?
4. How far can students in grade 6 retrieve the basic facts of multiplication from memory?

This mixed method study with a hermeneutic and pragmatic approach was carried out with, firstly, a quantitative worksheet with 40 students, secondly, a qualitative semi-structured interviews with 15 students focusing on the tasks in the worksheet. Our worksheet contains a 3-minute speed test with tasks about basic facts from multiplication with factors between 0-10. The second part contains tasks which challenges the students to show their strategies used when one factor is greater than 10. The last part is design towards levels of representations according to Bruner (1964, p. 2), including the iconic level, verbal-symbolic level, and non-verbal-symbolic level. This part focuses on revealing their conceptual understanding.

Results from this study reveals 6th graders have a lack of understanding the concept multiplication. The students that had a good conceptual understanding were able to show their understanding through the different levels of representation. Positively we found the students being able to use many different strategies when computing. Negatively we found only 20% being able to retrieve basic multiplication facts from memory, and students that only understand multiplication as repeated addition scores low for all the research questions.

Index

List of figures

1 Introduction

Mathematics is an important subject that all students meet in school. The function of mathematics education is not only to acquire mathematical competencies but also, above all, to develop students' personalities. The students develop an understanding of mathematical concepts through the subject, and this understanding is essential for problem-solving. Understanding is understood as connections between different types of representation, for instance the components of multiplication, models for multiplication, multiplicative calculations, and arithmetical properties. When the students show an understanding of these components, they can reason for the connections, and make mental representation. "Understanding multiplication is central to knowing mathematics" (Baek, 1998, p.151). The understanding of multiplication includes many aspects. It involves for instance the ability to reason multiplicatively. The students should justify for different calculation strategies and knowing what is the most efficient for every situation, explain different connection between operation and engage arithmetical properties (Nunes et al., 2009, p. 261; Young-Loveridge & Mills, 2009, p. 49). Without an understanding of mathematical concepts, they will meet difficulties regarding problem-solving. Therefore, it is very important that students develop a conceptual understanding, so they will become more robust when facing different mathematical problems.

The aim for this study is to understand more about students understanding of multiplication, how they use their knowledge to solve multiplication tasks, what strategies they use and if they are able to retrieve basic multiplication facts from memory.

1.1 Background and choice of topic

Multiplication and division are core contents of mathematics education both nationally and internationally. Competencies to be acquired by students concerning the operations of multiplication and division include basic knowledge, skills, and abilities whose importance results both from their use in situations of daily life and from their function as tools for mastering mathematics in later grades.

Mastery of mathematics always requires both conceptual and procedural knowledge. Students must first develop a comprehensive understanding of the content, even concerning multiplication and division. Based on this knowledge, skills can be acquired, abilities can be developed, and facts can be solidly memorized.

The learning objectives regarding the operation multiplication are that students:

- have a conceptual understanding of the operation multiplication,
- apply this conceptual understanding to describe and solve problems in their everyday life or in other subjects,
- can compute the value of every product using different strategies, and
- retrieve the basic multiplication facts directly from memory.

(e.g., Kling & Bay-Williams, 2015, CCSSI, 2010, p. 23)

At present, it is unknown precisely to what extent the stated goals regarding mastery of multiplication and division are achieved in grade 6 in Norway. Students in grade 6 have acquired in previous school years competencies regarding the operations of multiplication and division and are now challenged to apply them, e.g., when working with fractions. This school year is therefore of special interest for our research. Moreover, there is a lack of research about this topic in Norway (Ostad, 2013, p. 54). We have experienced students having problems applying the basic tasks of multiplication, for example, when they are supposed to find the value of terms like 23×7 or $117 : 3$, or shorten fractions. The results from PISA and TIMSS show that the mathematical domain "Number and Number Calculations" is the domain in mathematics which students in grade 1-7 are struggling most with and score lowest in (Grønmo et.al., 2017). These deficits require significant attention, further investigation, and conclusions for improvement because it concerns a domain that is a foundation for all further learning in mathematics.

One of the essential goals in mathematics is for students to acquire competencies regarding basic arithmetic operations. This applies especially to conceptual understanding of operations, developed skills and abilities to perform operations, and memorized basic facts. Students' competencies are a reflection and result of teaching. By identifying the students' competencies, we can address a possible need for a shift in mathematical teaching. Ojose states that schools in the United Kingdom have failed in making mathematical literate citizens, and there is a need for better mathematical understanding among citizens. This can also be the case in Norway. People need mathematical fluency in their daily life to interact in society (Ojose, 2011, p. 99). Examples of the use of mathematical competencies are estimating and extrapolation, for example when shopping. Examples are also recording, analyzing, and evaluating quantitative data, for example in diagrams, in statistics or in charts. School, home, and society are factors that influence citizens' mathematical literacy. If teachers conduct this responsibility in a good way, the more mathematical literate people will be.

Gleissberg and Eichler (2019, p. 2) states three observed problems in secondary school level in Germany, all grounded in students' lack of conceptual understanding of multiplication:

The first problem is that students cannot retrieve the results of multiplication tasks with two factors less than or equal to ten from memory. This becomes evident, for example, when students cannot recognize common factors in a fraction and, therefore, cannot shorten the fraction. The second problem is that students often have difficulties determining the value of products with a factor greater than ten. Third, students often have difficulties justifying the correctness of an equation like $5 \times 4 = 20$ because they do not know the meaning of the term 5×4. Students often argue that 5×4 is an abbreviation for the addition task $4 + 4 + 4 + 4 + 4$. In this project, we will explore these three problems and further investigate the reasons for students' lack of conceptual understanding.

1.2 Research questions and their justification

Terms of multiplication are concepts; they are mental reflections of classes of real situations. Consequently, students' conceptual understanding is the extent to which they are able to identify, realize, and systematize terms of multiplication. Therefore, a conceptual understanding of the operation multiplication means establishing and using the relations between terms and reality given by actions, images, or verbal descriptions. Students who manage the transfer between the enactive, iconic, verbal-symbolic, and non-verbal-symbolic level of representation, according to Bruner (1964, p. 2), have a conceptual understanding of the operation multiplication (Gleissberg & Eichler, 2019, p. 2). "*There seems to be an international consensus nowadays that students should both acquire a sound conceptual understanding of multiplication and eventually solve all basic tasks accurately and effortlessly*" (Gaidoschik, 2017, p. 347). Conceptual understanding means also to know and use the connections between operations and can integrate new facts into acquired knowledge.

Developing a conceptual understanding of multiplication is not a goal in itself. Students should be able to apply their conceptual understanding to describe and solve problems from daily life. Solving such problems requires that students can calculate the value of any product and efficiently use different multiplication strategies. The memorized knowledge of the basic facts of multiplication plays an important role. Retrieving the basic facts from memory is a prerequisite for successfully calculating term values such as 17×8, 70×80, etc. At the same time, memorizing the basic facts of multiplication is crucial regarding the work at the secondary level. Whether facts are retrieved from memory or constructed becomes evident by the amount of time needed from the task to the result. A student who calculates the value of a term like 6×8 as $5 \times 8 + 8$ or even $8 + 8 + 8 + 8 + 8 + 8$ require more time than a student who can immediately retrieve the fact $6 \times 8 = 48$.

The learning objectives regarding the operation multiplication, the problems mentioned, and the background presented above led us to the following research questions:

1. How far do students in grade 6 have a conceptual understanding of the operation multiplication?
2. How far can students in grade 6 apply their conceptual understanding of the operation multiplication to describe and solve problems in their everyday life?
3. How far can students in grade 6 compute the value of every product using different strategies?
4. How far can students in grade 6 retrieve the basic facts of multiplication from memory?

Due to the limited time, we restrict ourselves to the investigation concerning multiplication, and exclude division.

1.3 The structure of the thesis

For this master thesis we have structured the paper into different chapters. Chapter 2 clarifies the context of our thesis, by enlightening what is known about the subject and what is unknown. In chapter 3, we present the theoretical background of our thesis. The theoretical background of the present thesis firstly concerns the mathematical foundations regarding multiplication and its properties, secondly the objective goals regarding the acquisition and application of knowledge and skills regarding multiplication, and thirdly the essentials and central theories of the acquisition of the operation multiplication like Bruner (1964) and Anghileri (1989). The theoretical background presented will also be a tool for the discussion about the result in our paster project.

In chapter 4, we present our method in general and a few relevant philosophical aspects of our method too. In this chapter we present and justify how we selected our informants, which data we collected, and how we analyzed the data collected. A third part of this chapter is the discussion of the reliability and validity of our data as well as a discussion of some ethical considerations. In chapter 5, we present our findings from the worksheet and the interviews. In chapter 6, we discuss our findings considering our theory. In the last chapter, we present some conclusive remarks.

2 Context of our study

2.1 Importance of multiplication and the need for research

Mathematical education is unquestionably crucial. Together with reading and writing, calculating is considered as one of humankind's cultural techniques. Like all education, the importance of mathematical education results from two aspects. Firstly, the material aspect reflects the importance of mathematical competencies as a tool for solving problems in work, school, and everyday life. The second aspect is the formal aspect, which reflects that acquiring mathematical content develops the learner's personality.

The Educational Act of Norway reflects the importance of mathematics education. As teachers, we are obligated to follow what is written in the Educational Act of Norway, as it is part of the Norwegian law. §1-1 in the law states that the education must open doors for the students to encounter the world, as well as they need the skill to promote democracy, equality and scientific thinking (Opplæringslova, 1998, §1-1). A sufficiently high level of mathematical education is a precondition for achieving this goal. A strategy published by the Norwegian Department of Education and Research called "Close on mathematics and Science" had the goal to improve Norwegian students' competencies in mathematics and science. The Norwegian Government has tried for several years to improve students' mathematics and science competencies without achieving significant success (Kunnskapsdepartementet, 2015, pp. 11-12). People use mathematics in both work and everyday life. Low numeracy is linked with unemployment, poor health, and poor wages. If people know how to use mathematics in everyday life, they can use it as a tool to get a better way of living (National Numeracy, u.d.).

Arithmetic is a fundamental part of mathematics that includes the basic arithmetic operations with natural numbers and their properties as a basis. The basic arithmetic operations are addition, subtraction, multiplication, and division. In the early grades, students learn numbers and computing up to 100. The learning trajectory from the introduction of natural numbers until the calculation with fractions starts in kindergarten. Children have many experiences with different situations. They are able to describe situations with including numbers, from kindergarten. These experiences are context-based stories that reflect operations. In grades 1 and 2, they learn numbers and the basic operations addition and subtraction (Kunnskapsdepartementet, 2023). In year 3, the competence aims introduces multiplication and division. When the students start year 5, the competence aims introduces fractions (ibid).

Throughout the years, there are indicators that show strategies and quality in mathematical knowledge have coherence (Ostad, 2013, p. 52). Ostad states that there is a lot of research done comparing strategies in addition and subtraction with the quality of mathematical knowledge. However, when it comes to multiplication, there is a lack, especially in Norway. Previous research in Norway about strategies regarding basic multiplication facts concludes that most of the students in the research could not retrieve basic multiplication facts from memory (Ostad, 2013, pp. 54-59).

The Norwegian Ministry of Education and Research established a new Norwegian curriculum in 2020. With this, changes have been made in the school curriculum of mathematics. With this new curriculum, there is a shift in the goals of mathematics. Several new core elements have been introduced, which we also encounter in our research. The core element “mathematical areas of knowledge,” states that students must develop a good understanding of the concepts of numbers and operations and develop a variety of strategies when computing from an early age. This will help them have a good prerequisite when modelling and generalize in mathematics. The core element “reasoning and argumentation” focuses on the importance that the student learns to use mathematical argumentations and explanations by solving mathematical problems. Argumentation in mathematics means that the students can justify procedures, reasoning, and solutions, and prove that these are valid (Kunnskapsdepartementet, 2019). The new curriculum was implemented for 1st to 9th grade in the schoolyear 2020-2021(Kunnskapsdepartementet, 2023). This means that the 6th graders in our research have been working with the old curriculum, LK06, in grade 1, 2 and 3, and the new curriculum, LK20, in grade 4, 5 and currently year 6. LK20 is different from LK06 by having a much bigger focus on in-depth learning and interdisciplinarity. The core elements of the subject are also a focus in the newest curriculum.

The Norwegian curriculum of mathematics states multiplication as an important part in the mathematics education. Multiplication is one of the content related competence goals of grade 3 (Kunnskapsdepartementet, 2019). On one hand, the acquisition of competencies in multiplication has an independent function and enables students to describe and solve elementary multiplicative situations. The acquired knowledge, skills, and abilities are also a basis for solving a lot of other word problems with multiplicative contexts as well as for mathematics in higher grades. They are also necessary to solve problems fluently without repeatedly spending time on elemental tasks.

Addition and subtraction are the first two operations students learn. The usage of repeated addition can then be a start when calculating the value of a product. Often whole-number multiplication is introduced as repeated addition (Izsák, 2004, p. 40). Nevertheless, it is known that repeated addition is not suitable as a conceptual basis for multiplication, because the connection between terms and contexts is hidden for the students (Park & Nunes, 2001, p. 771). To be able to use and fully understand the concept of multiplication, a student should progress beyond the repeated addition strategy of multiplication. It is necessary to reconceptualize their thinking and get an understanding of the multiplicative concept. Multiplicative thinking is the capacity to work flexibly with the representation multiplication (and division). Siemon et al. (2012, in Hurst & Linsell, 2020, p. 5) characterize multiplicative thinking as the capacity to:

- Work flexibly with a variety of numbers, as large and small whole numbers, decimals, fractions, ratio, and percentage.
- Work conceptually with whole numbers and decimals in a range of representations.
- Show a conceptual understanding of multiplication situations, the relationship between multiplication and division, numbers of equal groups, factors and multiples, and the various properties of multiplication.

The Norwegian curriculum takes into account the objectively necessary level of mastery of multiplication (Kunnskapsdepartementet, 2019). As we will now show, there is a gap between the expectations and requirements of the curriculum and the results achieved in teaching. We now discuss studies that reflect the unsatisfactory results of teaching. These studies arouse our curiosity and motivate us to examine the inadequacies in more detail using the operation multiplication as an example.

Data shows when students start high school, they do not have the mathematical knowledge acquired till grade 10 to master mathematics at a higher level (Kunnskapsdepartementet, 2015, pp. 11-12). Of all the students who had exams in mathematics in grade 10, 40 % got a grade 1 or 2 (where 6 is the highest). As mentioned, a low understanding of mathematics can have severe consequences for later education and daily life. This is why the national strategy from 2015-2019 focused on improving mathematics and science skills in kindergarten and school (ibid, pp.11-12). This governmental strategy might then have given the students in the project a benefit of having a good mathematical education since it was a focus in their early school years.

PISA, or Programme for International Student Assessment, is an international assessment which evaluates 15-year-old students' understanding in reading, mathematics, and science. In the PISA assessment 2012, there was found a coherence between mastering mathematics, reading, science, and the ability to solve problems. Students who scored high in problem-solving also scored high in mathematics, science, and reading (Kjærnsli et al., 2014). Each time there is a PISA assessment, the focus subject switches. The last time there was an assessment in mathematics, was in 2012. Norwegian students have improved their mathematical skill since 2012, but the report from 2012 presents what mathematical domain Norwegian students master and not. The Norwegian 15-year-olds score the highest mean scores in interpreting mathematical problems and solutions. They are strongest in the content of uncertainty and data, which covers two closely related sets of issues: how to identify and summarize the messages that are embedded in sets of data presented in different ways, and how to appreciate the likely impact of the variability that is inherent in many real processes. The students have the lowest mean score in formulation situations mathematically and in employing mathematical concepts, facts and procedures and reasoning. They are also weakest in the content category of change and relationships (OECD, 2012, pp. 1-2). All these mathematical domains include also multiplicative understanding.

2.2 Previous research

As for our research to investigate in students understanding in multiplication, there is a need to present what previous research about this subject has shown, and what is unknown. In this sub-chapter we will present some relevant research on the topic.

Research done by Gleissberg and Eichler (2019) shows that the range of textbook tasks in grade 2 and 3 in Germany which supports conceptual understanding is insufficient in both grades. The transfer between the levels of representation, according to Bruner (1964, p. 2), can be used to develop the concepts of “product” and “multiplication” as well as justify the value of a term. Levels of representation according to Bruner (1964, p. 2) can be used when solving multiplication. The result of the study shows that most tasks are in the non-verbal symbolic level, and that there is a lack of tasks that required the transfer between the enactive, the iconic, the non-verbal-symbolic, and the verbal-symbolic level of representation. It is also shown that students struggle with word problems. The authors of the article raise an issue due to the representation of multiplication among students. The teachers need to use additional tasks from other resources to support students to develop conceptual understanding of multiplication. The way multiplication is represented can make a big impact on the students' number understanding and problem solving (Gleissberg & Eichler, 2019, p. 2). It is important that all the levels are represented through the tasks students are working on. Teachers all over the world mostly rely on textbooks during a mathematical lesson, therefore the books should provide with sufficient tasks which will help the students conceptual understanding (ibid, p. 3).

A part of our study will investigate students’ strategies when computing. Study done by Zhang et al. in 2014 investigated in multiplication strategy development with students with math difficulties in the United States. This study had the purpose of examine the effectiveness of a strategic training program for improving students in grade 3 performance when solving multiplication problems. The results of the study showed that the participating students increased their strategic development in multiplication during and after the study. It was based on the theory that the students themselves chose and argument for their own chosen strategy when computing, and thereafter make connections to more efficient and less time-consuming strategies. These results also revealed that it is important to carefully select task which fits the students (Zhang et al., 2014, pp. 25-27).

Mabbott and Bisanz carried out research in 2003 in Canada which investigated the differences in students’ knowledge and skills in multiplication from grade 4 and 6. The computational skills, conceptual knowledge, and working memory was measured in the two grades. This study showed that the use of retrieving from memory as a strategy when computing with factors less than ten was used by 88% of the children in grade 6 and by 67% of the children in grade 4. 99% of the children in grade 6 and 96% of the ones in grade 4 used it correctly. This shows that along the schooling, the memorization and performances in multiplication improves over ages (Mabbott & Bisanz, 2003, p. 1097). We therefore expect students in grade 6 are able to retrieve basic multiplication facts from memory.

As seen on this presented research, most if the studies have been carried out in lower grades in school and in other countries than Norway. This shows that it is still unknown to what extent the multiplication understanding among 6th graders in Norway is. Our study with the goals to investigate students' ability to retrieve from memory, strategy used when computing, and how far their conceptual understanding is, is necessary. This study is carried out in Norway and will add more research to help investigate in children's understanding in multiplication.

3 The theoretical background of our work

To be able to answer our research questions, we must give definitions of what it means to master the questions we ask. To investigate if the students in the project have mastered the research questions or not will be determined by the meaning and measuring we put in the terms used.

Our master thesis investigates students understanding of the concept multiplication. To get a comprehensive and nuanced description of the learning process for the students, we will include the relevant theoretical framework that is consistent with the studied phenomenon, and which is about understanding of mathematical concepts.

3.1 Mathematical background

3.1.1 Operation Multiplication

All two-digit operations * in $\mathbb{N}$ uniquely assign one natural number to each ordered pair of $\mathbb{N} \times \mathbb{N}$ which they are defined. $[a; b] \in \mathbb{N} \times \mathbb{N} \longrightarrow c \in \mathbb{N}$. Thus, every two-digit operation in $\mathbb{N}$ is at the same time a three-digit relation in $\mathbb{N}$, namely a subset of $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$:

$O = \{[a; b; c] \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: a * b = c\}$

Accordingly, the operation multiplication is a two-digit operation that uniquely assigns to each ordered pair $[a; b] \in \mathbb{N} \times \mathbb{N}$ the product c of the numbers a and b. The operation multiplication is, at the same time, a three-digit relation in $\mathbb{N}$:

$O_M = \{[a; b; c] \in \mathbb{N} \times \mathbb{N} \times \mathbb{N}: a \cdot b = c\}$ where c is the product of a and b.

