COHOMOLOGICAL COMPARISON THEOREM

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Abstract. If f is an idempotent in a ring Λ, then we find sufficient conditions which imply that the cohomology rings $\oplus_{n\geq 0}\operatorname{Ext}^n_\Lambda(\Lambda/\mathbf{r},\Lambda/\mathbf{r})$ and $\oplus_{n\geq 0}\operatorname{Ext}^n_{f\Lambda f}(f\Lambda f/f\mathbf{r}f, f\Lambda f/f\mathbf{r}f)$ are eventually isomorphic. This result allows us to compare finite generation and Gelfand-Kirillov dimensions of the cohomology rings of Λ and $f\Lambda f$. We are also able to compare the global dimensions of Λ and $f\Lambda f$.

1. Introduction

If M is a Λ-module for some ring Λ, knowledge of the cohomology ring of M, $\oplus_{n\geq 0}\operatorname{Ext}^n_\Lambda(M,M)$, is useful in the study of the representation theory of Λ. In view of this, connecting cohomology rings of modules over different rings can provide helpful information. The main goal of this paper is to find sufficient conditions so that the two cohomology rings $\bigoplus_{n\geq 0}\operatorname{Ext}^n_\Lambda(\Lambda/\mathbf{r},\Lambda/\mathbf{r})$ and $\bigoplus_{n\geq 0}\operatorname{Ext}^n_{f\Lambda f}(f\Lambda f/f\mathbf{r}f, f\Lambda f/f\mathbf{r}f)$ are eventually isomorphic, where f is an idempotent in the ring Λ and $\mathbf{r}$ denotes the Jacobson radical of Λ. For greater applicability, our results are stated in the more general setting of graded rings. The paper [6] contains results that are related to ours. Our work is in part inspired by [1], where the authors describe situations in which the cohomology groups of one ring split in the cohomology groups of the other.

To properly summarize the contents of this paper, we introduce some definitions and notation. Let G be a group and let $\Lambda = \oplus_{g\in G}\Lambda_g$ be a G-graded ring; in particular, if $g,h\in G$, then $\Lambda_g\cdot\Lambda_h\subseteq\Lambda_{gh}$. We denote the identity of G by $\mathfrak{e}$, the graded Jacobson radical of Λ by $\mathbf{r}$, and set $\mathbf{r}_\mathfrak{e} = \Lambda_\mathfrak{e}\cap\mathbf{r}$. A G-grading on Λ will be called a *proper G-grading* when it satisfies the following conditions: if $g\neq\mathfrak{e}$ then $\Lambda_g\cdot\Lambda_{g^{-1}}\subseteq\mathbf{r}_\mathfrak{e}$ and $\Lambda_\mathfrak{e}/\mathbf{r}_\mathfrak{e}$ is a semisimple Artin algebra over a commutative Artin ring C. We also fix the following notation: Given a Λ-module X, we let $\operatorname{pd}_\Lambda(X)$ and $\operatorname{id}_\Lambda(X)$ denote the projective dimension and the injective dimension of X over Λ respectively.

The main Comparison Theorem is Theorem 2.13 which we state below, omitting some technicalities.

Theorem (Theorem 2.13)**.** *Let G be a group and $\Lambda = \oplus_{g\in G}\Lambda_g$ be a properly G-graded ring. Assume that every graded simple Λ-module has a finitely generated minimal graded projective Λ-resolution. Suppose that e is an idempotent in Λ and*

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let $f = 1 - e$. Assume $f\Lambda e \subseteq \mathbf{r}$. Set Λ^* to be the ring $f\Lambda f$ and $\mathbf{r}^* = f\mathbf{r}f$. Assume that $\mathrm{pd}_{\Lambda^*}(f\Lambda e) = c < \infty$ and $\mathrm{id}_\Lambda((\Lambda/\mathbf{r})e) = b < \infty$. Then, for $n > b + c + 2$, there are isomorphisms $\mathrm{Ext}^n_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r}) \cong \mathrm{Ext}^n_{\Lambda^*}(\Lambda^*/\mathbf{r}^*, \Lambda^*/\mathbf{r}^*)$ such that the induced isomorphism

$$\bigoplus_{n>b+c+2} \mathrm{Ext}^n_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r}) \cong \bigoplus_{n>b+c+2} \mathrm{Ext}^n_{\Lambda^*}(\Lambda^*/\mathbf{r}^*, \Lambda^*/\mathbf{r}^*)$$

is an isomorphism of $\mathbb{Z} \times G$-graded rings without identity.