Thus, multiplication can be conceived once as a unique assignment with c = a · b and likewise as a three-digit relation: c is the (uniquely determined) product of a and b. This definition of the concept "multiplication in ℕ" presupposes the definition of the concept "product of two natural numbers."

3.1.2 Definition of the concept "product" - using sets

If the natural numbers are considered as cardinal numbers, i.e., as classes of finite sets, the product of two numbers can be explained by the so-called "combinatorial definition":

Def. 1: Let a and b be any two natural numbers represented by the sets A and B, i.e., a = | A | and b = | B |, then a × b = | A × B | is the product of the numbers a and b.

The name "combinatorial definition" refers to the applications of this definition.

With recourse to the natural numbers as cardinal numbers, the product of two numbers, a and b, can also be explained by the union of pairwise disjoint sets with the cardinality b.

Def. 2: Let a and b be any two natural numbers and B_1; B_2; B_3 ... B_a sets with
$| B_1 | = | B_2 | = | B_3 | = ... = | B_a | = b$,

where the sets B_x are chosen such that $\forall$ i, k $\in \mathbb{N}$ (i ≠ k): $B_i \cap B_k = \emptyset$.

Then $a \times b = | B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_a |$ is the product of the numbers a and b.

3.1.3 Properties of the operation multiplication and their application

Multiplication has several useful properties, especially in calculating the value of a term a × b. In this subsection, we introduce essential properties and show, illustrated by examples, why applying these properties is useful. We also show how to visualize these properties because visualization aids student insight.

Commutative law: $\forall\, a, b \in \mathbb{N}: a \times b = b \times a$

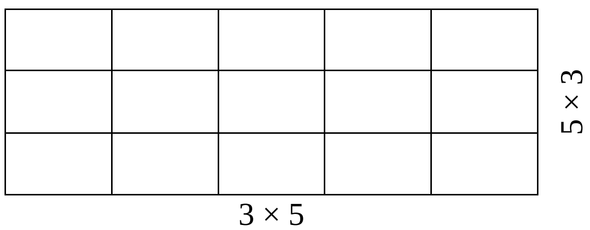


Figure 1: Commutative law

According to this property, the value of a product does not depend on the order of the factors, e.g.,

3 × 5 = 5 × 3. It is easy to see with this visualization above (Hinna et.al., 2016, p. 86).

Teachers mostly introduces the order of a mathematical expression as $a \times b$ meaning you have $a\ times\ b$. In a lot of countries, like England, Germany, and Norway, e.g., the meaning of 3×4 is mostly 3 fours. When students understand the commutative law of multiplication, the order of the factors does not necessarily need to be as important as when they are first introduced to multiplication. The meaning of the multiplication symbol will be less associated with the first interpretation a person has of it (Anghileri, 1989, p. 368).

Associative law: $\forall\ a, b, c \in \mathbb{N}: a \times (b \times c) = (a \times b) \times c$

Associativity means that if an expression contains the same associative operator two or more times, the order in which the operations are performed does not matter. That means rearranging the parentheses in an expression with an associative operator will not change the value of this expression.

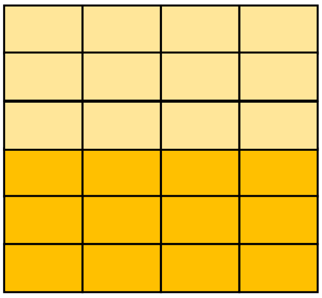


Figure 3: Associative law

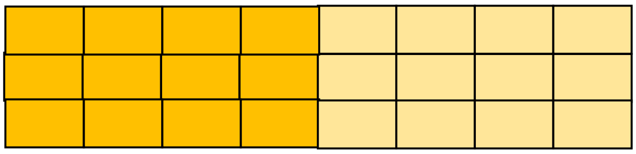


Figure 2: Associative law

Applying this law makes computing easier:

$$4 \times 28 = (2 \times 2) \times 28 = 2 \times (2 \times 28) = 2 \times 56 = 112$$

$$25 \times 36 = 25 \times (4 \times 9) = (25 \times 4) \times 9 = 100 \times 9 = 900$$

Distributive law: $\forall\ a, b, c \in \mathbb{N}: (a + b) \times c = (a \times c) + (b \times c)$

$$\forall\ a, b, c \in \mathbb{N}: (a - b) \times c = (a \times c) - (b \times c)$$

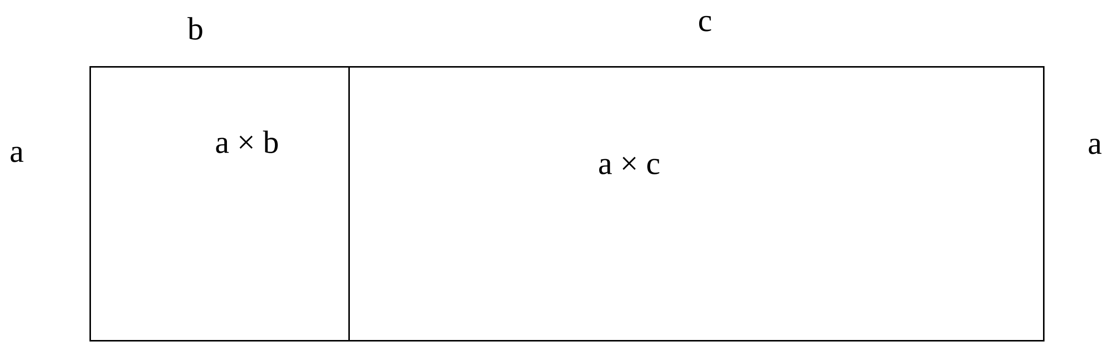


Figure 4: Distributive law

Applying this law makes computing easier:

$23 \times 7 = (20 + 3) \times 7 = 20 \times 7 + 3 \times 7$

$19 \times 7 = (20 - 1) \times 7 = 20 \times 7 - 1 \times 7$

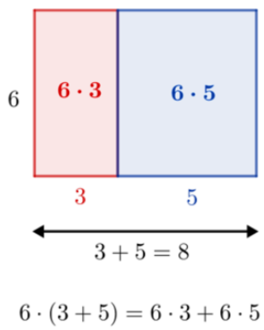


Figure 5: Distributive law expressed a connection between addition and multiplication.

For example, when calculating with two-digit numbers, it is possible to apply the distributive law to calculate the value of products whose two factors are sums of two numbers.

$$\forall\, a, b, c, d \in \mathbb{N}: (a + b) \times (c + d) = (a \times c) + (a \times d) + (b \times c) + (b \times d)$$

A lot of students are computing 32 × 25 = 610, because 32 × 25 = 30 × 20 + 2 × 5

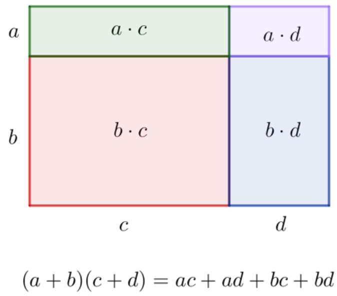


Figure 6: Distributive law illustrated when both factors have two terms.

A special case is: $(a+b)^2 = a^2 + 2ab + b^2$

One is the neutral element: $\forall\, a, b \in \mathbb{N}: (a+b)^2 = a^2 + 2ab + b^2$

Because of the importance of these properties the students are requested to apply these laws in our test: e.g., we expect them to be able to see connections between doubling and the associative law. We expect them to apply the distributive law when calculating tasks with factors over 10.

3.1.4 Axiomatic definition of the concept "product"

The Peano axioms are statements for the natural numbers presented by the mathematician Guiseppe Peano. The axioms are a set of basic statements considered to be true, which reflect what we all consider when counting correctly. The axioms are used as a foundation for a formalization of arithmetic and the basic properties of natural numbers. It contains five axioms,

(P1) $0 \in \mathbb{N}$

(P2) $\forall\, a \in \mathbb{N}: \exists!\, a' \in \mathbb{N}$

(P3) $\neg \exists a \in \mathbb{N}: a' = 0$

(P4) $\forall\, a, b \in \mathbb{N}: a' = b' \Longrightarrow a = b$

(P5) $[(S(0) \wedge (S(k) \Rightarrow S(k'))] \Longrightarrow \forall x \in \mathbb{N}: S(x)$

The axioms (P1) till (P4) describes the well-known process of counting, where every natural number x has one and only one successor x' and where from the successor of a number can also be concluded uniquely to the number.

The Peano axioms reflect the number in its property as a counting number, i.e., in its usability for counting things. Determining the value of a sum is possible by counting. Those who want to determine the value of $a + b$ must count on from the number a b steps.

Similarly, the value of a product can be determined by repeatedly adding the same summands. The following two axioms define the product of two natural numbers:

(M1) For all natural numbers b is: $0 \times b = 0$

(M2) For all natural numbers a and b is: $a' \times b = a \times b + b$

The use of M1 and M2 to determine the value of a product is shown as an example:
$2 \times 7 = 1' \times 7 = 1 \times 7 + 7 = 0' \times 7 + 7 = (0 \times 7 + 7) + 7 = 0 + 7 + 7$

Applying M1 and M2 as well as the addition it is proven that 2×7 is the same as $7 + 7$.

3.1.5 Aspects of multiplication

Multiplication is firstly an operation and is one of the basic operations in mathematics. To define what multiplication is one must see what the word means. Hinna et.al. defines:

"*Multiplication can be translated to diversification and experimentation. It is from the latin word* multiplication *which can be used with* multiplicare*, these two words means to multiplicate*" (Hinna et al., 2016, p. 81).

The most familiar situation with multiplication is tasks with a multiplicator and a multiplicand, the × or ·, where one should find the product. Multiplication goes beyond only simple calculation tasks; one also has the practical situations where repeated addition or combinations are multiplication. Anghileri (1989, p. 369) shows that there are several aspects of multiplication:

1. Repeated addition (using axioms)
2. Allocation/rate 1
3. Array /Area (connects sets and cartesian product)
4. Number line
5. Scale factor/rate 2
6. Cartesian product (using sets)

A more thorough explanation of the six aspects is defined in this paragraph. The repeated addition aspect counts how many sets, parts or groups that are involved and the size of each group, parts, or sets. Together they make a product which is the total of the two parts. The allocation/rate 1 aspect are parts that are given out from a bigger whole (allocation) to multiplicate parts together. The rate changes out of the allocation given. The area and array aspects are drawn into geometrical/symmetrical shapes, arranged by a set of objects into columns and rows (Pennant, 2012). Area has a product made of a unit different from the other aspects. Area is also a tool in measurement. The array models in rows or columns the product from an equal-group situation. The number line aspect encounters multiplication with "jumping" with the same amount on the number-line. A number grows with the same amount for a period until it stops after an amount of "jumps" which will give the product. The scale factor/rate 2 aspect scales an object. It would be to double or triple an amount, but the meaning is to scale the factor a given amount to get the product. For example, double lengths or time. The Cartesian product aspect shows combination problems. It shows the number of possible pairings that can be made between two or more sets (e.g., Anghileri, 1989, Hinna et al., 2016, pp. 82-84, Van de Walle et al., 2015, pp. 203-205)

3.2 Didactical background

From a mathematical point of view, terms are meaningful strings without a relation sign. According to their didactic nature, terms of multiplication are concepts and thus mental reflections of classes of reality. In reality, an infinite number of situations can be described with terms as 3×5.

Like all concepts, terms of multiplication can be extracted inductively by abstraction from concrete situations. There are two possibilities here.

- Spatially simultaneous (static): Whenever there are three groups of 5 objects each, this can be described as 3×5.
- Temporal successive (dynamic): Whenever three groups of 5 objects are placed one after the other, this can be described with 3×5.

Important facts about multiplication are the mathematical background of the operations with natural numbers, the didactical background of developing competencies regarding these operations. The didactical background contains the fact that terms like 3×4 or $12 \div 3$ are concepts. Concepts are fundamental "building blocks" of our thoughts and beliefs. Concepts arise as

- abstractions or generalizations from experience,
- results of transformations of existing ideas

Concepts are mental representations of classes of objects. Concepts play an important role in all aspects of cognition. Terms are concepts, for example, the term 3×4 is non-verbal-symbolic representation of concepts. Understanding the operation means understanding the meaning of concrete terms like 3×4.

Park and Nunes (2001) have investigated the development of the concept of multiplication in primary schools in England. They investigated students using repeated addition as a procedure for solving tasks versus students who used schema of correspondence in the solution for multiplicative thinking. Schema of correspondence means that the concept of multiplication is defined by an invariant relation between two quantities and not by repeated addition. The research was to give one group of students additive reasoning problems and another group of students multiplicative reasoning problems. There were a pretest and a posttest to see if the groups improved their multiplicative understanding (Park & Nunes, 2001, pp. 764-767). The result shows that students who acquired the concept of multiplication using the schema of correspondence made significantly better progress in multiplicative reasoning than those who acquired the concept of multiplication using repeated addition. This result shows that repeated addition is a strategy for solving multiplication problems, but not the procedure to use all the time when encountering multiplicative reasoning problems (ibid, pp. 771-772).

Continuously from Anghileri's six aspects, the findings from the research done are very interesting for this paper. The study investigated the development of understanding of multiplication from the early school years. The participating students were 4 till 12-year-olds, and they got different tasks from the six aspects to solve (Anghileri, 1989, pp. 367-369). The findings showed that each student possesses various meanings of multiplication. The skills that are developed from early ages, including the six aspects, will contribute to procedures that may be used when solving multiplication tasks. The link between the structure of a multiplication task and the solution strategy by the student can give some further information about the student's developing understanding of multiplication. The understanding here is then to be able to see multiplication as an operation which can be used to solve and represent tasks. Children see addition and multiplication linked through number relations and number patterns. At a higher age, student should be able to see multiplication facts and solve multiplication facts, not use addition, but this study showed that only students over above average ability managed to do this (ibid, pp. 383-385). Our study will also look at the strategies used by students; therefore it might be an addition to this previous research.

3.2.1 *Levels of representations*

Bruner (1964, p. 2) wrote about the “levels of representation”. These levels are the enactive, the iconic, and the symbolic level of representation. Following a suggestion by Bruner, Schmidt et al. (1994, p. 3) divided the symbolic level into the verbal-symbolic level and the non-verbal-symbolic level. The ability to switch between different representation of the same concept might indicate a deeper understanding

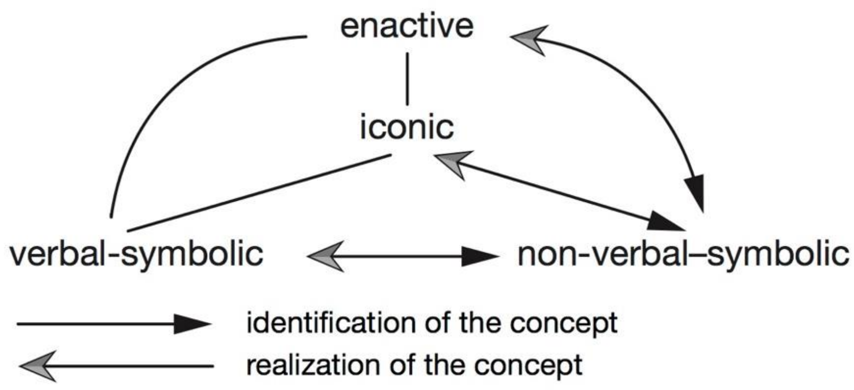


Figure 7: Levels of representation according to Bruner

(Bruner, 1964 p. 2; Schmidt et al., 1994, p. 3)

On the enactive level, students solve problems through external actions. Here, the central element is explorative action using materials. A distinction can be made here between actions with concrete objects, such as people, apples, and cars, and actions with semi-concrete objects, such as wooden cubes, which represent concrete objects.

The iconic level of pictorial representation and imagination is one first stage of internalization and abstraction. The external action is transferred - at first very close to reality, then more and more with simplifying symbols - into the pictorial. Two possibilities are to be distinguished:

1. drawing by the students as a modified activity

2. working on the completed picture as a “camera shot” of the activity.

On the symbolic level, during the primary school years, students should acquire the ability to understand and use terms and symbols from mathematical terminology. The distinction between the verbal-symbolic and the non-verbal-symbolic level is essential. Verbal-symbolic representations are, for example, all word problems. Especially with word problems, students often have difficulties finding the appropriate equation, i.e., the non-verbal-symbolic description. A transfer from the verbal-symbolic level to the iconic level can be helpful in this case: drawing a picture according to a verbal-symbolic fact provides this transfer. Afterward, starting from this picture, the transfer to the non-verbal-symbolic level can occur by forming a suitable term. When designing mathematics lessons, it is crucial to consider all three levels of representation. In particular, the enactive and iconic levels must not be viewed merely as “fleeting transit stages” on the way to the symbolic level. Instead, students must be able to make the transfer between the different levels of representation in each case in both directions. The ability to transfer a task posed at the symbolic level into the iconic or the enactive level is essential to ensure that students can successfully solve a non-verbal-symbolic task (Schmidt et al. 1994, pp. 2-5, Gleissberg & Eichler, 2019, p. 1-3). There are many typical situations for this: for example, every student who has understood the content of an operation should also be able to solve new tasks and, if necessary, sketch or use the material when solving. In general, students should be able to describe and justify solutions in this way. Helpful are tasks requiring intermodal transfer. Such tasks are:

- Lay out appropriate to the task (to the term).
- Draw according to the task (to the term).
- Tell a story that fits the task.
- Sketch appropriate to the text.
- Lay appropriate to the text (act out the story, etc.).

Going from non-verbal-symbolic level to the three other levels requires the students to *realize* the concept. An example can be going from non-verbal to iconic: draw a picture of 3×4. When going from the other levels to the non-verbal-symbolic level requires the students to *identify* the concept. It can be writing a task that fits the picture. When the students are *systemizing* the concepts, they find relationships between different terms.

The constructivist theory of learning focuses on the importance of the learner being an active agent. Students' concept of multiplication starts in their schema of correspondence in the learning process. Piaget (1965, in Park & Nunes, 2011, p. 764) was the first who highlight this view, and Bruner emphasizes the necessity of using existing schemata in the learning process. The learner constructs knowledge from the actions, physical or mental, done in their environment. Using terms and equations to describe reality instead of being told equations by a teacher is the most efficient way for students to acquire these symbols and grasp the meaning of them. Doing so supports an active form of learning, which again can bring a need for the learner to expand their schema. In the mathematical classroom, the constructivist view of learning has influenced the research investigating early number concepts and processes. It is believed that knowledge is actively constructed by the child, adapted to their environment (Mulligan, 1992, p. 25).

Although the ultimate goal is to recall multiplication facts from memory, it is essential for students to first develop a strong foundation in basic multiplication skills. Without a solid understanding of multiplication and the ability to calculate the value of terms, students cannot reconstruct forgotten multiplication facts. To reconstruct forgotten facts, they can find connections to other facts, such as using properties of multiplication, engaging in manipulative activities, or utilizing mental representations. Students can effectively retrieve and recall multiplication facts from memory by establishing a conceptual understanding of multiplication and enhancing their foundational skills.