We also obtain the following applications; see Theorem 3.5. To simplify notation, we write $E(\Lambda)$ for the cohomology ring $\oplus_{n\geq 0} \mathrm{Ext}^n_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r})$. Here $\mathrm{GKdim}(E(\Lambda))$ denotes the Gelfand-Kirillov dimension of $E(\Lambda)$.

Theorem (Theorem 3.5)**.** *Keeping the hypotheses of the above Theorem, the following hold.*

1. *Assume that $f\Lambda e$ has a finitely generated minimal graded projective Λ^*-resolution. The cohomology ring $E(\Lambda)$ is finitely generated over $\mathrm{Ext}^0_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r}) \cong Hom_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r}) \cong (\Lambda/\mathbf{r})^{op}$ if and only if the cohomology ring $E(\Lambda^*)$ is finitely generated as a $(\Lambda^*/\mathbf{r}^*)^{op}$-algebra.*
2. *Assume that Λ is K-algebra, where K is a field and that $\Lambda/\mathbf{r}$ is a finite dimensional K-algebra. Assume further that both $E(\Lambda)$ and $E(\Lambda^*)$ are finitely generated K-algebras. Then $\mathrm{GKdim}(E(\Lambda)) = \mathrm{GKdim}(E(\Lambda^*))$.*
3. *We have that $\mathrm{pd}_\Lambda(S) < \infty$, for all graded simple Λ-modules S if and only if $\mathrm{pd}_{\Lambda^*}(S^*) < \infty$, for all graded simple Λ^*-modules S^*.*

As already mentioned, the reason for choosing a graded setting is greater applicability. In particular, if we choose G to be the trivial group, ungraded Artin algebras Λ can be viewed as a special case of our set-up, see Example 2.3. In this case, slightly simplifying the theorems above, all simple Λ-modules have finitely generated minimal projective Λ-resolutions, and $f\Lambda e$ has a finitely generated minimal projective Λ^*-resolution. Moreover, in this case $\mathrm{pd}_\Lambda(S) < \infty$ for all simple Λ-modules if and only if Λ has finite global dimension.

2. Comparison theorem

Let G be a group and let $\Lambda = \oplus_{g\in G}\Lambda_g$ be a G-graded ring; in particular, if $g, h \in G$, then $\Lambda_g \cdot \Lambda_h \subseteq \Lambda_{gh}$. We denote the identity of G by $\mathfrak{e}$, the graded Jacobson radical of Λ by $\mathbf{r}$, and set $\mathbf{r}_\mathfrak{e} = \Lambda_\mathfrak{e} \cap \mathbf{r}$. By [5, Corollary 2.9.3], $\mathbf{r}_\mathfrak{e}$ is the Jacobson radical of the (ungraded) ring $\Lambda_\mathfrak{e}$.

A G-grading on Λ will be called a *proper G-grading* when it satisfies the following conditions: if $g \neq \mathfrak{e}$ then $\Lambda_g \cdot \Lambda_{g^{-1}} \subseteq \mathbf{r}_\mathfrak{e}$ and $\Lambda_\mathfrak{e}/\mathbf{r}_\mathfrak{e}$ is a semisimple Artin algebra over a commutative Artin ring C. If the G-grading is proper, then $\mathbf{r} = \mathbf{r}_\mathfrak{e} \oplus (\oplus_{g\in G\setminus\{\mathfrak{e}\}}\Lambda_g)$.

Let $\Lambda = \oplus_{g\in G}\Lambda_g$ be a properly G-graded ring. We denote the category of G-graded (left) Λ-modules and degree $\mathfrak{e}$ maps by $\mathbf{Gr}(\Lambda)$. We let $\mathbf{gr}(\Lambda)$ denote the full subcategory of finitely generated G-graded Λ-modules. We view Λ as a G-graded Λ-module with Λ_g living in degree g.

The shifts by elements of G induce a group of endofunctors on $\mathbf{Gr}(\Lambda)$. More precisely, the shift functor associated to an element $h \in G$ is defined as follows: if $X = \oplus_{g\in G}X_g$ is a graded Λ-module, we let $X[h] = \oplus_{g\in G}Y_g$, where $Y_g = X_{gh}$. Let $\Phi\colon \mathbf{Gr}(\Lambda) \to \mathbf{Mod}(\Lambda)$ denote the forgetful functor. If $X \in \mathbf{gr}(\Lambda)$ and Y is a graded Λ-module, then

$$\operatorname{Hom}_\Lambda(\Phi(X),\Phi(Y)) \cong \bigoplus_{g\in G} \operatorname{Hom}_{\mathbf{Gr}(\Lambda)}(X, Y[g]) \cong \operatorname{Hom}_{\mathbf{Gr}(\Lambda)}(X, \oplus_{g\in G} Y[g]),$$

see [5, Corollary 2.4.4].