3.2.2 The zone of proximal development

In Vygotsky's sociocultural learning theory, learning is seen in a context of situations together with others. The zone of proximal development describes what a student is able to do with the help of a teacher or a supervisor who has more knowledge than the student. A teacher must build "scaffolding" that support the student's learning attempts and makes the student able to carry out a task without help. Vygotsky believed going slightly outside the zone of what a student can, the student will eventually understand self. Students can also learn from each other (Vygotsky, 1978, p. 86-88). The language is a tool, and the teacher's task is to structure, provide help and support, and challenge the student. The teacher must also ensure that there is social interaction in the classroom through language and cooperation (Imsen, 2015, p. 72).

We took Vygotsky's theory of zones of development into account when we planned and conducted our study. We selected the tasks on the worksheet on the presumption that they would be in the zone of actual performance for most students. We therefore expected that most students would solve each task quickly, using ability-level solution methods (for example, associative or distributive law).

In planning and conducting the interviews, we started with the test results. In general, students did not perform as we expected on the test. Therefore, we planned the interviews so that students could receive that stimulation during the interview from our questions and hints that would enable them to solve tasks in the zone of proximal development. The tasks in the interview were chosen to be in the zone of proximal development for the individual student to be assumed according to his or her test results. In this way, we gave students who did not score well on the test the opportunity to solve the task in the interview.

3.3 Model of mathematical understanding

In this chapter we will clarify our view on "understanding" and how students can develop and demonstrate understanding. This will help us our research work and suggest how we will examine the students understanding of multiplication.

Understanding is a term used to describe an actual or potential experience. Understanding is a cognitive activity which gives meaning of a concept. People often say they have an understanding if they can answer the question "why" on a concept and find the explanation. To find patterns and relations between phenomena are also important aspects when defining understanding. Different pieces of information and the operation in the mind needs to be in place for an individual to make a statement about the act of understanding (Sierpinska, 1994, pp. 1-7).

In mathematics, understanding is to give meaning to patterns (Sierpinska, 1994, p. 2). Understanding of mathematical concepts can be seen as connections between different mental representations of the concept (Barmby et al., 2008, p. 219). Reasoning is connecting the different parts of understanding. To understand an operation means to identify, realize, and systemize a concept. Mathematical understanding entails knowing, perceiving, comprehending, and making sense of the meaning and connotation of mathematical knowledge (Yang et al., 2021, p. 1). Conceptual representations can also be referred as conceptual understanding which makes a mathematical analysis of a concept (Sierpinska, 1994, pp. 114-120). Skemp (2006, p. 95) emphasized the importance of internal connections and mental representations and describes the learning process as "building up a conceptual structure", to develop a relational understanding.

For students to develop robust mathematical understanding, mathematical teaching must aim for the students to achieve a relational understanding. If students have a relational understanding, they can understand the different procedures and why they are doing these procedures. This understanding takes longer to develop but it will give the students an opportunity to see connections between operations, and it will be easier to add knowledge to previous knowledge. To build a relational understanding, the student must use different strategies in different situations. This means that it is necessary to have an understanding about the concepts to be able to develop a relational understanding (Skemp, 2006, pp. 94-95). The conceptual understanding is the basis, and the aim for this study. If there are no conceptual understanding, the students will not be able to move past the lower levels. Instrumental understanding is an example of a lower understanding. With this understanding, the students are able to do a procedure, but they do not know why they are doing this (ibid, p. 91).

There are two essential types of knowledge that children acquire: conceptual and procedural knowledge (Rittle-Johnson & Siegler, 2001, p. 346). Procedural knowledge is knowledge of procedures and algorithms, and conceptual knowledge is knowledge that gives meaning to these mathematical procedures, and teaching mathematics must include both (Hartter, 2009, p. 201). Procedural knowledge is linked to specific problem types, like counting objects in rows or solving arithmetic computations (Rittle-Johnson & Siegler, 2001, p. 346). Rittle-Johnson & Siegler define conceptual knowledge as "implicit or explicit understanding of the principles that govern a domain" (2001, pp. 346-347). This knowledge is flexible and generalizable. Task with unfamiliar procedures requires the student to rely on their knowledge of relevant concepts to solve the task (ibid). These two types of knowledge are intertwined with each other, and once students develop knowledge of one type, the other develops as well (ibid, p. 347).

It is relevant to define what perception we have of the terms *skills*, *knowledge* and *ability* related to the concept of understanding. Schaeffer et al. (1974, pp. 357-358) write:

"*Skills are integrating during simultaneous processing in working memory. Their joint use comes about because: they generalize to a new stimulus situation, often during explicit problem solving, and they become automated, requiring only the limited processing capacity of working memory or less. After skills have been sufficiently automated, they can be simultaneously processed in working memory and integration can occur. Integration involves the focusing of attention on particular skill elements and the formation of a schema uniting them.*" To develop skills, one can solve series of tasks of the same type, e.g.:

$(5 \times 8;\ 6 \times 8)$ or $(3 \times 7;\ 6 \times 7)$. By practicing series of tasks, the students can develop skills realizing relations between the tasks fast, like these examples shows neighboring $(5 \times 8;\ 6 \times 8)$ and doubling $(3 \times 7;\ 6 \times 7)$.

We see *knowledge* as to realize the term, and to be able to reproduction of facts e.g.: $7 \times 8 = 56$. This focus only on reproduction, not calculation. Mathematical *ability* can be defined as “*the ability to obtain, process, and retain mathematical information*” (Krutetskii, 1976, in Vilkomir & O’Donoghue, 2009, p. 184) or as “the capacity to learn and master new mathematical ideas and skills” (Koshy et al., 2009, p. 215).

4 Method

Below we will present and justify the research method we have chosen. We will discuss the strengths and weaknesses of the different research methods and how they can help to answer our research question.

Our research covers all aspects of mastering the operation multiplication by students in grade 6. Below we will justify why choosing a mixed method approach is appropriate to clarify our research questions and presenting which results we can obtain through quantitative and qualitative approaches.

4.1 Mixed methods

The purpose of mixed methods is to understand a problem. Instead of having only one method for collection and analyzing data, mixed methods encounter several approaches (Creswell & Creswell, 2018, p. 10). We therefore use both qualitative and quantitative data to provide the best data for the research question. The *quantitative method* is the method known for measuring reality in numbers. Human behavior can be measured and analyzed as numbers in quantitative research. When applying quantitative methods, the focus is usually on observable facts and countable results. The advantage of quantitative research is that the data collection is divided and measured into numbers, which can be an effective way to analyze a phenomenon. The disadvantages are that this method means to collect and analyze a large quantity of data and it requires many informants. Another disadvantage is based on the result of an activity, it is usually not possible to make a reliable statement about the process of this activity. If one looks at the solution of a task, make assumptions about the approach to the solution. Statements about the approach towards solution are only possible if it is observed. The *qualitative method* collects data through language and words. Reality is measured though people’s texts, language, and observations. The advantage of using a qualitative method is that the researchers get very detailed information on the way to a solution and of the informants meaning, knowledge, and understanding. The disadvantages are that analyzing qualitative data demands coding, analysis, and interpretations (Høgheim, 2012, pp. 29-31). The researchers must be aware of what information is suitable for the research and what is not. A challenge with this method can occur in the analyzing process. Researchers can affect the results in a greater way compared to quantitative method, because of the coding and interpreting of empirical data.

Our study has been carried out by a mixed methods sequential explanatory design consisting of two distinct phases: quantitative data collection followed by in-depth qualitative data collection. The reason why we choose mixed methods sequential explanatory design is to understand the data at a more detailed level by using follow-up data collection to help explain the quantitate results. When we provide a more complete understanding to answer our research question, we have a better chance to provide a valid answer, than either quantitative or qualitative data alone (Creswell & Creswell, 2018, p. 17). We first collected and analyzed the quantitative (numeric) data. 40 students in grade 6 have done quantitative worksheet with several multiplication tasks. In the second sequence, which is based on the first, we collected and analyzed the qualitative data which was interviews with 15 students.

The qualitative data helped to explain and elaborate the quantitative results from the first phase in more detail. The result from the quantitative part helped us answer the research question regarding retrieving from memory. When we supplemented with this qualitative analysis, we got a more nuanced explanation of the 6^{th} graders understanding. This helped us answer the remaining research questions. When carrying out this method we explored weaknesses of the first method and got to investigate more about students' understanding.

4.2 Philosophical aspects

This chapter presents the core ideas and elements for the philosophical approach in this research. The approaches in our research come from the belief that mixed method is the fit method for our study. We will share the research's epistemology and ontology, and why the research is placed in the pragmatic paradigm. This continues to the abductive and hermeneutic approaches.

4.2.1 The research's ontology and epistemology

Given that our research aims to gain insight into 6^{th} graders' understanding of multiplication, we will present our ontology and epistemology.

The epistemological foundation of our research is constructivism in that we consider knowledge gained as an interpretation of reality. By interacting with students, with their working products, and with each other, constructing knowledge to answer our research question, our point of view could not be a completely neutral observer. We did intended to influence the students' results as little as possible. In principle, we could not avoid this. In the first part, the worksheet, just our presence in the classroom led to an interaction. In particular, students in any test naturally interpret the behavior of those taking it, especially their facial expressions. In the second part of our study, the interviews, we interacted with students by interviewing them. Here, we not did only interact in dialogues but also conducted these dialogues in a guided manner, thus influencing them.

Lastly, when evaluating the students' work and the transcripts of the videos, it was not possible for us to be completely neutral because our interpretations of what the students explained as multiplication were an essential part of the study. In this regard, Postholm & Jacobsen (2018, pp. 45-54) highlight that what the researcher sees as reality and knowledge will be crucial to the research outcome. From an ontological perspective, our research includes both the realist and constructivist approaches. In addition to what we presented above, we also assume at the same time that we can reliably represent the fundamental structures of reality with a certain degree of uncertainty and make statements about them.

4.2.2 Pragmatic approach

Our research is placed in the *pragmatic paradigm* because we are interested in understanding a problem. Pragmatists do not see the world as an absolute unity (Creswell & Creswell, 2018, p. 10). Pragmatic thinking acquires knowledge in the most efficient way which works and proves itself in practice. Knowledge is not evolved from causes as premises. The focus in pragmatism is on conclusions drawn from findings and the question of their consistency with practice. Explanations ultimately aim primarily to understand complex reality rather than to develop theory. The theory is seen as a useful tool (ibid, p. 43). The pragmatic approach states the fact that both numbers and words are equally important and suitable in social science research, but it is linked to different strengths and weaknesses of both the qualitative and quantitative (Postholm & Jacobsen, 2018, pp. 100-101). Creswell & Poth explains pragmatism as not committed to a system of philosophy or reality; it focuses on the best possible way to answer the research question with a focus on the practical implications for the research (2018, pp. 45-46).

As for our research, the pragmatic approach fits best to analyze 6th graders understanding in multiplication. The quantitative data collected in the worksheet solved by the students will not be sufficient to answer the research questions. We have therefore added a qualitative interview which will be a supplement to answer the research questions.

4.2.3 Abduction as theoretical perspective

There are three main theoretical approaches for research: induction, deduction, and abduction. In the *inductive approach,* a large amount of empirical data is analyzed, essentials are positively abstracted, and statements are made by generalization. The researcher collects information with an open mind and later systemizes the data collected towards a theory (Postholm & Jacobsen, 2018, pp. 100-103). This approach requires a large amount of data collection; therefore, this approach will not fit our research due to time constraints and work amount. Moreover, in our opinion, it is not necessary to collect that much data. We expect that there will be a relatively rapid saturation of the sample, especially when recording errors or exploring students' strategies to solve particular tasks. This means that investigating new data will not lead to new insights.

In the *deductive approach,* new statements are obtained from existing true statements only by logical reasoning. In mathematics, for example, new universal statements are deductively derived. The deductive approach is not an option in our case because we do not have any statements available to deduce the answers to our research questions.

In our research, we follow the approach of abduction, a problem-solving process that combines deductive and inductive approaches (ibid). Abduction is a pragmatic approach to conclude a phenomenon to the best possible explanation for it. Based on the observation, a hypothesis explaining this phenomenon is made. Predictions are made deductively from this hypothesis. Finally, the extent to which these predictions are true is tested inductively.

In this sense, we collected data from informants to answer our research questions and our theory. The data collected to answer the research questions will be used in an abductive way because the informants' perspectives are a part of the research. We started from the solutions shown by the students in the worksheet. For example, we looked for solution strategies and error patterns as well as a matching theoretical explanation. For this purpose, we used the theoretical background concerning the acquisition of the operation "multiplication". From the obtained hypotheses, we deductively obtained expectations regarding the other solutions made by the same student as well as regarding the statements of the student in the interview. The hypotheses established after the evaluation of the test were an important tool for us to prepare for the interviews. Although all interviews followed a common guideline, each interview was specified in this sense concerning the particular student.

4.2.4 Hermeneutic

One part of our investigation was analyzing students' conceptual understanding of the operation of multiplication. We analyzed the students' understanding based their worksheets and the speech in the interviews.

We transcribed the interviews and made interpretations of both the worksheets and the transcripts. In interpreting the worksheets and the transcripts of the interviews, we applied the principles of hermeneutics. Hermeneutic explains that we always understand or interpret verbal and non-verbal symbols, especially sequences of mathematical signs, diagrams, written texts and spoken words based on the foundation of our prior knowledge and prerequisites. We never encounter the world with no prior perquisites. Humans have the benefit of making their own meaning and interpretation of a phenomena. We had to bear in mind that every interpretation and every understanding of a phenomenon is based on the subjective presuppositions of the interpreter. Thus, the researcher can affect the results of a study by the coding and interpreting of empirical data. Those who investigate human phenomena must always be aware of this when doing research. Prior experience and knowledge, which differ from person to person, play an important role and ultimately lead to differences in how different people interpret and evaluate one and the same object. In order to objectify these subjective interpretations and evaluations, the so-called double hermeneutics was established. Double hermeneutics means that the researchers always interpret and research something which is already interpreted by another human, but at the same time implement research and reconstruct the other human's interpretation. The researcher must reconstruct the words and interpretations with scientific background and terms (Gilje & Grimen, 1993, pp. 142-148). Our challenge was to interpret and judge the student's work and not to be influenced by personal subjective experiences. When we analyzed the interviews of the students, we had different interpretations of what understanding of multiplication is and how it is shown in the interviews. This is understandable because we interpreted the students' work with different personal experiences. We were aware of these differences and made a consensual validation of our interpretations. We tried to establish an interpretation and evaluation of the student's work in the dialogue as objectively as possible.

The hermeneutic circle is a process which we have been through. In the hermeneutic circle the researcher's understanding of the whole is established by the understanding of the parts of the whole (Gilje & Grimen, 1993, pp. 153-155). In our study the hermeneutic circle encounters our research when we first make meaning of the worksheet, but new meaning and context was added when we made new meaning from interviews. This therefore shows that the mixed method gave supplemental information for helping get an understanding of 6th graders understanding of the concept multiplication.

4.3 Quantitative part of our research

As one tool to answer the research questions, we have designed a worksheet. The worksheet can be found in attachment 2.

The worksheet contains:

a. As part one, a speed test, with 30 tasks in 3 minutes (paper-pencil-test). The results of this test will show in which degree they can retrieve the facts from memory.
b. Part two was carried out for 5 minutes and contains tasks where the students are challenged to compute the value of terms like 19×7 and 13×4. The result will show how far the students are able apply their understanding of multiplication.
c. Part three was carried out for 30 minutes and contains tasks to check the students conceptual understanding. These tasks challenge students to perform intermodal transfers referring to Bruner's levels of representation (1964, p. 2). For example, “draw 3×8”, “write a text that fits the task”.

It is important that the worksheet give an insight in students conceptual understanding and how they use this understanding to compute every-day problems. The worksheet is designed so it is possible for the students to show the strategies they use, and how far they are able to retrieve multiplication facts from memory.

The front page of the worksheet is a description of what the worksheet contains, how it will be done prior to time and why the two of us are having this research. It also contains a thanks to all the participants. This front page was also explained orally to all the students when the researchers arrived the schools.

4.3.1 *Explanation of our tasks*

Part one of the worksheet is designed for the students to write answers of multiplication tasks 0-10.

Explanation:

- Relatively easy entry
- Yellow font: commutativity, you can see if the student has memorized both tasks, or only one of them.
- colored background: Tasks that belong together: Neighboring tasks (e.g., 5×8 ⟹6×8 and 10×3 ⟹ 9×3) and doubling (e.g., 3×7 ⟹ 6×9)
- One number in the task with red font: Multiplication by 0, by 1 and doubling.
- Two numbers with red font: Tasks with two equal factors

3 · 4 =			2 · 7 =		
5 · 5 =			7 · 4 =		
6 · 5 =			6 · 9 =		
5 · 8 =			0 · 8 =		
7 · 8 =			8 · 7 =		
6 · 6 =			3 · 7 =		
10 · 3 =			8 · 4 =		
9 · 7 =			4 · 9 =		
6 · 1 =			6 · 7 =		
9 · 3 =			9 · 9 =		
9 · 6 =			10 · 5 =		
8 · 9 =			4 · 8 =		
7 · 9 =			9 · 5 =		
8 · 8 =			7 · 7 =		
6 · 8 =			8 · 6 =		

Figure 8: Speed test

We expected that most of the students will be able to retrieve these basic facts from memory and finish the worksheet. That is why we did not have a particular order of the tasks.

Part two of the worksheet is designed with four tasks which ask the students to show computations to their solution. The meaning of these four tasks was to see what methods students used when calculating with one factor greater than 10, and what different strategies are used for the tasks.

Part three of the worksheet is designed according to the intermodal transfer between different levels of representation, according to Bruner (1964, p. 2). This part also contains tasks that reflect every-day problems. We designed the tasks after what we believed students in grade 6 in this region of Norway can encounter. Using a conceptual understanding to describe and solve daily life problems means to use the strategies in multiplication in an efficient way. In our research, students who can understand and answer the questions asked correctly knows how to use multiplication in daily life. It was important to write the formulation, so it was easy for the students to understand, not using difficult words or long complicated sentences. The students' solutions show their level of conceptual understanding.

To retrieve from memory means to retrieve the answer directly from memory without computing or using a strategy. E.g. “3 × 4 = 12 just because I know” (Ostad, 2013, p. 52). Students who have answered all questions correct on the speed test shows that they can retrieve multiplication facts from memory. One mistake or two is considered the ability to retrieve from memory as well.

Working with memorizing the multiplication facts is done in mathematical teaching all over the world, but if the students are to develop conceptual understanding in multiplication, it is not acquired by only retrieving by memory (Gaidoschik et al., 2017, pp. 346-347) . Therefore, it was important to include part two and three in our worksheet. Part two includes four tasks asked to show and explain their solutions for tasks with one factor greater than 10. As a product is the value of the term $a \times b$, to compute the value using different strategies means to find the correct answer using a fitted strategy. Some students might use several strategies, which can in some cases be beneficial. Our expectations are that students who show a conceptual understanding of multiplication will presumably use more efficient strategies. When solving different multiplication tasks, it can be expedient using either distributive law or doubling. Use of repeated addition in these types of tasks are typical for lower-achieving students in higher grades (over grade 4) (Park & Nunes, 2001, pp. 771).

An explanation of what can challenge the student in each task for part two is shown as:

a) Task 1: 13×4. This task challenges the student to apply the distributive law:13×4 = 10×4+3×4 or 13×4 = 13× (2+2) = 13×2+13×2.
   More effective is the application of the associative law: 13×4 = (13×2) ×2 because doubling is often easier to compute than multiplication and addition.
b) Task 2: 19×7. In this task, the student is in favor if it realizes that 19 is near 20. In this case the student can compute 19×7 = (20×7)-7 otherwise the student can apply the distributive law regarding the addition: 19×7 = (10×7) + (9×7)
c) Task 3: 43×2. This task challenges the student to double.
d) Task 4: 27×4. This task challenges the student again to use the distributive law or double twice.

The tasks in part three make it possible for the students to show is they can *realize, identify, and systemize* the concepts of multiplication, by going from one level to another.

An explanation of what can challenge the student in each task for part three is shown as:

a) Task 1 challenges the student to go from iconic to non-verbal-symbolic representation.

b) Task 2 challenges the student to go from non-verbal-symbolic to iconic representation.
c) Task 3 challenges the student to go from verbal-symbolic to non-verbal-symbolic representation.
d) Task 4 challenges the student to go from non-verbal-symbolic to verbal-symbolic representation.
e) Task 5 challenges the student to go from verbal-symbolic to iconic representation.

All the tasks which involve the verbal-symbolic representation demands an understanding in word problems. Word problems involving multiplication and division has shown to be somehow challenging for students. Word problems are important for two reasons. First, they help identifying the difficulties students might encounter when they work on mathematical problems. Second, they help the student analyze complex problems. The students must therefore understand the use of strategies and problem-solving. Therefore, word problem-tasks are a useful tool for evaluating students conceptual understanding (Pallavi, 2015, pp. 69-70). In addition, when students are solving word problems, we can see how far the students are able to connect the real world to mathematics. We expect that most of the students can solve the word problem, as it is a common way of representing multiplication task.

4.3.2 Possible solutions and expectations

In this sub-chapter we will explain different solution strategies for each task and what expectation we have for the solutions. To discuss possible solutions for part one is not relevant, as the answers are either correct or not. We expect that almost all students can solve tasks with the factors 5 and 10, but that it might be more challenging with the tasks that include the factors 7, 8 or 9.

For part two, we expect most of the students to be able to solve all four tasks within the time limit, but there might be a variety of solution strategies. Possible solutions for the four tasks are shown below (D means doubling):

Task 1: 13×4

1. $3 \times 4 + 10 \times 4 = 12 + 40 = 52$
2. $(13 \times 2) \times 2 = 26 \times 2 = (25 \times 2) + (1 \times 2) = 50 + 2 = 52$
3. $13 \times 4 = 4 \times 13 =$ D(D(13))
4. $(13 \times 2) \times 2 = 26 \times 2 = 52$
5. $(13 \times 2) \times 2 = 26 \times 2 = 26 + 26 = 52$
6. $(15 \times 4) - (2 \times 4) = 60 - 8 = 52$
7. $13 + 13 + 13 + 13 = 52$

Task 2: 19×7

1. $(20 \times 7) - (1 \times 7) = 140 - 7 = 133$
2. $(9 \times 7) + (10 \times 7) = 63 + 70 = 133$
3. $(20 \times 5) + (20 \times 2) - (1 \times 7) = 100 + 40 - 7 = 133$
4. $19 + 19 + 19 + 19 + 19 + 19 + 19 = 38 + 38 + 38 + 19 = 133$

Task 3: 43×2

1. $43 + 43 = 86$
2. $(40 \times 2) + (3 \times 2) = 80 + 6 = 86$
3. $43 \times 2 = 2 \times 43 = \text{D}(43)$
4. $40 + 40 + 3 + 3 = 80 + 6 = 86$

Task 4: 27×4

1. $(7 \times 4) + (20 \times 4) = 28 + 80 = 108$
2. $(25 \times 4) + (2 \times 4) = 100 + 8 = 108$
3. $(30 \times 4) - (3 \times 4) = 120 - 12 = 108$
4. $(27 \times 2) \times 2 = 54 \times 2 = \text{D}(54)$
5. $27 + 27 + 27 + 27 = 54 + 54 = 108$

The strategies are placed in the order what we believe is the most efficient way, 1 being the most, and the higher the number being less efficient. We expect students with good conceptual understanding to use efficient strategies, like distributive law or doubling, and students with less conceptual understanding to use more time-consuming strategies like repeated addition.

For the final part of the worksheet, their conceptual understanding will be shown in a greater way. Possible solutions for the five tasks in part three are shown or described below:

Task 1:

1. $3 \times 7 = 21$
2. $7 \times 3 = 21$
3. $3 + 3 + 3 + 3 + 3 + 3 + 3 = 21$
4. $6 + 6 + 6 + 3 = 21$
5. $6 \times 3 + 3$ (if the configuration "six boxes" is dominant)

Students who understand sets (boxes) and elements (apples) will most likely understand that there are 7 sets with 3 elements. We expect that some students who have misconceptions about sets and their elements will not solve this correctly.

Task 2:

Correct answers:

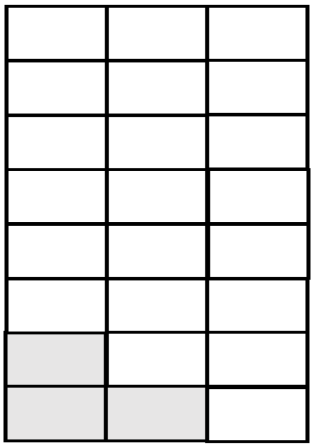


Figure 9: Solution strategy 1

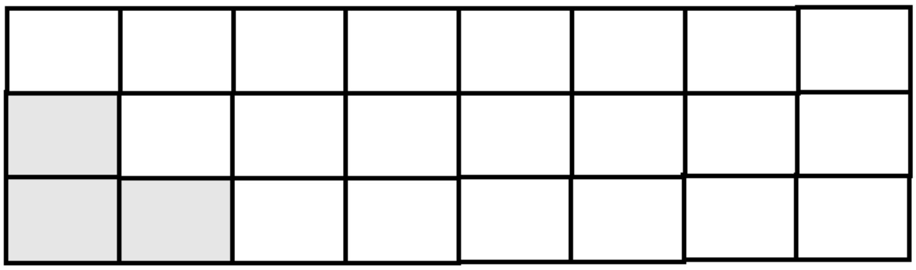


Figure 10: Solution strategy 2

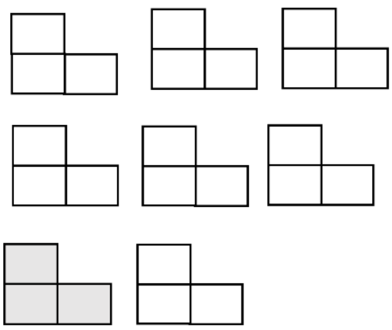


Figure 11: Solution strategy 3

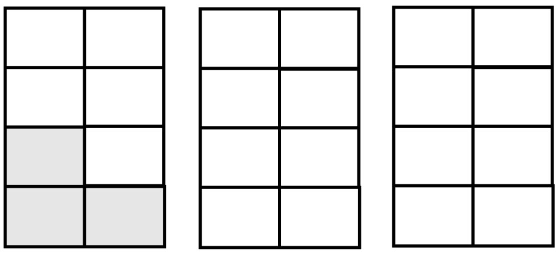


Figure 12: Solution strategy 4

Incorrect answers:

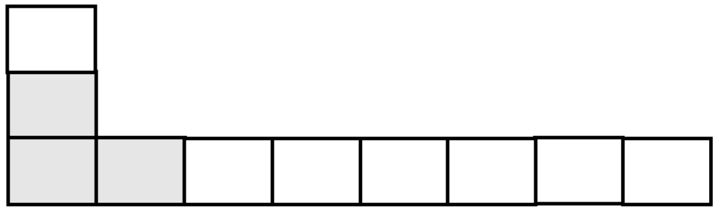


Figure 13: Solution strategy 5

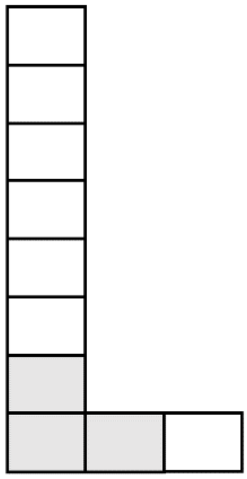


Figure 14: Solution strategy 6

The students are asked to make an iconic representation of 3×8. This task is an offer to use the array representation in multiplication. Research made by Barmby et al. shows that students in grade 6 had no problem with using the array when computing multiplication (2009, p. 17). The study showed that the array representation is a support for calculating strategies through identifying groups within the array. Both commutative and distributive law can be shown in the array representation (ibid, p. 9).

Task 3:

1. $11 \times 3 = 33$
2. $3 \times 11 = 21$
3. $11 + 11 + 11 = 33$
4. $(3 \times 1) + (3 \times 10) = 3 + 30 = 33$

Like mentioned earlier, we expect most of the students being able to solve word problems, like this task. This is a relatively easy task, and we believe it unlikely that the students will give incorrect answer.

Task 4:

1. Making a word problem of a situation that fits 5×4, either static or dynamic
2. Making a word problem where the operation is division (incorrect)
3. Making a word problem where the operation is addition (incorrect)

For this task, we expect that it can be time-consuming and maybe unknown to solve word problems for some students. They must make a task by themselves, and reassure it fits to the original task. That is one of the reasons we planned 30 minutes for part three to give students enough time.

Task 5:

1. A sketch where it is clearly six sets (people) of four elements (lobsters) in the sets.
2. A sketch where it is unclear how many items there are in each set.

The sketch can also be time-consuming if the students are very engaged in being precise in their drawing. This can "steal" time from other tasks, and therefore it is the reason why this is in the end. If it was the first task in part three, it could be too time-consuming for them to finish all the other tasks.

4.4 Qualitative part of our research

Qualitative interviews are the second part in our study. Interviews are a flexible method which gives detailed answers. The qualitative research interview is described as a professional conversation that can elicit unprejudiced descriptions of the interviewees' opinions, experiences, and perceptions of their world. An interview aims to construct significant knowledge by eliciting a diversity of different points of view through interactions between interviewer and informant (Kvale & Brinkmann, 2009, p. 21). We see interviews as a well-suited tool in our research.

4.4.1 Semi-structure interview

Interviews can be structured, semi-structured, and unstructured. The purpose of a semi-structured interview is to understand the participants' perspective. The interview has a set of themes to be uncovered, but it gives the freedom to order the questions differently in each interview, and the interviewer can ask additional/follow-up questions during the interview. This gives an opportunity to investigate the phenomenon deeper (Kvale & Brinkmann, 2015, p. 46). For our research we have chosen to do the semi-structured interview with 15 of the students. We wanted to go deeper into their understanding of multiplication and have them explain as much as possible about the tasks we gave. The interviews were done with one student at a time with the both of us present, as well as video recordings.

4.4.2 The process of interviews

As we chose to do a semi-structured interview, we designed the interview guide after the solutions of the worksheet the students had done. The interview was an expansion of the worksheet with questions where we asked the students to explain what they have done, how they were thinking when solving the tasks, and if they can solve some tasks different ways. This was to map out as much as possible of their understanding.

We made our interview guide according to the ideas from the interview guide in book of Johannessen et al. (2010, p. 141-142) (see attachment 3). With these guidelines we started our interviews with an introduction; we introduced ourselves and our project. In the introduction, it was important to explain that the participation of the research is voluntarily, and the recorded videos are kept confidential and only available by the researchers in the project. We thereafter asked questions about their hobbies, what they like in the subject mathematics, and what they think when they hear the word “multiplication”.

To start an introduction of the key questions, we asked the students about their thoughts about the worksheet and what tasks they found challenging. It was important that we did not say yes or no during the whole interview, to confirm or deny any answers, because it may have affected their answers. To start the key questions, we asked the students basic multiplication facts. It was an expansion of the speed test. We wanted to notice the time when answering the tasks to check if they retrieved from memory or not. The multiplication tasks we chose for each student was related to what we evaluated as challenging within the worksheet. Further were questions about part two of the worksheet, where we asked them to explain their computing. This was to see what strategies they used and if they were able to solve with another strategy. Additional questions were investigating if they can double. For part three we had the students explain their answers. The students that had either no or wrong answer were asked to solve it one more time. The focus on part three was to have the students show their understanding and have them explain how their answers fit the task.

If there were still time left, we asked additional tasks, e.g., explain and justify the answer of 5×0 or 7×0 , and 5×1 or 7×1. Finally, we summed up, and asked how they felt about the experience and if they had some feedback.

4.5 Selections of informants

We chose to research among students in grade 6 from two schools in our county which was relatively easy to visit and interested in our research project. We started the research after having parents’ approval. For the worksheet we wanted a lot of informants so we could get a large quantity of data. In total we had 40 informants.

To best answer our research questions, we selected a variety of informants for the interviews. We categorized students according to their different levels of understanding. After looking at the worksheet, we categorized students into who we believed had a good conceptual understanding with efficient strategies used and many correct answers. We also wanted students with fairly good conceptual understanding with some efficient strategies and some correct and incorrect answers. Lastly, we wanted students with poor conceptual understanding with inefficient strategies and many wrong answers. We selected five students from each group, which had given consent to be interviewed.

4.6 Transcribing and analysis of data

In mixed methods sequential explanatory design there are two data analyses and their integration. First reporting the quantitative results, second report the qualitative results and lastly connecting the quantitative results to the qualitative data collection. This integration focuses on how the qualitative findings help to explain the quantitative results (Creswell & Creswell, 2018, p. 222).

Kvale & Brinkmann (2015, p. 223) illustrates different forms of interview analyses, and the fitted analytical method for the interview is analyses with focus on meaning and interpretation. These analyses have the focus on what is being said. After the interviews were done, we transcribed the recordings immediately within a 3-week time limit. To remember the context of the interviews, body movements, and facial expressions of the informants, we transcribed as quickly as possible. We only transcribed the parts of the interviews relating the research questions to reduce data.

Further for the analytic method concerning both the worksheet and the interviews, we chose to do a *consensual validation*. This method is particularly suitable for us, as we are two researchers. Consensual validation is the process of checking with collaborative researchers whether there is a correspondence when both analyze the data separately (Oblak, 2021, p.471). We both individually analyzed the data collected for the quantitative and qualitative part of the study. After we assumed a result of the data, we compared and validated our interpretations. There must be a consensual validation as we have different aspects of what knowledge and reality is, but at the same time find results evaluated as valid to answer the research question. By finding the consensus of the data, we avoid having one-sided interpretations of the results.

The analytical method done for the first phase was sorting different information into different categories and conduct different overviews. We were also interested to find common obstacles/misconceptions among the students. For part one, the results were categorized into:

correctness, solving of special tasks (multiplying with 2, 0 and two equal factors), using relations (doubling, neighboring, commutative law). When sorting the information, we were looking if they were able to answer correctly or not for the different categories. For part two, we categorized the results into: correct, incorrect and no answers for the 40 students, strategies used for the tasks, and a summary of strategies. For part three, we categorized the results into: correctness (master the intermodal transfer between the levels of representation), solutions, and how many with these solutions. Finally, we also looked at the 15 students' correctness before and after the interview in both part two and three.

After analyzing the databases, we interpreted the results in a discussion. Our result is interpreted in three stages: First reporting the quantitative results, then the qualitative results. Lastly there is a discussion about what specifies how the qualitative results help to expand the quantitative results. In this interpretation it is important to remember not to merge the two databases, as it is a common misstep (Creswell & Creswell, 2018, p. 223).

4.7 Reliability and validity

In this section we will discuss the study's quality through its reliability and validity. To ensure reliability and validation, we will explain in detail the various phases of the research. *Reliability* has to do with the consistency of the research results, and credibility (Kvale & Brinkmann, 2015, p. 276). Reliability is also connected to questions about whether the same study can be reproduced by other researchers, if our informants would give different answers to other researchers, or if our findings would be transcribed and categorized differently. To achieve a good reliability, we had to stay objective and remember our role as researchers.

For the first part of the study, we had 40 students to answer the worksheet. This selection is a small selection in a quantitative study. Greater selection leads to greater reliability. Because this is a master thesis with a time limit of a few months and limited with manpower, it was not possible to make a greater selection in combination with qualitative research. The benefit of our study is that we went to two schools, instead of one. This means they have two different math teachers with different learning environment.

Our prior knowledge and personal experience will in some way affect the qualitative part of the study. We have a hermeneutic approach where we make meaning when analyzing the interviews with the students. Our interpretations have therefore played a role in the analysis. Nothing will be completely objective, which is something that affects the reliability of the research. Another matter regarding the reliability is the possibility of misinterpretation of the student answers. If we misinterpret their answers, we will not get the correct meaning of the situation and it will weaken the study's reliability. To avoid misinterpretations among the students we used easy vocabulary during both worksheet and interview.

Due to our philosophical theory, we will focus on *pragmatic validation*. This means the truth is what helps us act towards a desired outcome. Pragmatic validation depends on observations and interpretations, where one commits to action based on the interpretations (ibid, pp. 285-286). Validation also reflects on whether the chosen method is suitable.

Advantages with our method are that it opens to explore the weaknesses of the quantitative part and explore more perspectives with the qualitative part. The results from the quantitative study give a general answer to the research question, while a thoroughly qualitative analysis can nuance, expand, and explain the general picture (Johannessen et.al. 2010, p. 262). Challenges with our design are several. For example, it is very time-consuming to analyze both data collections and for us to become familiar with both quantitative and qualitative methods. To address these challenges, we made a clear schedule to follow and made time to learn the two methods. Our advantage regarding the amount of data collection is that we are two researchers. Another challenge with our study is that we had to plan adequately what quantitative results to follow up and which informants to gather qualitative data from (Creswell & Creswell, 2018, p. 222). A validity concern is the accuracy of the overall findings may be compromised because the researcher does not consider and weigh up all the options for following up the quantitative research results. It was important to not overlook any information in the worksheet which could be helpful to answer our research questions (ibid, p. 223).

Generalizability of this research is to what extent these findings can be transferred to other similar situations. If other teachers or researchers find similar results with their students, the study is generalizable (Kvale & Brinkmann, 2009, pp. 264-265). The generalizability for this study is ensured through the worksheet and the interview guide. A criterion for the worksheet when we made it, was that all teachers should be able to do the same test. It needs to be objective to ensure better reliability. Because the questions in the interview were adjusted to each student, it must also be done by the next researcher or teacher.

4.8 Ethical considerations

There were many ethical considerations to encounter in this study. Most important was to always secure the informants' personal information and keep it confidential. Our role was to always protect the informants and have approval to conduct the research. The informants got an information paper about the research and a voluntarily sign-up (Anker, 2020, pp. 104-107). Since the age group for this research is 6^{th} graders, an approval from the parents was needed. Confidentiality is mandatory during and after the research (Personopplysningsloven, 2018, §18). We got an approval from Sikt (attachment 1) before we could start our research.

Each student's name was changed to a number in the worksheet and the interview transcriptions. By doing this, it guaranteed the students anonymity. For the interviews, there were several ethical considerations to bear in mind when being a researcher. Kvale & Brinkmann (2015, p. 174-175) presents several factors the interviewer must consider when interviewing children. One factor is how the researcher asks the question. Our goal was that the questions we asked were fit for the age group, not leading, and made the students feel confident answering. Another factor is the balance of powers. The students see teachers with the right answer for the questions. Children often seek the right answers; therefore it was important that we did not confirm or deny their answers. A third factor was creativity with the questions. The students can be hesitant to answer fully detailed answers on questions from researchers who are strangers. Therefore, it was necessary to have extra questions prepared prior the interview that we used if we found it necessary (Kvale & Brinkmann, 2015, pp. 174-175).