We need one further assumption; namely, if $\bar{\epsilon}$ is an idempotent element in $\Lambda_{\mathfrak{e}}/\mathbf{r}_{\mathfrak{e}}$, then there is an idempotent $\epsilon \in \Lambda_{\mathfrak{e}}$ such that $\pi(\epsilon) = \bar{\epsilon}$, where $\pi\colon \Lambda_{\mathfrak{e}} \to \Lambda_{\mathfrak{e}}/\mathbf{r}_{\mathfrak{e}}$ is the canonical surjection. If a graded ring Λ has this property, we say *graded idempotents lift*. Assume that graded idempotents lift in Λ. It follows that if S is simple graded Λ-module, then $S \cong (\Lambda_{\mathfrak{e}}/\mathbf{r}_{\mathfrak{e}})\epsilon[g]$, for some primitive idempotent $\epsilon \in \Lambda_{\mathfrak{e}}$ and some $g \in G$. We also see that the canonical surjection $\Lambda\epsilon[g] \to (\Lambda_{\mathfrak{e}}/\mathbf{r}_{\mathfrak{e}})\epsilon[g]$ is a projective cover. The next three examples provide important classes of graded rings satisfying our assumptions.

Example 2.1. Let K be a field, $\mathcal{Q}$ a finite quiver, G a group, and $W\colon \mathcal{Q}_1 \to G \setminus \{\mathfrak{e}\}$. We call W a *weight function*; see [2]. Setting $W(v) = \mathfrak{e}$ for all vertices v in $\mathcal{Q}$, and, if $p = a_n \cdots a_1$ is a path of length $n \geq 1$, with the $a_i \in \mathcal{Q}_1$, then set $W(p) = W(a_n)W(a_{n-1})\cdots W(a_1)$. In this case, we say p has *weight* $W(p)$. We G-grade the path algebra $K\mathcal{Q}$ by defining $(K\mathcal{Q})_g$ to be the K-span of paths p of weight g. Let I be an ideal in $K\mathcal{Q}$ such that I can be generated by elements x_i, such that, for each i, the paths occurring in x_i are all of length at least 2 and all have the same weight. We assume there is an integer t such that all paths of weight $\mathfrak{e}$ and length greater than t starting and ending at the same vertex belong to I. Let $\Lambda = K\mathcal{Q}/I$. The G-grading on $K\mathcal{Q}$ induces a G-grading on Λ.

Note that if $a \in \mathcal{Q}_1$ with $W(a) = g$, then $a + I$ is a nonzero element in Λ_g. Using that $g \neq \mathfrak{e}$, one can show that $\mathbf{r}$ is the ideal generated by $\{a + I \mid a \in \mathcal{Q}_1\}$. It follows that $\Lambda/\mathbf{r}$ is the semisimple ring $\prod_{v\in\mathcal{Q}_0} K$, which is a semisimple Artin algebra over K. Furthermore, one may check that $\mathbf{r}_{\mathfrak{e}}$ is the ideal in $\Lambda_{\mathfrak{e}}$ generated by the elements of the form $p + I$, where p is a path of length ≥ 1 in $\mathcal{Q}$ of weight $\mathfrak{e}$. Thus the G-grading on Λ is a proper G-grading. It is also clear that graded idempotents lift.

Example 2.2. Let $G = \mathbb{Z}$ and let $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \Lambda_2 \oplus \cdots$ be a positively $\mathbb{Z}$-graded ring such that Λ_0 is an Artin algebra. It is immediate that Λ is a properly $\mathbb{Z}$-graded ring in which graded idempotents lift.

Example 2.3. Let Λ be an Artin algebra over a commutative Artin ring C. Let G be any group and $\Lambda_{\mathfrak{e}} = \Lambda$, and, for $g \in G \setminus \{\mathfrak{e}\}$, $\Lambda_g = 0$. We see that Λ, as a G-graded ring, is properly G-graded and graded idempotents lift. One choice for G is the trivial group $\{\mathfrak{e}\}$.

We recall some known results about graded projective resolutions over properly graded rings in which graded idempotents lift. We leave the proof to the reader.