The interviews were recorded on a video camera owned by the university and the videos were saved on a OneDrive connected to the university. Nothing was saved on our private devices due to the students' security. The videos of the students were deleted within 3 weeks. There is a lot of responsibility when recording children on camera. It is an unnatural setting for students to go out of the classroom to be recorded by two unknown people. Being recorded while being interviewed can be a somehow scary experience. A close look at these challenges helped us reduce reactivity.

5 Findings

In this chapter we present the empirical data that we have collected through the worksheets and the interviews. After every part of the study, we have divided the students into groups according to their solutions/results from the tasks. The findings in our research are presented in this chapter.

5.1 Results from the worksheets

5.1.1 Results from part one of the worksheet

Numbers of correct answers	*n*	%
10-14	8/40	20
15-19	7/40	17,5
20-24	12/40	30
25-27	5/40	12,5
28-29	3/40	7,5
30	5/40	12,5

This table shows the number of correct answers of how many students. Even though only five students had 30/30 correct answers. Three students had 28-29 correct answers with very simple mistakes, which we consider insignificant. That is why we evaluate eight of 40 (20%) students can retrieve basic multiplication facts from memory.

Findings according to special tasks:

Doubling (multiply by 2)	*n*	%
Can	38/40	95
Did not answer	2/40	5

Multiplying with 0	*n*	%
Can	33/40	82,5
Wrong answer	7/40	17,5

Two equal factors	*n*	%
Can	19/40	47,5
Cannot	6/40	15
Did not answer	15/40	37,5

The task 2×7 in the worksheet was the task that our results of doubling with multiplying 2 is based on. 38 out of 40 students had this task correct, and two students did not answer. This result shows that most students master to double.

When evaluating if students are able to multiply with 0, the task 8×0 was the task they had to solve correctly. 33 students had the correct answer, and seven students had a wrong answer.

The criteria for evaluating if students are able to retrieve tasks with two equals factors from memory are based on their answers for the five tasks: 5×5, 6×6, 7×7, 8×8, 9×9. Students who can solve four or five tasks correctly are evaluated to “can”. This means that they are able to solve tasks with two equal factors. The six students what are evaluated to “cannot” had one wrong answer or only one correct answer. The two tasks 5×5, 6×6 were placed in the beginning of the worksheet. Because of this, it is most likely that the student solved these tasks before the three others. If student had not answered these two tasks correctly, they are categorized as “cannot”. The students with only two or three correct answers out of the five, are categorized as “did not answer”. This is because it is hard to evaluate if they can or cannot. The three tasks 7×7, 8×8, 9×9 are further down on the worksheet and it is hard to evaluate if the students who did not answer these tasks can or cannot solve them, or if they did not have time to solve them. Because of this, we place these students in “did not answer”.

Findings according to mastery of related tasks:

Doubling	*n*	%
Can	16/40	40
Did not answer	24/40	60

Neighboring tasks	*n*	%
Can	24/40	60
Cannot	1/40	2,5
Did not answer	15/40	37,5

Commutative law	*n*	%
Can	17/40	42,5
Cannot	2/40	5
Did not answer	21/40	52,5

These tables show how many students solved related tasks successfully. When sorting the different answers, we had some criteria. For doubling, we looked at the following six pairs:

- 2×7 and 7×4,
- 3×7 and 6×7,
- 7×4 and 7×8,
- 9×3 and 9×6
- 4×9 and 8×9,
- 8×4 and 8×8.

This is the order we evaluate to be most important for the students to complete. If the students have a wrong or no answer for the first pair, they are not evaluated as "can". The reason for this order is based on how the tasks are placed in the worksheet, as well as being relatively easy entry tasks. The first pair is on the upper part of the worksheet and next to each other. The next pair is also relatively close, and in the middle of the worksheet. The last four pairs have more space in between and some are also on the opposite side of the worksheet. The placement of the tasks plays a role because of the time limit and the ability to see the relation between the pairs. For the 16 students that are evaluated "can" do doubling, had either five or six pairs correct. Additionally, we include the students who had four pairs correct, and where it was clear that it was because of time limit that the remaining pairs were not solved. These students solved one task of the remaining pairs correct, but it was clear that the student ran out of time. This is shown by not answering the bottom tasks in the worksheet. The students who we evaluate as "did not answer" are based on the following fact: they did not answer enough tasks to give us a sufficient amount of data to tell if they can or cannot double. It is difficult to evaluate that students "cannot" double when they many tasks not answered.

For neighboring tasks, we looked at the following five pairs:

- $2 \times 7 \Longrightarrow 3 \times 7$,
- $5 \times 5 \Longrightarrow 6 \times 5$,
- $8 \times 9 \Longleftrightarrow 9 \times 9$,
- $10 \times 3 \Longrightarrow 9 \times 3$,
- $7 \times 7 \Longleftrightarrow 6 \times 7$.

The arrows represent the common way of mental calculation going from one task to the neighboring task.

The two first pairs were essential to have correct in order to continue looking at the next pair of tasks and evaluate further if the students are able to do neighboring tasks. This is because these four tasks have a relatively easy entry as well as being in the top left corner of the worksheet. When they are being placed there, it is most likely for the students to begin to fill out the answers of these tasks. Multiplying with low numbers like 2 and 3, and with 5, are considered easier to learn and memorize because of their properties. Looking further at the next three pairs, we accepted that they had only four out of five pairs correct. These tasks are placed further apart from each other and therefore there is not necessarily easy to see the connection between them. One student is evaluated as "cannot" do neighboring tasks. This is because this student wrote wrong answers for two pairs and did not answer all the tasks on the remaining pairs.

This student answered the following:

$$5 \times 5 = 50$$
$$6 \times 5 = 30$$
$$10 \times 3 = 3$$
$$9 \times 3 = 21$$

15 students are evaluated as "did not answer" because they did not answer enough tasks to give us a sufficient amount of data to evaluate if they "can" or "cannot" do neighboring tasks.

For commutative law, the evaluation of the student answers is based on the following five pairs:

- 8×7 and 7×8,
- 8×6 and 6×8,
- 8×4 and 4×8,
- 9×7 and 7×9,
- 9×6 and 6×9.

If they have four or five pairs correctly, they are evaluated as "can". Two students answered a pair wrong, but still showed the ability to use the commutative law, e.g., one student answered $9 \times 6 = 56$ and $6 \times 9 = 56$. These two are evaluated as "can". Two students cannot use the commutative law as they gave two different answers for two tasks with the same factors.

If the students had less than four pairs correct or not answered, there is not a sufficient amount of data to evaluate of their ability to apply the commutative law, and therefore placed in “did not answer”.

5.1.2 Results from part two of the worksheet

Results of tasks 2.1 – 2.4:

	Task 2.1 13×4		**Task 2.2** 19×7		**Task 2.3** 43×2		**Task 2.4** 27×4	
	n	%	*n*	%	*n*	%	*n*	%
Correct answer	34/40	85	25/40	62,5	35/40	87,5	28/40	70
Incorrect answer	3/40	7,5	11/40	27,5	3/40	7,5	8/40	20
No answer	3/40	7,5	4/40	10	2/40	5	4/40	10

This table gives an overview how many students who have correct answers for the four tasks in part two. This table gives an overview only of the results. Various possible ways and reasons can lead to these results. Therefore, it is crucial to investigate the students’ ways of solving the task. The interviews will add more meaning and explanations of the students’ ways of solving these tasks. Some students did not answer the different tasks. The reason for this could be that they used time-consuming strategies for the first tasks, and then the 5-minute time-limit was out before they managed to solve the remaining tasks. Task 2.2 seems to be more difficult for the students, as 27,5% got a wrong answer. For task 2.4, 20% got an incorrect answer, which also shows that this task could be difficult.

An overview of the used strategies:

	Task 2.1 13×4	**Task 2.2** 19×7	**Task 2.3** 43×2	**Task 2.4** 27×4
	n	*n*	*n*	*n*
Distributive law:	22/34	19/25	20/35	21/28
Repeated addition:	8/34	3/25	10/35	3/28
Doubling:	0/34	0/25	3/35	0/28
Neighboring task:	1/34	2/25	0/35	1/28
Doubling twice:	2/34	0/25	0/35	0/28
Correct answer, no visible strategy shown	1/34	1/25	2/35	3/28

This table shows what strategies the students with correct answers used. A closer look at different solutions for every task is presented below.

Task 2.1 was correctly solved by 34 students. They applied the following strategies:

22 students applied the distributive law: $13 \times 4 = 10 \times 4 + 3 \times 4$
8 students applied repeated addition: $13 \times 4 = 13 + 13 + 13 + 13$
1 student applied neighboring: $13 \times 4 = 13 \times 5 - 13 \times 1$
2 students doubled twice: $13 \times 4 = (13 \times 2) \times 2$

Three students solved it incorrectly:

2 students added wrong: e.g., $12 + 40 = 62$
1 student multiplied 13 with 3: $13 \times 3 = 39$

Task 2.2: 25 students correctly solved this task. They applied the following strategies:

19 students applied the distributive law: $19 \times 7 = 10 \times 7 + 9 \times 7$
3 students applied repeated addition: $19 \times 7 = 19 + 19 + 19 + 19 + 19 + 19 + 19$
2 student applied neighboring: $19 \times 7 = 20 \times 7 - 1 \times 7.$

Eleven students solved it incorrectly:

4	students added wrong:	e.g., $19 \times 7 = 10 \times 7 + 9 \times 7 = 70 + 63 = 143$
3	students multiplied wrong:	$19 \times 7 = 10 \times 7 + 9 \times 7 = 70 + 72 = 142$
1	student with neighboring mistake	$19 \times 7 = 20 \times 7 - 10 = 130$
3	students started calculation but did not finish or wrote an answer.	

Task 2.3: 35 students correctly solved this task. They applied the following strategies:

20	students applied the distributive law:	$43 \times 2 = 40 \times 2 + 3 \times 2$
10	students applied repeated addition:	$43 \times 2 = 43 + 43$
3	students applied doubling:	43×2 is the double of 43

Three students solved it incorrectly:

1	student with a multiplication mistake:	$43 \times 2 = 40 \times 2 + 3 \times 2 = 80 + 4 = 84$
2	students that did not finish or wrote an answer.	

Task 2.4 was correctly solved by 28 students. They applied the following strategies:

21	students applied the distributive law:	$27 \times 4 = 20 \times 4 + 7 \times 4$
3	students applied repeated addition:	$27 \times 4 = 27 + 27 + 27 + 27$
1	student applied neighboring:	$27 \times 4 = 30 \times 4 - 3 \times 4$.

Eight students solved it incorrectly:

2	students added wrong:	$27 \times 4 = (27 + 27) + (27 + 27) = 34 + 34 = 68$
2	students multiplied wrong:	e.g., $27 \times 4 = 20 \times 4 + 7 \times 4 = 100 + 28 = 128$
4	students that did not complete the task	e.g., $27 \times 4 = 27 + 27 = 54$.
3	students started calculation but did not finish or wrote an answer.	

Summary of the strategies used when computing:

Strategy	*n*
Using distributive law	29
Repeated addition	11
Doubling	3
Doubled twice	2
Neighboring	2
No strategy shown	3

For this table, we added together all strategies used for every student. If a student used several strategies, it was counted in those strategies. Because of this, the total will not add up to 40 students. Most of the students used the distributive law, e.g., $19 \times 7 = (10 \times 7) + (9 \times 7)$, which is an efficient method. Eleven students used repeated addition, which is a very time-consuming strategy. Three students used doubling, e.g., 43×2 is the double of 43, and two students was doubling twice for the task 13×4. Two students used the neighboring task.

5.1.3 Results from part three of the worksheet

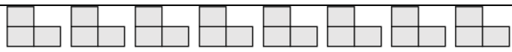
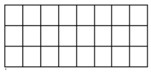
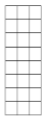
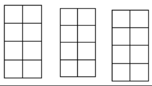
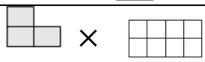
How many students master the intermodal transfer between the levels of representation:

	n	%
Transfer from iconic to non-verbal-symbolic level	31/40	77.5
Did not do the task	0/40	0
Mistakes	9/40	22.5
Transfer from non-verbal-symbolic to iconic level	17/40	42.5
Did not do the task	1/40	2.5
Mistakes	22/40	55
Transfer from verbal-symbolic to non-verbal-symbolic level	39/40	97.5
Did not do the task	1/40	2.5
Mistakes	0/40	0
Transfer from non-verbal-symbolic to verbal-symbolic level	21/40	52.5
Did not do the task	1/40	2,5
Mistakes	18/40	45
Transfer from verbal-symbolic to iconic level	27/40	67,5
Did not do the task	0/40	0
Mistakes	13/40	32,5

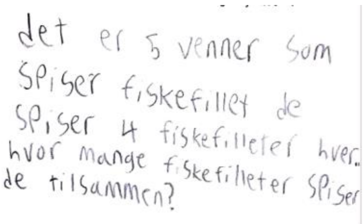
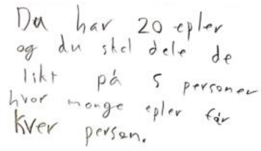
This table shows how many students can go from the different levels of representation according to Bruner (1964, p. 2) in the worksheet. This includes all 40 students. Task 3.2 where the students were requested to go from non-verbal symbolic level to the iconic level seems to be challenging. Only 40% of the students have a correct answer for this task. Only 47,5% are able to go from the non-verbal symbolic level to the verbal symbolic level in task 4. 97,5% of the students are able to go from the verbal symbolic level to the non-verbal symbolic level.

The tables below show all student answers, and how many with the same answer:

Task 3.1	Student solutions	*n*
Correct answers	$3 \times 7 = 21$	16/40
	$7 \times 3 = 21$	9/40
	$3 + 3 + 3 + 3 + 3 + 3 + 3 = 21$	3/40
	$9 \times 2 = 18 \rightarrow 18 + 3 = 21$	1/40
	$9 + 9 + 3 = 21$	1/40
	$12 + 9 = 21$	1/40
Wrong answers	$3\times4=12$	2/40
	9×12	2/40
	$3\times3=9\rightarrow9\times9=81\rightarrow81:3=27$	1/40
	3×6	1/40
	7×21	1/40
	$3+3=6$	1/40
	$3\times2, 6\times7, 3\times3$	1/40

Task 3.2	Student solutions	*n*
Correct answers		7/40
		4/40
		2/40
		4/40
Wrong answers		11/40
	4 × 6 box	3/40
	2 × 4 box	1/40
		1/40
	6×7 box, 2×9 box, 2×6 box, 2×3 box (different types of wrong boxes)	6/40
No answer		1/40

Task 3.3	Student solutions	*n*
Correct answers	$11 \times 3 = 33$	22/40
	33	9/40
	$10 \times 3 \rightarrow 1 \times 3 = 33$	5/40
	$3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 + 3 = 33$	2/40
	$11 + 11 = 22 \rightarrow 22 + 11 = 33$	1/40
Wrong answers		0/40
No answer		1/40

Task 3.4	Student solutions	*n*
Correct answers	Correct and creative task e.g.,  "There are 5 friends that eats fish filet. They eat 4 fish filet each. How many fish filet do they eat together?"	21/40
Wrong answers	Makes a division task, e.g.,  "You have 20 apples, and you are sharing equally for 5 people. How many apples does each person get?"	7/40
	Makes an addition task, e.g., 11+9	
	Did not include a question.	4/40
	12×2	4/40
	3×1, 4×1, 5×5, 5×10, 6×10	1/40
	"Take 5, and plus 5 four times"	1/40
		1/40
No answer		1/40

Students who wrote a word problem showing how it fits the task are evaluated as "correct". This includes an informative text about the concept 5×4 and a question asked in the end. For the ones who had answers connected to another operation, e.g., addition, are evaluated as "wrong answer", because the word problem does not include the concept multiplication. Some students did not write a word problem, and some did not include the question in the end. An example of this is: "Hendrik and his friends eat cake and there is four friends and five pieces of cake".

Task 3.5	Student solutions	*n*
Correct answers	Correct drawing with necessary information, and answered the task	27/40
Wrong answers	Unclear drawing with a lack of essential information Draw only 24 fish	10/40 3/40

For this task, we evaluate correct answers based on their information in the sketch. If the drawing shows six people (sets) with four lobsters (elements) each, the drawing included the essential information requested for the task. Students who only drew lobsters and costumers, but e.g., did not include that each costumer got four, are evaluated to "wrong answer".

5.2 Results of the interviews

Our interviews were an expansion of the worksheet, asking students to explain, justify, and look for different strategies to the tasks in the worksheet they have done. The interviews had a few obstacles relating technical and time issues. When interviewing the first student, we got some technical issues with the video camera which did not give us time to ask question from part three. As the interviews were 20 minutes, it was not enough time to ask about all tasks for the students that used time-consuming strategies. Because of this, we do not have data for all 15 students for every task, especially tasks in part three.

5.2.1 Results from part one of the interview

The students were asked questions about basic multiplication facts. The students who did not finish the worksheet were asked about their thought around why they did not finish. One student said, "because it did not come to the head right away" and that the student found the situation stressful. Another student answered that it was stressful and prioritized the easiest tasks first, if there was more time, the student would have tried to solve the more difficult tasks. This same student also answered the same question with: "… Sometimes I had to count them, somehow multiply them. It was a little hard. I couldn't solve them in 3 minutes". Another student who did not finish answered "because multiplication is hard for me". Several students mentioned that they thought the reason for not answering all 30 task was because of the stressful situation.

The questions that were asked was in line with their worksheet answers. If they had some tasks skipped or incorrect answers, we asked these tasks again for the interview. Many students could answer the tasks correctly but used a lot of time answering. Two students showed, as we interpret it, that they were clearly counting. They were nodding their heads in a rhythm and using their fingers hidden under the table. Some used 30 seconds or more to answer. One student was asked 6×8, and after not saying anything for 45 seconds, the interviewer asked if we would just skip that task. This student had 10/30 correct answers from the worksheet and is clearly not retrieving from memory.

Many students could give the correct answer within seconds. Some could answer all questions correctly within 1-3 seconds, some could answer the easiest tasks fast and around 15 seconds to answer the higher number tasks, e.g., 8×9.

One student explained thoroughly how the student was calculating to find the answer for 7×8. The student was using the distributive law by breaking 7 into 5+2 and multiplying with 8. When multiplying 5×8, the student used repeated addition by adding the fives eight times. After finding 40 and then 16, the student was adding these together and found 56. Altogether, this student was using 1 minute and 15 seconds for this task. This student had 10/30 correct answers in the worksheet. This strategy shown, along with the speed test, reveals that the student is clearly not retrieving from memory. Therefore, the student could not finish all 30 tasks in the worksheet.

Three of the students we interviewed had an incorrect answer for the task 8×0 in the worksheet. We asked these students the answers to 0×7 or 3×0. One student was incorrect again and answered 7. Another student explained that it is not possible and that you must borrow in order to find the answer:

Interviewer:	*What is* 0×7*?*
Student:	*Uhm, in a way, I don't know if the 0 is possible to multiply.*
Interviewer:	*Okey?*
Student:	*Uhm, so I think you have to, in a way, uhm… It is hard to explain. You have to, uhm, borrow.*
Interviewer:	*Okey, what do you mean by borrow?*
Student:	*Uhm, if it was only 7, then I had to borrow from 7. It would be… 1, uhm… on the 7. So, it would be 6 on the 7. So, it would be* 6×1*, which is 6.*
Interviewer:	*Okey, so if it was written the multiplication* 0×7*, you would have answered 6?*
Student:	*No, I would have answered 7.*
Interviewer:	*You would have answered 7?*

Student: *It is almost the same as 1… Just less.*

This student does not understand how to multiply with 0, or do not have a clear thought about the concept of multiplying with 0. The last student got the chance to think about its own answer, as the student firstly answered that $3 \times 0 = 3$, but after six seconds the student was realizing the mistake and answered 0.

Regarding the commutative law, one student who was placed at “cannot” use the commutative law answered this in the interview:

03:06 Interviewer: *What is* 6×8*?*

03:10 Student: *36*

03:13 Student: *No*

03:16 Student: *48*

…some more questions being asked.

03:35 Interviewer: *Okey,* 8×6*?*

03:41 Student: *46*

03:44 Student: *Or 48 or something.*

As seen from this example, the student hesitates and is unsure about the answers it gives. The two tasks are 20 seconds apart, but the student does not see the connection between them. We interpret this as the student is either guessing or computing the answer, and therefore having no understanding of the commutative law.

Students with all correct answers from the worksheet was first asked a few tasks about basic multiplication facts, after answering correctly within 1-5 seconds, they were challenged with the two tasks 11×11 and 20×21. These tasks do not reflect directly if they are able to retrieve from memory but was given as a challenge. Of the four students we asked, only one had both tasks correct. This student answered within six seconds for each task and computed mentally. One student did written computation and found one correct answer.

5.2.2 Results from part two of the interview

Results of tasks 2.1 – 2.4, before and after the interviews:

Task 2.1	Before		After	
13 × 4	*n*	%	*n*	%
Correct answer	12/15	80	13/15	87
Incorrect answer	2/15	13	2/15	13
No solution	1/15	7	0/15	0

Task 2.2	Before		After	
19 × 7	*n*	%	*n*	%
Correct answer	10/15	67	14/15	93
Incorrect answer	3/15	20	1/15	7
No solution	2/15	13	0/15	0

Task 2.3	Before		After	
43 × 2	*n*	%	*n*	%
Correct answer	13/15	87	15/15	100
Incorrect answer	1/15	7	0/15	0
No solution	1/15	7	0/15	0

Task 2.4	Before		After	
27×4	*n*	%	*n*	%
Correct answer	12/15	80	13/15	87
Incorrect answer	1/15	7	1/15	7
No solution	2/15	13	1/15	7

These tables show the correctness of the four tasks in part two of the 15 students that were interviewed. For every task, we asked the students how they solved it in the worksheet and made them explain their thinking. When explaining, some of the students realized their mistakes, and had the chance to correct it. For the students that did not solve the task, were given another chance to solve it. The percentages for the correct answers are increased for every task, after the interviews.

As presented from the worksheet, it is known that the students in this project used the distributive law, repeated addition and doubling most when solving the tasks with one factor greater than 10. From the interview of the 15 students, we saw a coherence of the ones using the distributive law scored higher on the speed test in part one than the students using repeated addition.

Nine students from the interview used the distributive law for solving the tasks and explained the solution well. Most of these students scored 22 or above correct on the speed test. One student had 15 correct only on the speed test but managed to solve the tasks in part two using the distributive law correctly. Another student used both distributive law and repeated addition in the worksheet, but explained the following in the interview:

Student: *Oh, that is what I did! First 10×7 and then 9×7 and then I took those two answers and added.*

Interviewer: *Yeah, then you added them.*

Interviewer: *Could that one be solved another way?*

Student: *Yes, it could so. But this is the easiest way.*

Interviewer: *Can you tell if there is a difference in the method you used here (task 2.2) and the method you used here (2.1)?*

Student: *Yes. I could have taken 10×4 and 3×4.*

This student used repeated addition for task 2.1 (13 × 4) and the distributive law for task 2.2 (19 × 7) and 2.4 (27 × 4). This example shows that the student can reason that the distributive law is the most efficient way of solving the tasks.

For the students who used repeated addition as strategy, either writing and explaining the addition, or calculating mentally, they all scored 22 or below correct on the speed test. Several of them were asked during the interview if they could solve the tasks with another strategy, three of them could explain the distributive law as well. One student answer was: "I took 10+10+10+10. That is 40. Then I took 3+3+3+3 and that is 12. Then I added 40+12, which became 52". When interviewing this student, the student explain that it was possible to multiply 10 with 4. Another student showed repeated addition on the worksheet and did not manage to solve the task another way during the interview.

Student answer for task 2.2 in worksheet:

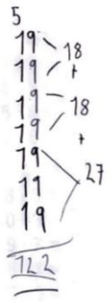


Figure 15: Student solving 19×7

Interviewer: *Here the tasks say you must calculate 19×7. What did you do here?*

Student: *I took 9+9 and that is 18. Then I took the same again, and then I took 9 three times, and that is 27. Then I added all these together, and that is 52. Then I took 5…I added all these (points at the 5 and all the 1s to the left) together. 7 plus 5 is 12…yeah.*

Interviewer: *Then you wrote the answer was…*

Student: *122*

Interviewer: *122 yes. Okay so when you think of 19×7, then you think that you must add 19 seven times? Could you have solved this task another way? Is that possible?*

Student: *I do not know.*

Another student who used repeated addition was cut off by the 5-minute time cap because of the time-consuming strategy. Here are the two answers the student answered for part two:

Task 2.1: 13×4

$4+4+4+4+4+4+4+4+4+4+4+4+4=52$

Task 2.2: 19×7

$7+7+7+7+7+7+7+7+7+7+7+7+7+7+7+7+7+7+7=133$

The student adds 4 thirteen times and 7 nineteen times. This is a mentally repeated addition with no calculation shown to how the student found the correct answers. When being asked in the interview, the student did not mention that repeated addition could have disadvantages, and it did not seem that the student knew other strategies. The transcription below is from the interview with the same student, talking about task 2.2 (19×7):

Interviewer: *You have written seven 19 times. Can it be solved another way?*

Student: *Yes.*

Interviewer: *How?*

Student: *19 seven times.*

Interviewer: *Are there any disadvantages with the way you have calculated it, do you think?*

Student: *What are disadvantages?*

Interviewer: *That it might be something bad with this strategy.*

Student: *(whispers something we cannot hear) You can just add 7s.*

The student does not reflect on the fact that this strategy could have any disadvantages, as missing a 7 or being time-consuming.

The third most used strategy in the worksheet was doubling. This was mostly shown in task 2.3 with 43×2. During the interview a lot of students showed the ability of doubling and halving. 9 of 15 students showed the ability to double during the interviews. They were asked questions which gave the opportunity to double the answer they have already written. E.g., additional question for task 2.3 (43×2):

Interviewer: *You have the answer to 43×2. If I ask you to solve 43×4, how can you find the answer?*

Student: *I can either double this (points at 86) easily, or it will be exactly like this one more time (points at computing done for the task).*

The student above used the distributive law when computing the tasks in part two, but only score 15/30 in the speed test. Another student who scored 29/30 in the speed test also used the distributive law for part two but did not show the ability to double. E.g., task 2.4 (27×4):

Interviewer: *What if I ask you to solve 27×8?*
Student: *I want to write here (student writes 20×8=160 and 7×8, crosses it out). No, I would rather do this.*

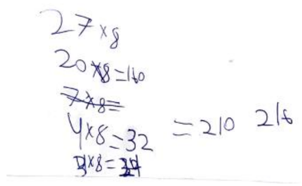


Figure 16: Student solving 27×8

Because of the well result from the speed test, we had already evaluated this student to be able to double, therefore this result surprises us that the student could not reason the answer of 27×8 by doubling the answer of 27×4. In addition, it was surprising that the student could not retrieve 7×8 from memory in this task, when in the speed test, the student solved correctly both 7×8 and 8×7.

The last strategy we investigated, was the ability to use relation between factors. The result of the worksheet shows that only one student used this strategy. This was a strategy we expected to see more of from the worksheets. Six students showed the ability to use neighboring tasks in the interview. The most frequently asked question was 21×7. The most common answer was to add 14 to 133 (133 being the answer to task 2.3). An example from a student showing this strategy correctly within 10 seconds:

Interviewer: *Can you find the answer of 21×7 when you know the answer of 19×7?*

Student: *Eeh… I would just take plus 14.*

Interviewer: *Plus 14?*

Student: *It is just two more 7s and 7+7=14.*

Some students computed 21×7 with either distribute law or repeated addition. One student used repeated addition when answering 21×7:

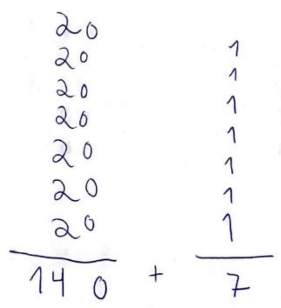


Figure 17: Student solving 21×7

This student used 34 seconds to find the answer. If we compare the two student answers and their strategy, it is clear that using neighboring task to compute the value is a more sufficient and less time-consuming strategy than repeated addition.

5.2.3 Results from part three of the interview

Results from informants:

	Results in the worksheet		Results after the interviews	
	n=15	%	*n*=15	%
From iconic to non-verbal	11	73	13	86
From non-verbal to iconic	5	33	11	73
From verbal to non-verbal	15	100	15	100
From non-verbal to verbal	5	33	9	60
From verbal to iconic	10	66	13	86

This table shows if the students had correct answers for the five tasks in part three, and if they are able to go from the different levels of representations according to Bruner (1964). There were some challenges during the interviews that have affected the results. One interview had technical issues, and we decided to end the interview before asking any questions from part three. Another challenge was the 20-minute time cap. We had to stop the interviews when interviewing students that computed slowly and used time-consuming strategies. This have therefore affected the table above because not all students were asked some of the tasks a second time, therefore we do not have data to show if they could get a correct answer after the interview.

For all tasks in part three we were interested to have a deeper insight their explanations of their answers. For task 3.1, eleven students had a correct answer in the worksheet. When explaining how these fit, they all could explain that there were seven boxes with three apples in each box. One of these eleven students had a correct answer, but a calculation with addition instead of multiplication: $9 + 9 + 3 = 21$. The explanation from the student was counting the seven boxed like this:

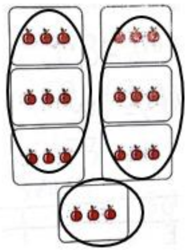


Figure 18: Student answer for task 3.1

The interviewer asked further about these seven boxes, and if the student could make a calculation knowing there are seven boxes. The student then wrote $3 \times 7 = 21$, and could explain that this calculation fits because there are seven boxes with three apples in each. These same eleven students, except one, were given either one or two extra tasks. They were asked to change the picture so it fits 8×3 and later 8×4. All these students could do it correctly. An example of a student answer:

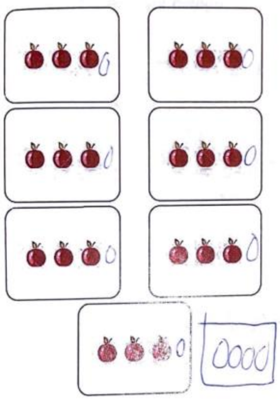


Figure 19: Student solving 8×4

This student was first asked to change the picture, so it fits 8×3. The student drew an extra box with three apples in, to the right of the bottom box. The student said that it is now eight boxes with three apples in each, and therefore it fits the calculation. After that, the student was asked to change the picture again, so it fits 8×4, and the student drew an additional apple in each box.

Four students had incorrect answers in the worksheet. Two students answered: $3 \times 4 = 12$. They were asked how their calculation fits the picture, and they answered that there are three and four boxes in the picture like this:

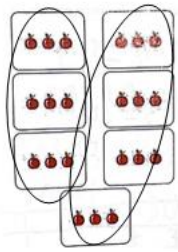


Figure 20: Student thinking of task 3.1

Instead of thinking about how many apples there are in total, they thought about the boxes. Therefore, we asked them how many apples there are in total, and they both answered 21. One of them answered $9 \times 2 + 3 = 21$, which is correct, but not a multiplication calculation. The other student could not use the information about the total of 21 apples to make a new calculation. This student gave a new incorrect answer after 22 seconds: 9×12. When explaining this answer, the student says that there are 9 apples + 12 apples. This explanation is correct but is not written the same way as explained. When the interviewer asks about a clarification between the written computation and the explanation, the student hesitates and answers: “I don’t know, I don’t understand”.

One student with the incorrect answer: $3{\times}3{=}9{\rightarrow}9{\times}9{=}81{\rightarrow}81{:}3{=}27$, explained the calculation like this:

Student:	*There are 3 in each box. And it is three on each side, and then it gets 3×3 which is 9. And then it becomes 9x9 which is 81. And the 81 divided by 3, which is 27.*
Interviewer:	*Can you try to count all apples in the picture?*
Student:	*21*
Interviewer:	*Can you make a calculation that fits the fact that there are 21 apples in total?*
Student:	*7 times...7 divided…*7×3
Interviewer:	*Can you write it?*
Student:	*(Student writes* 7×3*) I would do that because there are three apples in each box and there are 7 boxes.*

This student was also asked an additional task, to make the drawing fit the calculation 8×3. The student drew this:

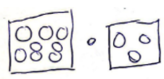


Figure 21: Student drawing 8×3 incorrectly

The interviewer then asked about the original task, how it looked and that the calculation 7×3 fits the picture, like the student had already claimed. The interviewer asked how many apples there are in total in this new picture, and the students answered 11. The student said that $8 \times 3 = 24$, and the student now drew a new picture:

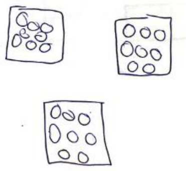


Figure 22: Student drawing 8×3 correctly

The student explains that this drawing "shows 3 boxes and 3 in each". This explanation was most likely just a mistake. The student could explain the difference in the pictures of 8×3 and 3×8: "They would not look the same because then one would have three boxes with eight in each, and the other would have had eight boxes with three in each".

The last incorrect answer was:

Figure 23: Student answer for task 3.1

During the interview this student only calculated and explained the task with addition, and not multiplication. The student understood that there are 21 apples, 7 boxes and 3 apples in each box. Even though the student showed an understanding of this, the student was unable to make a multiplication calculation which fits the information. In total, the student explains this adding method three times, for four minutes. After this, the interviewer continues with the next task. An explanation the student gave:

"The first thing I did was to multiply 3×3, because this is three boxes. And there are three apples in each. Then I would have taken 3×3, because there are three apples there and there is—uhm yeah..three apples. Then I would multiply them. And then I would have taken $3 \times$, uhm.. which is, in a way, 9. Or you could just add all the apples in a box. So you could have taken this (points at the left boxes) which is 9, and the whole here (points at the right boxes) which is also 9. Uhm, and then you just take 9+9 which is 18. Then I would have taken the last box and added it to 9… no 18. Which is 21".

The explanation is very inefficient and time-consuming. It reveals the students understanding of multiplication is only connected to repeated addition.

For the next task, task 3.2, only five students gave a correct answer. When justifying for their answers, they all could justify their drawing, either it was a drawing of three boxes eight times, eight boxes three times, or a rectangle divided into three rows of eight boxes or the other way around. One student used the knowledge of a 4×4 Rubik's cube to explain the answer. Five could also explain another correct way of solving the same task.

Eight students had incorrect answers, one student had not answered this task, and one interview had technical issues and therefore we don't have data. In total, there were six different incorrect answers from the students we interviewed. Four students drew boxes divided into either 4×6, 3×2, 7×6 or 6×2. One student drew this:

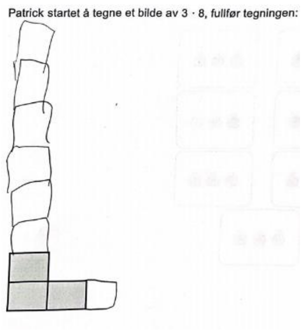


Figure 24: Student answer for task 3.2

This student could explain that 3×8 means 3 eight times. Later the student explained: "So, if that has been a box then it would be 3 out on each up here (pointing at the boxes upwards), 3, 3, 3, 3, 3, 3, 3, 3". The student says that it is a total of 24 boxes and makes a new drawing. The drawing is now correct, and the student explains: "In the first drawing I only show what is necessary. But on the second I show kind of the whole thing".

One student answer was 6×2 in the worksheet. This student adjusted the drawing and gave the correct answer for the task the when being asked, "how does your drawing fit the calculation 3×8?". This student corrected the answer self by drawing new boxes.

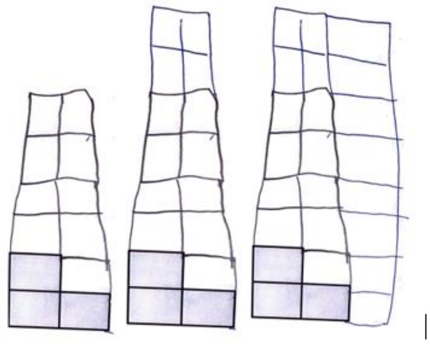


Figure 25: Stages of a student drawing

The last incorrect answer shown was “three boxes × eight boxes”, looking like this:

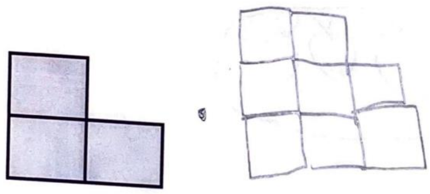


Figure 26: Student answer for task 3.2

Three students answered the same and explained that the drawing fits because “there are 3 and 8 boxes. And there is a multiplication sign there”, while pointing at the dot between the boxes.

Five student who answered incorrectly on task 3.1 were given two additional tasks: a picture of two boxes divided into 3×2 and 4×2. Our findings here is that these two tasks were a tool to help answer the original task. Three students solved it correct, and the remaining two students solved it incorrect by only thinking addition. An example of an incorrect answer for the 3×2 box: “Uhm, 3×3…Because there are three boxes here and three boxes underneath. And if that would have been a calculation, that must be the only answer.” The student is clearly thinking addition but saying “times”.

For task 3.3, all 15 students had this task correct in the worksheet. The students we asked could all explain well how their answer fits the task. One student explains that the strategy used was mental calculation. This student also said: “I could have also written 11 three times, but that takes longer time, and it is not necessary”. Three of the students was explaining the strategy repeated addition. One of these students answered:

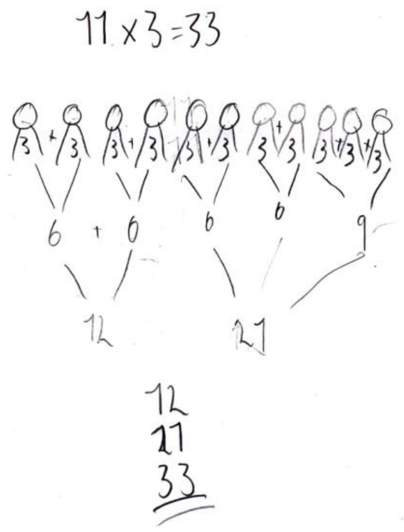


Figure 27: Student illustrating task 3.3

This student explained that are eleven players with three fish each. The student explains how the fish are added together to find the answer, using repeated addition.