Lemma 2.4. *Let $\Gamma = \oplus_{g\in G}\Gamma_g$ be a properly G-graded ring in which graded idempotents lift and let $\mathbf{r}_\Gamma$ denote the graded Jacobson radical of Γ. Suppose X is a finitely generated graded Γ-module and $X/\mathbf{r}_\Gamma X \cong \oplus_{i=1}^n S_i$, where each S_i is a graded simple Γ-module. Let $P_i \xrightarrow{\alpha_i} S_i$ be graded projective covers for each i and let $P = \oplus_{i=1}^n P_i$. Then*

(1) *For each $i = 1, \ldots, n$, $P_i \cong \Gamma\epsilon_i[g]$, for some idempotent $\epsilon_i \in \Gamma$ and $g \in G$.*
(2) *The map $P \xrightarrow{\oplus_{i=1}^n \alpha_i} \oplus_{i=1}^n S_i$ is a graded projective cover.*

(3) *If $P \xrightarrow{\beta} X$ is a graded map such that the following diagram commutes*

$$\begin{tikzcd} X \arrow[r, "\pi"] & X/\mathbf{r}_\Gamma X \arrow[d, equal] \\ P \arrow[u, "\beta"] \arrow[r, "\oplus_i \alpha_i"] & \oplus_{i=1}^n S_i, \end{tikzcd}$$

where π is the canonical surjection, then $\beta\colon P \to X$ is a graded projective cover. Moreover, $\ker(\beta) \subseteq \mathbf{r}_\Gamma P$.

(4) *If $0 \to K \xrightarrow{\sigma} P \xrightarrow{\beta} X \to 0$ is a short exact sequence in $\mathbf{Gr}(\Gamma)$ with P finitely generated, such that $\sigma(K) \subseteq \mathbf{r}_\Gamma P$, then β is a graded projective cover.*

(5) *Suppose that*

$$\mathcal{P}^\bullet\colon \cdots \to P^2 \xrightarrow{\delta^2} P^1 \xrightarrow{\delta^1} P^0 \xrightarrow{\delta^0} X \to 0$$

is a graded projective Γ-resolution of X with each P^n finitely generated. Then $\mathcal{P}^\bullet$ is minimal if and only if, for $n \geq 1$, $\delta^n(P^n) \subseteq \mathbf{r}_\Gamma P^{n-1}$.

(6) *If P and Q are finitely generated graded projective Γ-modules and $\alpha\colon P \to Q$ is a map in $\mathbf{gr}(\Gamma)$, then there are primitive idempotents ϵ_i and ϵ'_j and elements g_i and h_i of G, $i = 1, \dots, m$ and $j = 1, \dots, n$, for some integers m and n such that*
   (a) $P \cong \oplus_{i=1}^m \Gamma\epsilon_i[g_i]$,
   (b) $Q \cong \oplus_{j=1}^n \Gamma\epsilon'_j[h_j]$, *and*
   (c) *viewing (a) and (b) as identifications, α is given by an $n \times m$ matrix $(\gamma_{j,i})$, where $\gamma_{j,i} \in \epsilon'_j \Gamma_{h_j^{-1} g_i} \epsilon_i$.*

(7) *Keeping the notation and assumptions of part (5), we see that $\mathcal{P}$ is a minimal graded projective resolution of X if and only if the matrices that give the δ^n, $n \geq 0$, all have entries in $\mathbf{r}_\Gamma$.*

(8) *The forgetful functor Φ is exact, preserves direct sums, and, if Y is a graded Γ-module, $\Phi(Y)$ is a projective Γ-module if and only if Y is a graded projective Γ-module. Thus, Φ takes graded projective Γ-resolutions to projective Γ-resolutions.*

□

Let e be an idempotent in $\Lambda_\mathfrak{e}$. We say that (e, f) is a *suitable idempotent pair* if $f = 1 - e$ and $f\Lambda e \subseteq \mathbf{r}$. Note that if (e, f) is a suitable idempotent pair, then, since e and 1 are homogeneous of degree $\mathfrak{e}$, so is f. Furthermore, if (e, f) is a suitable idempotent pair, then $\mathrm{Hom}_\Lambda((\Lambda/\mathbf{r})e, (\Lambda/\mathbf{r})f) = \mathrm{Hom}_\Lambda((\Lambda/\mathbf{r})f, (\Lambda/\mathbf{r})e) = 0$. Note that if (e, f) is a suitable idempotent pair, then (f, e) is also a suitable idempotent pair.

For the remainder of this section, we fix a suitable idempotent pair (e, f). Let $\Lambda^* = f\Lambda f$ and $\mathbf{r}^* = f\mathbf{r}f$. The G-grading of Λ induces a G-grading on Λ^* and it is not hard to show that $\mathbf{r}^*$ is the graded Jacobson radical of Λ^*.