Our findings from task 3.4 in the interview reveal that the students with correct word problems show a conceptual understanding and could justify their answer from the worksheet. Eight of the students with incorrect word problems, made word problems that fits either an addition calculation, division calculation or a task that fits another multiplication calculation e.g., 12×2. Two students wrote a word problem that fits an addition calculation. An example of a student answer is:

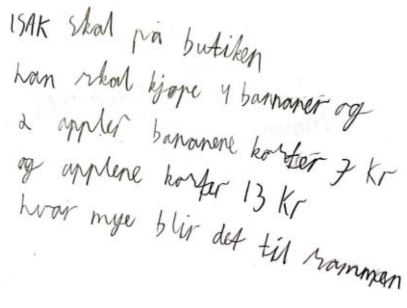


Figure 28: Word problem

This says: "Isak is going to the store. He buys 4 bananas and 2 apples. The bananas costs 7kr and the apples costs 13kr. How much is it all together?" The calculation that fits this is $7 + 13$. From the interview the student explained this:

Interviewer: *How does this fit* 5×4*?*
Student: *Because* $7 + 13$ *is …. 20*
…

Interviewer: *If I ask you to make a word problem that fits the calculation* 5×4. *Can du try to make another word problem than what you have written here?*
Student: *I don't know how I could do it…*
Interviewer: *Okey, so if you look at the previous one where we read a word problem with Glimt players. They were 11 players that ate three fish each. Could you try and make one with the five coaches maybe?*
Student: *five coaches and four players*
Interviewer: *Or what the five coaches are doing with something that is four? … Is it difficult?*
Student: *Yes*
Interviewer: *Maybe make one where the five coaches eat four fish each? How many fish do they eat in total? Does that fit?*
Student: *Noo…*
Interviewer: *No, why?*
Student: *Because five coaches eat four fish, is not in total 20*

This shows that this student still could not manage to make a new word problem in the interview. The interviewer gave a lot of hints and helped the student try to see connections to another task.

Two students made word problems that fits a division calculation. An example is:

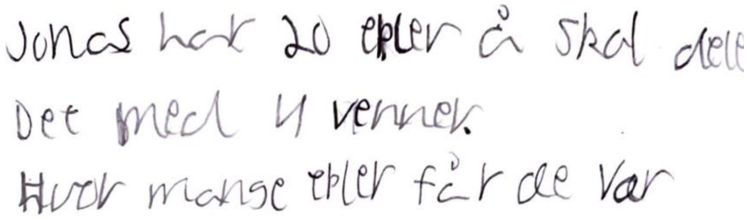


Figure 29: Student answer for task 3.4

This says: “Jonas has 20 apples and is sharing with 4 friends. How many apples do they get each?” When the interviewer asks how this fit 5×4, the student answers: “I don't really know if it fits the calculation. But they get 5 apples each”. The interviewer asks what kind of calculation fits this word problem, and the student answers division. The student gets another chance to make a word problem that fit 5×4, and the student gives a correct word problem. The other student with a division task also managed to make a new word problem that fits 5×4.

The student that made a word problem fitting 12×2, is answering the question about why the word problem 5×4 fits like this: “There was 12 students, they get two each and then…” The student talks with a very low voice, seemingly hesitant. The interviewer asks what the answer to this word problem is, and the student answers after 44 seconds hesitantly 22, which is wrong. The interviewer asks why 22, and the student to write this:

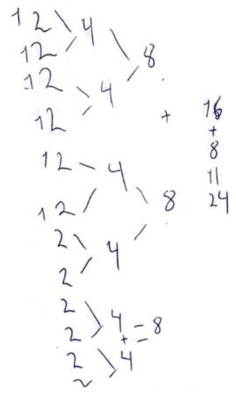


Figure 30: Student computing 12×2

The student used 1 minute and 36 seconds to write this and therefore the time of the interview was running out. We stopped the interview when the student answered 24.

The results of task 3.5 shows that ten students had a correct sketch before the interview and five with either wrong sketches or not finished. Three students with incorrect answers were asked to change their sketch so it could include essential information that was missing. An example below shows how the drawing was originally:

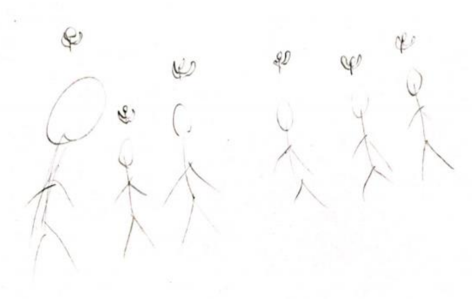


Figure 31: Incorrect drawing of task 3.5

There are six people, and one lobster each. The interviewer read the task one more time, and the student said: "If there are 24 lobsters and 6 customers. $24 \div 6$ which is… $24 \div 6$… I do not know what it is right now". The interviewer suggests: "Maybe one can divide the 24 lobsters equally for each person. How can you do that?" The student says it is possible to draw, and starts distributing the 24 lobsters equally for each person. The sketch is now:

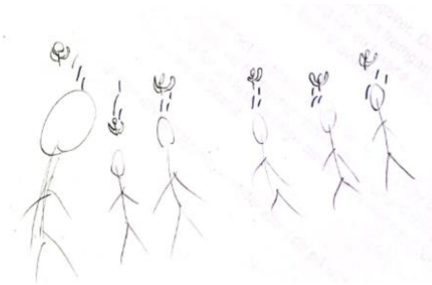


Figure 32: Correct drawing of task 3.5

The student is now able to argue why each person gets four lobsters each. The similar happens with another student that only drew six people and did not show how many lobsters each person gets. This student gets on the other hand, more help from the interviewer. The type of help is more instructing: "Okey. So, when we have 24 lobsters here, and everyone should be shared equal, can you try and draw and distribute to them? Do you want to try that?" After this comment, the student draws circles illustrating lobsters, and finds the same answer as the student above did.

The ten students with correct sketches from the worksheet, could explain well how their sketch fits the task. We asked them how they found their answer, and one student said: "I took 24 ÷ 6, and I know that 6 × 4 is 24" and the student used this information to draw the sketch:

Figure 33: Correct sketch for task 3.5

The student distinguishes the lobsters and the people using bigger and smaller dots. After making 24 dots as lobsters, the student is grouping them into six groups, representing the customers.

6 Discussion

In this chapter, we discuss and value every part of our findings in the light of the theoretical background.

6.1 Part one

Part one of the worksheet is designed towards our last research question: How far students in grade 6 are able to retrieve the basic multiplication facts from memory. Many students did not finish the speed test and used several seconds answering when being asked multiplication facts in the interview. We therefore conclude that these students cannot retrieve all basic facts from memory even though they answer correctly. When retrieving from memory, it is clear that the answer must be given quickly. If the student must calculate, either mentally or written, to find the answer they are not retrieving from memory.

There are two possible explanations of why they are not able to retrieve these facts from memory. They either never memorized the basic multiplication facts, or they memorized these facts but have over the time forgotten it. It is known from the mathematics curriculum from grade 3, 4, and 5 that these students have learned multiplication earlier (Kunnskapsdepartementet, 2019), but we do not know the quality of this learning. Maybe they have learned inefficient strategies for memorization, or do not have enough training and repetition in grade 4 and 5 regarding the multiplication facts. It is a big possibility the students have forgotten these facts as one of the functions of the brain is forgetting. The teacher should then enable the students to reconstruct the multiplication facts.

Our results from part one are not satisfying. Only 20% of the students can retrieve the basic facts from memory. It is expected in a much higher degree that they know these facts at this age. The interviews confirmed that there are too many student not being able to memorize the basic multiplication facts. In absence of knowledge, these students must count, compute, and often count and compute using fingers. Some students just used 1-3 seconds to answer the questions, but these students also scored high in the speed test in the worksheet. We are surprised that such a low amount of 6th graders are able to do so. From the curriculum they start to learn multiplication in grade 3, so therefore we expected a much higher percentage. The previous research shows that a higher percentage of students are able to retrieve basic multiplication facts in other countries. E.g., Mabbot and Bisanz (2003) showed that Canadian students in grade 6 had to a much bigger extent the ability to retrieve from memory. The practicing and learning of multiplication are developing throughout the years. We must put in context of what can be done better in the mathematical classroom to improve this. The governmental strategy from 2015-2019 (Kunnskapsdepartementet, 2015) focused on science and mathematics, and the latest results from PISA showed a small improvement (OECD, 2012, pp. 1-2). Our results reflect a need to make improvements, and teachers in both early and upper grades adjust their teaching to improve this result.

The findings from the special tasks (see chapter 5.1.1) showed mostly a positive result. The students showed the ability to double with the factor 2 which means that the factor 2 is easy to memorize, also this shows that counting with even numbers which is done often from an early age in school help memorize multiplication facts. To multiply with the factor 0 also shows a good result, but we also find it concerning that seven students had incorrect answers for 8×0. According to the axioms of multiplication for all natural numbers b is: $0 \times b = 0$. This should be relatively easy for a 6th grader to remember. In the interviews, three students corrected their mistake. The ability to compute with the factor 0 is important for solving other task, e.g., to carry out the procedure of written multiplication, like an algorithm. To multiply two equal factors with each other seemed somehow more challenging for the students. We experience students often learn songs, riddles, and highlight the two equal factors when practicing the basic multiplication facts. This is a concerns which can be a consequence for students when they encounter square numbers later in mathematics education.

The tasks which demanded the students to see relations between task were doubling, neighboring, and commutative law. When investigating all of these tasks we got a large number of students who did not answer, therefore it is difficult to conclude with a specific number to tell if they can do it or not. We do find it concerning that only 16 out of 40 students show they can double. We expected a larger number of students knowing how to double because of the connection to the associative law. Students in the interview were asked about the connections between certain tasks (part two), which included both doubling and halving. An example was the relation between the tasks 27×4 and 27×8. Nine out of the 15 students could explain this relation, using the term doubling. The distance between these two results could be due to the factor of time limit and stress. In the interview, they were not pressured regarding time like in the speed test. For the speed test, the students might have to prioritize what task to answer first, and maybe skipped the six pairs we looked at regarding doubling.

For the neighboring tasks, a look at successors can be a helpful tool. Peano's axioms presents that every number has a unique successor. This fact is related to neighboring tasks, and an efficient strategy to find the solution. The most frequently asked question regarding this was 21×7 after they solved 19×7. Here we found that the student using the neighboring task and added 14 to the previous used less time than others. One of Anghileri's aspects was using the number line. This can be put in context due to the “jumping” with the same amount in a sequence. With this method, it is clear that the relation between the factors is the *n* amount of the same number either added or subtracted. Teachers must present numbers as a line to help student visualize the concept of numbers jumping when changing factors.

It is known that once students show an understanding of the commutative law, the importance of the interpretation of the multiplication symbol is less important (Anghileri, 1989, p. 368). We can for a certainty say that the 17 students who can use the commutative law in our study has gone beyond the interpretation of the multiplication sign. Only two students showed that they do not understand the commutative law. The advantages of being able to understand the commutative law is that it is effective when computing with natural numbers as well as computing with fractions. The teachers should introduce the commutative law as early as possible to help the students understand that the order of the factors does not matter to give the correct solution.

There are some uncertainties in our results. We divided the data for part one in “can do”, “cannot do”, and “not valid data”. We do not know if students did not answer a task because of time or did not know the answer. With a lot of students who did not answer tasks, we do believe that to some extent there are more students than presented that cannot do the tasks. It is an obstacle with these unanswered tasks to present numbers of the results without errors.

6.2 Part two

We investigated if the students answered correctly on the tasks or not. Their strategies used to solve the tasks can reflect if they know important facts. If the students are forgetting these facts, it is important that they are able to reconstruct these. The strategies used can reflect if they are able or not to reconstruct what they have forgotten. If the student is unable, the student will continue to forget. From the results, most student were able to find the correct solution using the distributive law. This is a good result, as it is an efficient strategy to figure out the value of a term. Because of this, we would say that most student are able to solve the tasks using efficient strategies. Despite this result, there were also students using less efficient strategies, like repeated addition.

A closer look at the correctness for each task, for the question 19×7, 27,5% of the student answered wrong. This is a much larger percentage compared to the other tasks. This task included the single-digit factor 7, but the other three tasks included the single-digit factors 2 or 4. This finding shows that computing with 7 seems to be more difficult than computing with 2 and 4.

Altogether, there are five different strategies used to solve the tasks. The most used strategy is the distribute law. This is an efficient strategy by breaking the numbers into ones, tens, hundreds and so on, so it easier to compute the value. The students using the distributive law in the worksheet could easily justify why their strategy was correct during the interview. Of the students being interviewed, the ones using the distributive law in part two, scored 22 or above correct on the speed test. We therefore found a coherence between retrieving from memory and using the distributive law correctly. A reason why the students have a big trust in this strategy could be that it is the most represented strategy by teachers and textbooks.

The repeated addition strategy of high factors is very time-consuming. Repeated addition is often the entrance of learning multiplication in early ages, but as the students develop an understanding of the concept multiplication, they should be able to use more efficient strategies. From our findings we saw that students applying repeated addition used more time computing and could easier get adding mistakes. They also score 22 or lower on the speed test, therefore we can say that the students that preferred repeated addition as strategy cannot retrieve basic multiplication facts from memory. From Park and Nunes (2001) we know that students who are being taught and use repeated addition as the only multiplication strategy show less progress in multiplication than students who used schema of correspondence to learn multiplication. Repeated addition is a fitted strategy if the factors are low, but teacher must clarify that when it comes to larger factors, the students should have the ability to use more efficient strategies (Park & Nunes, 2001, pp. 771-772). If students do not develop, use, or fluently apply more efficient strategies beyond repeated addition, they will not develop their instrumental understanding to a higher level.

Students used repeated addition more often in task 2.1 (13×4) and 2.3 (43×2) compared with the two other tasks $(19 \times 7), (27 \times 4)$. The reason for this could be the fact that these numbers are easier to add, e.g., $13 + 13$ is easier than $27 + 27$ because they are lower numbers. We expected the students who had a good conceptual understanding would use the strategies as distributive law, neighboring and doubling where it fits. The less efficient strategy is repeated addition. Some students used several strategies throughout the four tasks, which can be very efficient, and strengthen their conceptual understanding. To know the fitted strategy regarding the task is a useful ability.

The result from the interviews shows that there is an improvement in the answers of the students. Before the interviews many of the students had correct answers on the tasks in part two, and there is an improvement after the interview as well. This is shown as 13 to 15 students have a correct answer for every task, and therefore they do have strategies to compute with one factor greater than 10. As presented earlier, Zhang et al. (2014) showed that when student themselves can chose the strategies and thereafter make connections to more efficient methods, they will improve their strategic development (Zhang et al., 2014, pp. 25-27). The same result can to some extent also be seen here, when the students can choose strategy and have no time pressure during the interview, they will succeed. At the same time, we expected that most of the students were able to answer the tasks correctly.

6.3 Part three

As for part three of the worksheet, we see the results are different depending on the intermodal transfer. The most surprising result regarding all the different levels, are the number of mistakes, except task 3.3, are very high. The students' mistakes for these tasks are between 22,5% and 55%. In the light of the Norwegian curriculum and also in comparison with our expectations these results are too weak.

The tasks starting at the non-verbal-symbolic level showed the most incorrect answers. There were a lot of misconceptions when going from either non-verbal-symbolic to iconic level, or non-verbal-symbolic to verbal-symbolic level. When interviewing students with the misconceptions, we found the students using repeated addition as multiplication strategy struggled going from the iconic to non-verbal-symbolic level. When we take a look at the different wrong answers going from the non-verbal-symbolic to verbal-symbolic level, many students make word problems regarding other operations, such as addition or division. They start with the product of the task to make a word problem with this fact, rather than the actual multiplier and the multiplicand. A reason for the low number of correct answers can be linked to students' unfamiliarity to making a word problem themselves.

Searching an explanation, we have seen the results from Gleissberg and Eichler (2019, p. 6) regarding German textbooks. They found that most of the tasks in the textbooks are only in the non-verbal-symbolic level, where students must compute the value of a term. The German students are not often challenged to perform an intermodal transfer between different levels of representation. If the Norwegian textbooks are similar, we have *one* explanation.

Tasks that challenge students to represent a term verbal-symbolically are extremely rare (ibid), and therefore they are not given the same opportunity to grasp the meaning and consolidate this knowledge compared to other representations. The teachers must give the students this same opportunity.

Task 3.2, where the students were going from the non-verbal-symbolic to iconic level had most mistakes. The most common answer was “three boxes × eight boxes” (figure 26). The reason for the original task only including three boxes was strategic there to reveal this misconception. If there were four boxes in the original drawing, the students would maybe not believe that this answer fits. The student that only drew three boxes out and eight up (figure 24), could have a misconception regarding understanding array and grid structure. This misconception is common among students (Tan Sisman & Aksu, 2016, p. 1296) and it is often caused by the way the teacher is explaining the concepts of multiplication with arrays. As it is a common misconception (ibid), we expected to see more of this answer, so therefore it was surprising that only one student answered this.

After the interviews of the two tasks starting at the non-verbal-symbolic level, students showed an improvement. They realized their misconceptions and corrected their answer. Several students got some guidance from us, as scaffolding. It was not essentially too much help. This reflects that the students were in their zone of their proximal development as they used the guidance to solve the task. In the sociocultural learning theory according to Vygotsky, learning happens when the students are going slightly outside the zone of what they can do, and then eventually they will understand by themselves (Vygotsky, 1978, p. 86-88). Students who were not able to solve the tasks without help from us are not considered to be in the zone of their proximal development for these tasks.

For the tasks having the students going from either iconic or verbal-symbolic to non-verbal-symbolic levels shows the two best results. A reason for this might be that the students are more familiar with these types of tasks. These findings show according to the levels of representations (Bruner, 1964 p. 2; Schmidt et al., 1994, p. 3), students in this study are more able to identify the concept of the non-verbal-symbolic level but cannot realize it to the same extent. The intermodal transfer between verbal-symbolic to non-verbal-symbolic level showed 97% correct answers in the worksheet and students in the interview could justify their answer well. Going from the iconic to non-verbal-symbolic level showed 77,5% correct answers in the worksheet. We expected a higher number of students being able to do this task correctly, but the interviews showed an improved result of the students being able to go to the non-verbal-symbolic level with some guidance from us.

The task asking the students to go from the verbal-symbolic to the iconic level also shows a low number of students being able to do so. The mistakes in this task were 32,5% which we evaluate as a high percentage. It is known that word problems can be challenging for students, but word problems can be a useful tool for developing a conceptual understanding (Pallavi, 2015 pp. 69-70). Therefore, these findings are proof that word problems can be difficult. These findings also show that practicing word problems is necessary to help the students develop conceptual understanding. From incorrect answers, the students missed essential information in the drawings, not showing sets and elements. After the interviews three students improved their answers to make a correct drawing.

We have observed in the results from the worksheet and the interviews students who only have a perception about multiplication being repeated addition. Park and Nunes (2001, p. 771) emphasize that this strategy is not a conceptual basis for multiplication, only a procedure. We have seen the same students that have this understanding, is in some cases successful and in some cases not. One student that was interviewed showed in every task how the student is understanding multiplication as only repeated addition. The student only answers 14 out of 30 correct in the speed test, adds the two-digit number *n* times, e.g., 19 seven times for part two, and meets trouble for part three. When the student is explaining the strategy, the student admits to only adding, as well as we observed counting using fingers. The student had a wrong answer for every task in part three, except task 3.3. This is the same result also after the interview. We can observe that the student is clearly coming up short only using this strategy. This confirms what Park and Nunes claims (2001, p. 771), repeated addition as only a procedure. If the students do not develop a conceptual understanding regarding multiplication, including different levels of representation, the student will not develop further understanding and will meet difficulties.

Due to the fact that we have these results, there are some possible factors that may have affected them. The students were given a time-limit when solving the worksheet, but this stress factor is not what they are used to in a regular mathematics lesson. It was also commented by several students in the interviews that especially part one was particularly stressful. Another factor for the interviews is the videotaping. We wanted to make the interviews as natural as possible, but the recordings is not natural. This is another possible stress factor.

6.4 Implications for teaching

It is important to discuss the facts of what the teachers can do to either improve these bad results or prevent similar results.