The main tool in this section is the functor $F\colon \mathbf{Gr}(\Lambda) \to \mathbf{Gr}(\Lambda^*)$ given by $F = f\Lambda \otimes_\Lambda -$. Let $H\colon \mathbf{Gr}(\Lambda^*) \to \mathbf{Gr}(\Lambda)$ be given by $H = \mathrm{Hom}_{\Lambda^*}(f\Lambda, -)$. Note that for X in $\mathbf{Gr}(\Lambda)$ and Y in $\mathbf{Gr}(\Lambda^*)$, the Λ^*-module $F(X) = f\Lambda \otimes_\Lambda X$ and the Λ-module $H(Y) = \mathrm{Hom}_{\Lambda^*}(f\Lambda, Y)$ have induced G-gradings obtained from the gradings of X, Y and $f\Lambda$. The next result is well-known.

Proposition 2.5. *Keeping the above notation, we have that*

(1) *the functor F is exact,*
(2) *(F, H) is an adjoint pair, and*
(3) *$f\Lambda \cong \Lambda^* \oplus f\Lambda e$, as left Λ^*-modules.*

□

The functor H is exact if and only if $f\Lambda$ is a left projective Λ^*-module, and, by Proposition 2.5(3), H is exact if and only if $f\Lambda e$ is a left projective Λ^*-module. Note that $F(\Lambda e) \cong f\Lambda e$ does not, in general, have finite projective dimension as a left Λ^*-module, as the example below demonstrates.

Example 2.6. Let $\mathcal{Q}$ be the quiver

$$\begin{tikzcd} \overset{u}{\circ} \arrow[r, "a"'] & \overset{v}{\circ} \arrow[loop right, "b"] \end{tikzcd}$$

Let I be the ideal generated by ba and b^2 and let $\Lambda = \mathcal{Q}/I$. Taking $e = u$ and $f = v$, we see that $f\Lambda e$ has infinite projective dimension viewed as a left Λ^*-module where $\Lambda^* = f\Lambda f$.

We abuse notation by denoting the forgetful functor from $\mathbf{Gr}(\Lambda^*)$ to $\mathbf{Mod}(\Lambda^*)$ also by Φ. We also use F to denote the functor $f\Lambda \otimes_\Lambda -$ from $\mathbf{Mod}(\Lambda)$ to $\mathbf{Mod}(\Lambda^*)$. The meaning of both F and Φ will be clear from the context.

We note that if X is a graded Λ-module, then $F(\Phi(X)) \cong \Phi(F(X))$ and if $\mathcal{P}^\bullet \colon \cdots \to P^2 \xrightarrow{\delta^2} P^1 \xrightarrow{\delta^1} P^0 \to X \to 0$ is a graded projective resolution with syzygies $\Omega^n_\Lambda(X)$, then $\Phi(\mathcal{P}^\bullet)$ is projective resolution of $\Phi(X)$,

$$\Phi(F(\Omega^n_\Lambda(X))) \cong F(\Omega^n_\Lambda(\Phi(X))),$$

where $\Omega^n_\Lambda(\Phi(X))$ denotes the n-th syzygy of $\Phi(X)$ in the projective resolution $\Phi(\mathcal{P}^\bullet)$.

The next result is quite general and will allow us to apply the functor F and keep control of the cohomology if $\mathrm{pd}_{\Lambda^*}(f\Lambda e) < \infty$. One does not need that the G-grading is proper.

Theorem 2.7. *Let G be a group and Λ be a G-graded ring and let (e, f) be a suitable idempotent pair in Λ. Set $\Lambda^* = f\Lambda f$. Suppose that $\mathrm{pd}_{\Lambda^*}(f\Lambda e) = c < \infty$. Let X be a graded left Λ-module and $\Omega^i_\Lambda(X)$ (respectively, $\Omega^i_{\Lambda^*}(F(X))$) denote the i-th syzygy of X (resp., $F(X)$) in a graded projective Λ-resolution of X (resp., a graded projective Λ^*-resolution of $F(X)$). Then, for $t > c + 1$ and $n \geq 0$,*

$$\mathrm{Ext}^t_{\Lambda^*}(\Phi(F(\Omega^n_\Lambda(X))), -) \cong \mathrm{Ext}^t_{\Lambda^*}(\Phi(\Omega^n_{\Lambda^*}(F(X))), -).$$

Proof. For $n = 0$ the result is clear and hence we assume $n \geq 1$. Without loss of generality, we may start with a graded projective Λ-resolution of X in which each graded projective module is a direct sum of copies of graded projective modules of the form $\Lambda[g]$, for $g \in G$. Since $1 = e + f$, this resolution has the form:

$$\cdots \to P^2 \oplus Q^2 \to P^1 \oplus Q^1 \to P^0 \oplus Q^0 \to X \to 0,$$

where P^i is a direct sum of copies of graded modules of the form $\Lambda f[g]$ and Q^i is a direct sum of copies of graded modules of the form $\Lambda e[g]$, for $i \geq 0$. Setting $F(P^i) = L^i$ and $F(Q^i) = M^i$, we note that L^i is a graded projective Λ^*-module and M^i is a direct sum of copies of graded modules of the form $(f\Lambda e)[g]$. Applying the exact functor F to the resolution above, we obtain an exact sequence of graded Λ^*-modules

$$\cdots \to L^2 \oplus M^2 \to L^1 \oplus M^1 \to L^0 \oplus M^0 \to F(X) \to 0.$$

For $i \geq 1$, note $F(\Omega^i_\Lambda(X)) = \mathrm{Im}(L^i \oplus M^i \to L^{i-1} \oplus M^{i-1})$ and L^i is a graded left projective Λ^*-module. For ease of notation, we let $Z_i = F(\Omega^i_\Lambda(X))$, for $i \geq 1$ and $Z_0 = F(X)$.

For $n \geq 1$, we have a short exact sequence of graded Λ^*-modules

$$0 \to Z_n \to L^{n-1} \oplus M^{n-1} \to Z_{n-1} \to 0.$$

Let $P(M^{n-1}) \to M^{n-1} \to 0$ be exact sequence of graded Λ^*-modules with $P(M^{n-1})$ a graded projective module. Then we obtain the following exact commutative diagram:

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & Z_n \arrow[u] \arrow[r] & L^{n-1} \oplus M^{n-1} \arrow[u] \arrow[r] & Z_{n-1} \arrow[r] & 0 \\
0 \arrow[r] & \Omega^1_{\Lambda^*}(Z_{n-1}) \arrow[u] \arrow[r] & L^{n-1} \oplus P(M^{n-1}) \arrow[u] \arrow[r] & Z_{n-1} \arrow[u, "="'] \arrow[r] & 0 \\
0 \arrow[r] & \Omega^1_{\Lambda^*}(M^{n-1}) \arrow[u] \arrow[r, "="] & \Omega^1_{\Lambda^*}(M^{n-1}) \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

The first column yields the short exact sequence

$$0 \to \Omega^1_{\Lambda^*}(M^{n-1}) \to \Omega^1_{\Lambda^*}(Z_{n-1}) \to Z_n \to 0.$$

Taking graded projective Λ^*-resolutions of the two end modules, applying the Horseshoe lemma, and taking syzygies, we obtain short exact sequences

$$0 \to \Omega^{j+1}_{\Lambda^*}(M^{n-1}) \to \Omega^{j+1}_{\Lambda^*}(Z_{n-1}) \to \Omega^j_{\Lambda^*}(Z_n) \to 0,$$

for $j \geq 0$. Hence we obtain short exact sequences of Λ^*-modules

$$0 \to \Phi(\Omega^{j+1}_{\Lambda^*}(M^{n-1})) \to \Phi(\Omega^{j+1}_{\Lambda^*}(Z_{n-1})) \to \Phi(\Omega^j_{\Lambda^*}(Z_n)) \to 0.$$

Note that $\Phi(\Omega^{j+1}_{\Lambda^*}(M^{n-1})))$ is a projective Λ^*-module if $j \geq c$ since $c \geq \mathrm{pd}_{\Lambda^*}(\Phi(M^{n-1}))$. It follows that, for $j \geq c$ and $t \geq 2$,

$$\mathrm{Ext}^{t+j}_{\Lambda^*}(\Phi(Z_n), -) \cong \mathrm{Ext}^t_{\Lambda^*}(\Phi(\Omega^j_{\Lambda^*}(Z_n)), -) \cong \mathrm{Ext}^t_{\Lambda^*}(\Phi(\Omega^{j+1}_{\Lambda^*}(Z_{n-1})), -) \cong$$
$$\cong \mathrm{Ext}^t_{\Lambda^*}(\Phi(\Omega^{j+2}_{\Lambda^*}(Z_{n-2})), -) \cong \cdots \cong \mathrm{Ext}^t_{\Lambda^*}(\Phi(\Omega^{j+n}_{\Lambda^*}(Z_0)), -) \cong$$
$$\cong \mathrm{Ext}^{t+j}_{\Lambda^*}(\Phi(\Omega^n_{\Lambda^*}(Z_0)), -).$$

Finally, we note that $Z_n = F(\Omega^n_\Lambda(X))$ and $Z_0 = F(X)$ and the result follows. $\square$

The next result is immediate and we only provide a sketch of the proof.