First, it is important to pay attention for a conceptual understanding of multiplication. Second, it is important the students meet different representations of terms in tasks. If students mainly work with tasks going from the same levels of representations, then they will be trained to solve the same kinds of task. A variety of tasks with different levels of representation can be efficient for the student to grasp the concept of multiplication.

Second, teachers should focus their teaching for student to work on different ways to compute the value of a term. Very important is a classroom discussion about what strategy is the most efficient and why, should be implemented. Students should be able to choose a suitable and efficient strategy. The students need to know what strategy is the most efficient regarding different tasks, and it is important to discuss this with the teacher. By enacting the students, they will take more ownership of the different methods. They must also develop a sense of different terms and numbers in these terms. For instance, 19 and 29 are close to 20 and 30. These properties are in many cases helpful to be aware of when computing. After highlighting the efficient strategies, the students should train these, so they become confident in every strategy. Training of using the different strategy is important. As our results shows, we assume many students are not trained or confronted with these discussions.

Lastly, a focus on students' ability to retrieve basic facts from memory should be highlighted by teachers. It is very important for students in further teaching and daily life to know these basic facts. If the student should be able to memorize important facts, the teacher must know how the brain works, and organize the lessons taking the function "forgetting" into account. There are two ways of doing this: one is that the students should develop the ability to reconstruct forgotten facts, the other is to organize the lessons so the students are confronted with the basic facts repeatedly. This could be done by daily exercises. A variety of different approaches to learn the basic multiplication facts, such as games, quiz etc. can also be done. This will be a helpful tool for the students.

7 Conclusive remarks

For this project we have investigated 6^{th} graders' understanding in multiplication. As a conclusive remark, we want to give comments about each research question and an overall remark about this topic. Lastly, we will arise some suggestions for further research.

The aim for this study was to understand more about students understanding of multiplication, how they use their knowledge to solve multiplication tasks, what strategies they use and if they are able to retrieve basic multiplication facts from memory.

For our first research question "*how far do students in grade 6 have a conceptual understanding of the operation multiplication?*", we found some different results regarding this question. Our results are more or less too low and not satisfying when investigating this. As for showing a conceptual understanding through the different directions of intermodal transfer we find the result going from non-verbal-symbolic to iconic, from non-verbal-symbolic to verbal, and from verbal to iconic very poor. The solutions and explanations improved to some extent during the interviews, but we do find these results still low. A large number of students in this study did not manage to show a good conceptual understanding of the operation multiplication. This was not expected and reflects that there is a need for improvement of the teaching to build more sense in the teaching. Our result shows that 97% of the students are able to go from the verbal-symbolic level to non-verbal-symbolic level. In the interview the participants were able to justify their answer. Students with difficulties to realize a term going from the non-verbal-symbolic level to another level of representation have a lack in their conceptual understanding. As for the opposite, the tasks which starts with either iconic or verbal-symbolic level of representation going to non-verbal-symbolic level of representation is where they scored the highest. The students in our study gave more correct answers regarding the task which challenged them to go from the verbal-symbolic level to the non-verbal-symbolic than going to the iconic level. The results show a difference if the students can go from the different levels of representations, and not a satisfying number of students can show a good understanding in all of them. Therefore, we must say that a large number of students still need to learn more to understand the concept of multiplication.

To answer our second research question "*how far can students in grade 6 apply their conceptual understanding of the operation multiplication to describe and solve problems* *in their everyday life?"* we take into account different findings. From part three, demanding going from the different levels, we found a lot of students answering incorrect, which showed a low conceptual understanding. The tasks were designed to what students in grade 6 will encounter or solve similar problems in their daily life. It is important to master multiplication for solving daily life problems, and the student must be able to find a connection between a multiplication task and daily life problems. When they are not able to do so, they can make mistakes, use time-consuming strategies, and miss essential information. We found a lot of students being able to use efficient strategies for computing, but the ones using inefficient ones spent a lot of time solving relatively easy tasks. Most of the students did not show the ability to retrieve the basic multiplication facts from memory, which really makes computing with two factors less than 10 difficult. This is also a time-consuming activity in the daily life. They did though show a good ability to double.

For our third research question "*how far can students in grade 6 compute the value of every product using different strategies?",* our results show students that know different strategies and being able to use an efficient strategy regarding the task, are able to compute the value of every product. The most common strategy was using the distributive law, which is in many cases is a good strategy. Students using repeated addition had more problems computing the correct value of every product. These students will struggle with the concept multiplication if they do not grasp an understanding of other strategies beyond repeated addition.

For our last research question "*how far can students in grade 6 retrieve the basic facts of multiplication from memory?*", the results showed that only 20% of the participants were able to retrieve basic multiplication facts from memory. The remaining 80% did not show the ability to do so. 80% is a very high percentage and surprised us when it is very important to have these facts in memory when working with mathematics both in school and daily life. If we compare to the previous research done in 2003 in Canada, 99% of the students in grade 6 managed to do it correctly (Mabbott & Bisanz, 2003, p. 1097), then the 6^{th} graders in our study scored very low.

Overall, we have found that the students who answered correctly on the worksheet and justified their answer in the interview showed a good conceptual understanding in multiplication. These students understand the different procedures in multiplication and why they are doing them. Their conceptual understanding can help develop a higher level of understanding; a relational understanding where they can see connections between operations (Skemp, 2006, p. 92). The ability to see connections and patterns in mathematics is highlighted in the Norwegian curriculum (Kunnskapsdepartementet, 2019). The core element "reasoning and argumentation" in mathematics focuses that the students should be able to justify and argument for why their solution is correct (Kunnskapsdepartementet, 2019). And the core element "mathematical areas of knowledge" states that the students must develop a good understanding in the concept of numbers.

This study was limited to two schools in Nordland, Norway with only 40 informants. Our findings shows that there is a need for a deeper search with more participants from several schools in this field. Our suggestions are to look deeper into other regions of Norway, with a wider scope. A possible reason for our results could be linked to the students' textbooks. This was not researched in our study, and it can therefore be relevant to take a closer look at Norwegian mathematical textbooks.

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Illustration credits

Figure 5 Hoffmann, A. (2019). *Den distributive lov uttrykker en sammenheng mellom addisjon og multiplikasjon.* University of Southeast-Norway: den distributive lov – Store norske leksikon (snl.no)

Figure 6 Hoffmann, A. (2019). *Den distributive lov illustrert når begge faktorene har to ledd.* University of Southeast-Norway: den distributive lov – Store norske leksikon (snl.no)

Figure 7 Bruner, J. (1964). The course of cognitive growth. *American Psychologist*, *19*(1), 1–15. https://doi.org/https://doi.org/10.1037/h0044160.

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Attachment 1 – Approval from Sikt

Sikt

Meldeskjema / Forskning til Masteroppgaven GLU 5-10 / Vurdering

Vurdering av behandling av personopplysninger

Referansenummer	Vurderingstype	Dato
759687	Standard	28.12.2022

Prosjekttittel
Forskning til Masteroppgaven GLU 5-10

Behandlingsansvarlig institusjon
Nord Universitet / Fakultet for lærerutdanning og kunst- og kulturfag / Grunnskole

Prosjektansvarlig
Klaus-Peter Eichler

Student
Ingvild Reinfjord

Prosjektperiode
23.01.2023 - 15.05.2023

Kategorier personopplysninger
Alminnelige

Lovlig grunnlag
Samtykke (Personvernforordningen art. 6 nr. 1 bokstav a)

Behandlingen av personopplysningene er lovlig så fremt den gjennomføres som oppgitt i meldeskjemaet. Det lovlige grunnlaget gjelder til 15.05.2023.

Meldeskjema

Kommentar
OM VURDERINGEN
Sikt har en avtale med institusjonen du forsker eller studerer ved. Denne avtalen innebærer at vi skal gi deg råd slik at behandlingen av personopplysninger i prosjektet ditt er lovlig etter personvernregelverket.

UTDYPENDE OM LOVLIG GRUNNLAG FOR UTVALG 1 og UTVALG 2
Utvalg 1 og 2 er barn under 16 år. Prosjektet vil dermed innhente samtykke fra foresatte til deltakerne (barn under 16 år) til behandlingen av personopplysninger. Vår vurdering er at prosjektet legger opp til et samtykke i samsvar med kravene i art. 4 nr. 11 og 7, ved at det er en frivillig, spesifikk, informert og utvetydig bekreftelse, som kan dokumenteres, og som den registrerte kan trekke tilbake.

For alminnelige personopplysninger vil lovlig grunnlag for behandlingen være samtykke fra foresatte for barn under 16 år, jf. personvernforordningen art. 6 nr. 1 a.

FORSKE PÅ EGEN ARBEIDSPLASS
Ettersom en av dere jobber samme sted som dere ønsker å rekruttere informanter fra er det flere forhold dere bør være bevisst på, deriblant dobbeltroller og frivillighet. Mer informasjon finner dere på våre nettsider: https://sikt.no/forske-pa-egen-arbeidsplass

FØLG DIN INSTITUSJONS RETNINGSLINJER
Vi har vurdert at du har lovlig grunnlag til å behandle personopplysningene, men husk at det er institusjonen du er ansatt/student ved som avgjør hvilke databehandlere du kan bruke og hvordan du må lagre og sikre data i ditt prosjekt. Husk å bruke leverandører som din institusjon har avtale med (f.eks. ved skylagring, nettspørreskjema, videosamtale el.).

Personverntjenester legger til grunn at behandlingen oppfyller kravene i personvernforordningen om riktighet (art. 5.1 d), integritet og konfidensialitet (art. 5.1. f) og sikkerhet (art. 32).

MELD VESENTLIGE ENDRINGER
Dersom det skjer vesentlige endringer i behandlingen av personopplysninger, kan det være nødvendig å melde dette til oss ved å oppdatere meldeskjemaet. Se våre nettsider om hvilke endringer du må melde: https://sikt.no/melde-endringar-i-meldeskjema

OPPFØLGING AV PROSJEKTET
Vi vil følge opp ved planlagt avslutning for å avklare om behandlingen av personopplysningene er avsluttet.

Lykke til med prosjektet!

Attachment 2 – Worksheet

NAVN:______________________________

Multiplikasjonsoppgaver

Du vet sikkert allerede mye om multiplikasjon. Vi skal bli lærere og vil gjerne vite hvor mye dere elever kan allerede i 6.klasse. Derfor har vi laget dette heftet.

Oppgavene er delt inn i 3 deler:

1. Første del er en hurtigtest med 30 oppgaver. Du vil få 3 minutter til å gjøre så mange oppgaver som du klarer.

2. Andre del består av 4 oppgaver. Du har 5 minutter til å gjøre oppgavene. HUSK: vis fremgangsmåten din, det vil si det skal være mulig for en leser å følge trinnene du har gjort for å komme fram til svaret.

3. I tredje del er det 5 oppgaver. Du får 30 minutter til denne delen. HUSK: vis fremgangsmåten din, det vil si det skal være mulig for en leser å følge trinnene du har gjort for å komme fram til svaret.

Tusen takk for din deltagelse, vi setter stor pris på ditt arbeid og din innsats!

Del 1

Oppgave 1: Regn ut flest mulig oppgaver på 3 minutt

$3 \cdot 4 =$	$2 \cdot 7 =$
$5 \cdot 5 =$	$7 \cdot 4 =$
$6 \cdot 5 =$	$6 \cdot 9 =$
$5 \cdot 8 =$	$0 \cdot 8 =$
$7 \cdot 8 =$	$8 \cdot 7 =$
$6 \cdot 6 =$	$3 \cdot 7 =$
$10 \cdot 3 =$	$8 \cdot 4 =$
$9 \cdot 7 =$	$4 \cdot 9 =$
$6 \cdot 1 =$	$6 \cdot 7 =$
$9 \cdot 3 =$	$9 \cdot 9 =$
$9 \cdot 6 =$	$10 \cdot 5 =$
$8 \cdot 9 =$	$4 \cdot 8 =$
$7 \cdot 9 =$	$9 \cdot 5 =$
$8 \cdot 8 =$	$7 \cdot 7 =$
$6 \cdot 8 =$	$8 \cdot 6 =$

Del 2

Oppgave 1

Regn ut og vis fremgangsmåte: $13 \cdot 4$

Del 2

Oppgave 2

Regn ut og vis fremgangsmåte: $19 \cdot 7$

Del 2

Oppgave 3

Regn ut og vis fremgangsmåte: $43 \cdot 2$

Del 2

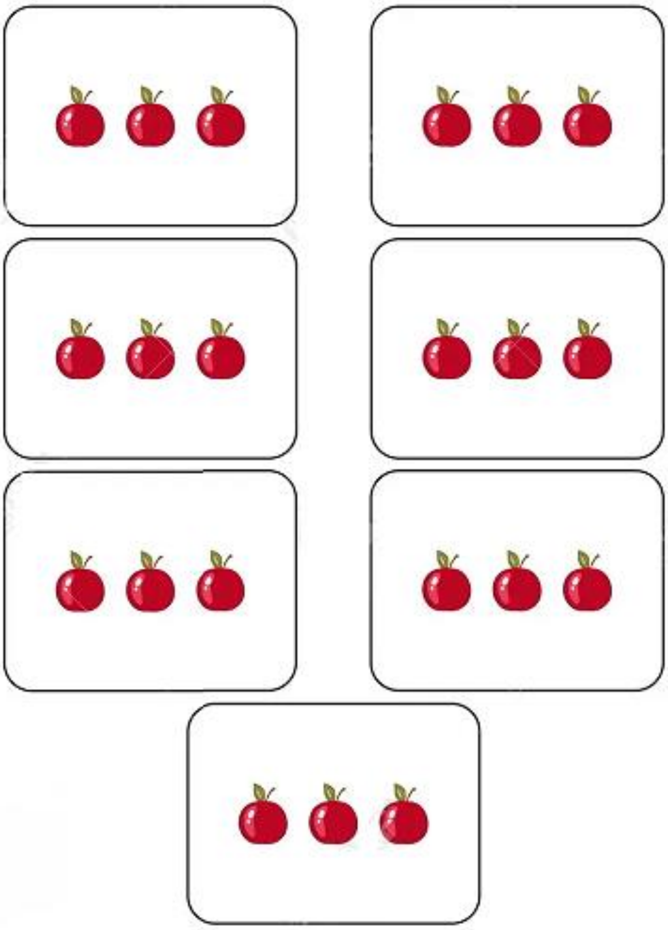
Oppgave 4

Regn ut og vis fremgangsmåte: $27 \cdot 4$

Del 3

Oppgave 1

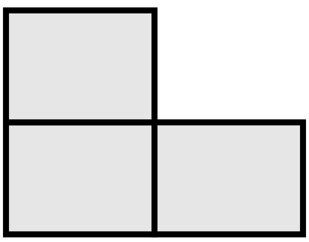
Skriv regnestykket som passer til bilde



Del 3

Oppgave 2

Patrick startet å tegne et bilde av 3 · 8, fullfør tegningen:



Del 3

Oppgave 3

Bodø/Glimt spillerne skal spise fisk. Alle 11 spillere spiser 3 fisker hver. Hvor mange fisk spiser de til sammen?

Del 3

Oppgave 4

Skriv en tekstoppgave som passer til regnestykket $5 \cdot 4$

Del 3

Oppgave 5

En sjark har vært på lofotfiske. Med i garnet kom det 24 hummere også. 6 kunder på kaia ville ha hummer.

Hvor mange hummer får hver kunde hvis det skal bli fordelt likt?

Lag en enkel skisse som passer til oppgaven:

Attachment 3 – Interview guide

Innledning:

Presentere oss selv, ønske om å bli bedre lærere slik at vi kan hjelpe elevene. Forklare at vi ser på eleven som en viktig ressurs for vår læring. Elevens feil kan lære oss som lærere å hjelpe andre elever som opplever samme vanskeligheter.

Informere om prosjektet: Dette intervjuet baserer seg på multiplikasjonsheftet du fikk delt ut tidligere. Dette intervjuet vil hjelpe oss forstå enda mer av hva 6.klassinger forstår i multiplikasjon.

- Forklare kort om hva vi kommer til å stille spørsmål om.
- Utrykke ønske om at eleven svarer ærlig og etter beste evne på spørsmålene.
- Fortelle om dokumentasjon og personvern ang. intervjuet.
- Garantere anonymitet
- Forklare elevens rettigheter knyttet til forskningen.
- Antyde hvor lenge intervjuet vil vare

Faktaspørsmål:

"Hvordan har du det i dag?", "Hva liker du å gjøre på fritiden?", "Hva liker du best i matematikk?"

Hvis elevene svarer at de ikke liker/hater matematikk: Se tilbake på all framgangen, for eksempel husker du da du begynte på skolen og ikke ante hva pluss eller minus var. Vi er sikre på at det er mye i matematikk du forstår og vet.

Introduksjonsspørsmål:

"Hva tenker du på når jeg sier ordet multiplikasjon eller ganging?"

Mulige elevsvar: Kan komme med et eksempel. Kan komme med en definisjon, Kan si at man ikke vet eller ikke tenker på noe : "Kan du komme med et eksempel?"

Overgangsspørsmål:

"Hvordan følte du du løste matematikkheftet?". Mulige elevsvar: Godt/dårlig, Det var lett/vanskelig . "Hvilken oppgave syntes du var mest utfordrende?": Oppfølging: "Hvorfor?"

Hoveddel, nøkkelspørsmål:

Spørsmål som krever utdypning - spørsmål som skal gi informasjon om problemstilling og formål med intervjuet.

De konkrete spørsmålene vi skal stille vil være tilpasset hver enkelt elev ut ifra hvordan de løser multiplikasjonsheftet. Disse vil vi utforme i samarbeid med veilederen vår i prosjektet etter vi har sett på oppgaveheftene. Vi ser på hvilke utfordringer elevene hadde i oppgaveheftet, og om vi oppdager noe vi syns er spesielt interessant. Hvis vi oppdager en utfordring, vil vi gi dem en ny oppgave, eller gir dem den oppgaven de hadde i oppgaveheftet, og stiller spørsmål som retter fokuset på at eleven skal forklare fremgangsmåten deres. Forklaringen er i hovedfokus i alle spørsmålene.

Eksempler:

1. Forklar hvorfor dette bilde passer med regnestykket
2. Hvordan kom du fram til svaret ditt?
3. Konfrontere elevene med to problemer fra dagliglivet
4. Ett spørsmål innenfor lille gangetabell
5. Ett spørsmål hvor den ene faktoren er større enn 10
6. Bruk et bilde og spør spørsmål ut ifra bildet.

Oppgaver “på lur”:

- Ha et rutenett og spør hvilket multiplikasjonsstykke som passer til dette? (2 x 4)

- Kjappe gangespørsmål
- Hva betyr det når det står 5 x 0 eller 7 x 0? Hvorfor blir svaret alltid 0?
- Hva med 5 x 1 eller 7 x 1? Hvorfor blir svaret det som blir?

Avslutning:

Informere om at intervjuet går mot slutten, for eksempel “nå er det bare to spørsmål igjen”. Setter av tid til avsluttende kommentarer til å oppklare eventuelle uklarheten. Spør informanten om h*n har noen spørsmål eller kommentarer til andre ting intervjuet burde ta opp. Gir rom for innspill om informanten har noe på hjerte som ikke tidligere har kommet frem.

Mulige tilbakemeldinger / respons: “Det var kreativt tenkt”, “Du løste oppgaven fort!”, “Så flink du er”, “Bra jobba”, “fin tegning”

- Ikke bruk tilbakemeldinger som ja/ nei til riktig svar. Gjelder også nikking