Proposition 2.8. *Let G be a group and Λ a properly G-graded ring with graded Jacobson radical $\mathbf{r}$ and suitable idempotent pair (e, f). If $\mathrm{id}_\Lambda((\Lambda/\mathbf{r})e) \leq a < \infty$, then for every graded Λ-module X,*

$$\bigoplus_{n>a} \mathrm{Ext}^n_\Lambda(\Phi(X), (\Lambda/\mathbf{r})f) \cong \bigoplus_{n>a} \mathrm{Ext}^n_\Lambda(\Phi(X), \Lambda/\mathbf{r})$$

as $\mathbb{Z}\times G$-graded modules over the $\mathbb{Z}\times G$-graded ring $\oplus \operatorname{Ext}^n_\Lambda(\Lambda/\mathbf{r},\Lambda/\mathbf{r})$. Furthermore, if $\operatorname{pd}_\Lambda((\Lambda/\mathbf{r})e)\le a<\infty$ and $\operatorname{id}_\Lambda((\Lambda/\mathbf{r})e)\le a<\infty$, then

$$\bigoplus_{n>a}\operatorname{Ext}^n_\Lambda((\Lambda/\mathbf{r})f,(\Lambda/\mathbf{r})f)\cong\bigoplus_{n>a}\operatorname{Ext}^n_\Lambda(\Lambda/\mathbf{r},\Lambda/\mathbf{r})$$

as $\mathbb{Z}\times G$-graded rings without identity.

Proof. Since $\Lambda/\mathbf{r}=\Lambda_0\cong\Lambda_0e\oplus\Lambda_0f$,

$$\operatorname{Ext}^i_\Lambda(X,\Lambda/\mathbf{r})=\operatorname{Ext}^i_\Lambda(X,(\Lambda/\mathbf{r})e)\oplus\operatorname{Ext}^i_\Lambda(X,(\Lambda/\mathbf{r})f)$$

and

$$\operatorname{Ext}^i_\Lambda(\Lambda_0,\Lambda_0)=\operatorname{Ext}^i_\Lambda(\Lambda_0e,\Lambda_0e)\oplus\operatorname{Ext}^i_\Lambda(\Lambda_0e,\Lambda_0f)\oplus\operatorname{Ext}^i_\Lambda(\Lambda_0f,\Lambda_0e)\oplus\operatorname{Ext}^i_\Lambda(\Lambda_0f,\Lambda_0f)$$

the result follows. □

If X is a graded Λ-module and $\mathcal{P}^\bullet\colon\cdots\to P^2\xrightarrow{\delta^2}P^1\xrightarrow{\delta^1}P^0\xrightarrow{\delta^0}X\to 0$ is a graded projective Λ-resolution of X, then we say that $\mathcal{P}^\bullet$ *is finitely generated* if P^n is a finitely generated graded Λ-module for $n\ge 0$. For $c\ge 0$, we let $\mathcal{P}^{>c}$ denote the resolution of $\Omega^{c+1}(X)$,

$$\mathcal{P}^{>c}\colon\cdots\to P^{c+2}\xrightarrow{\delta^{c+2}}P^{c+1}\xrightarrow{\delta^{c+1}}\Omega^{c+1}(X)\to 0,$$

obtained from $\mathcal{P}^\bullet$.

Let ϵ be an idempotent element of $\Lambda_{\mathfrak{e}}$. We say a graded simple module S *belongs to* ϵ if $\epsilon S\neq 0$. Equivalently, S belongs to ϵ if S is isomorphic to a summand of $(\Lambda/\mathbf{r})\epsilon[g]$, for some $g\in G$. We say a graded projective Λ-module P *belongs to* ϵ, if $P/\mathbf{r}P$ is a direct sum of graded simple Λ-modules with each summand belonging to ϵ. We now state a useful result.

Lemma 2.9. *Let X be a graded Λ-module and assume that $\mathcal{P}^\bullet\colon\cdots\to P^2\xrightarrow{\delta^2}P^1\xrightarrow{\delta^1}P^0\xrightarrow{\delta^0}X\to 0$ is a graded projective Λ-resolution of X such that, for $n>c$, P^n belongs to f. Then*

1. *$F(\mathcal{P}^{>c})$ is a projective Λ^*-resolution of $F(\Omega^{c+1}X)$, where $(\Omega^{c+1}X)$ is the $(c+1)$-st syzygy in $\mathcal{P}^\bullet$.*
2. *If $\mathcal{P}^\bullet$ is a finitely generated minimal graded projective Λ-resolution of X, then $F(\mathcal{P}^{>c})$ is a finitely generated minimal graded projective Λ^*-resolution of $F(\Omega^{c+1}X)$.*

Proof. The functor F is exact. We need to show that if P belongs to f, then $F(P)$ is a projective Λ^*-module. Since P belongs to f, P is a direct sum of indecomposable projective modules, each of which is a summand of $(\Lambda f)[g]$, for some $g\in G$. Thus, it suffices to show that, for $g\in G$, $F((\Lambda f)[g])$ is a graded projective Λ^*-module. But $F((\Lambda f)[g])=(f\Lambda\otimes_\Lambda\Lambda f)[g]\cong(f\Lambda f)[g]=\Lambda^*[g]$ and part (1) follows.

By minimality and our assumptions, the maps $F(\delta^i)$, viewed as matrices (as in Proposition 2.4), have entries in $f\mathbf{r}f$. But $f\mathbf{r}f=\mathbf{r}^*$, the graded Jacobson radical of Λ^*, and (2) follows. □

The following is an immediate consequence of the above lemma.

Corollary 2.10. *Assume that $\operatorname{id}_\Lambda((\Lambda/\mathbf{r})e)=b<\infty$. Suppose that X is a graded Λ-module and let*

$$\mathcal{P}^\bullet\colon\cdots\to P^2\xrightarrow{\delta^2}P^1\xrightarrow{\delta^1}P^0\xrightarrow{\delta^0}X\to 0$$

be a finitely generated minimal graded projective Λ-resolution of X. Then, for $n > b$, P^n belongs to f and $F(\mathcal{P}^{>b})$ is a finitely generated minimal graded projective Λ^*-resolution of $F(\Omega_\Lambda^{b+1}(X))$.

Proof. Let $n > b$ and consider P^n. If there is an indecomposable summand of P^n belonging to e, then $\operatorname{Ext}_\Lambda^n(X, (\Lambda/\mathbf{r})e) \neq 0$, contradicting $\operatorname{id}_\Lambda((\Lambda/\mathbf{r})e) = b$. Hence, P^n belongs to f and the result follows. $\square$

Using the above result we have one of the main results of this section.

Theorem 2.11. *Let G be a group and $\Lambda = \oplus_{g\in G}\Lambda_g$ be a properly G-graded ring in which graded idempotents lift. Let $\mathbf{r}$ denote the graded Jacobson radical of Λ and (e, f) be a suitable idempotent pair. Set Λ^* to be the ring $f\Lambda f$ and $\mathbf{r}^* = f\mathbf{r}f$. Assume that $\operatorname{pd}_{\Lambda^*}(f\Lambda e) = c < \infty$, and that $\operatorname{id}_\Lambda((\Lambda/\mathbf{r})e) = b < \infty$. Then, for a graded Λ-module X having a finitely generated minimal graded projective resolution and for $n > b + c + 2$, the functor $F = f\Lambda \otimes_\Lambda -\colon \mathbf{Gr}(\Lambda) \to \mathbf{Gr}(\Lambda^*)$ induces isomorphisms*

$$\operatorname{Ext}_\Lambda^n(\Phi(X), (\Lambda/\mathbf{r})f) \cong \operatorname{Ext}_{\Lambda^*}^n(\Phi(F(X)), \Lambda^*/\mathbf{r}^*).$$

Moreover, assuming that every graded simple Λ-module belonging to f has a finitely generated minimal graded projective resolution, then the induced isomorphism

$$\bigoplus_{n>b+c+2} \operatorname{Ext}_\Lambda^n((\Lambda/\mathbf{r})f, (\Lambda/\mathbf{r})f) \cong \bigoplus_{n>b+c+2} \operatorname{Ext}_{\Lambda^*}^n(\Lambda^*/\mathbf{r}^*, \Lambda^*/\mathbf{r}^*)$$

is an isomorphism of $\mathbb{Z} \times G$-graded rings without identity. Furthermore, identifying $\oplus_{n>b+c+2} \operatorname{Ext}_\Lambda^n((\Lambda/\mathbf{r})f, (\Lambda/\mathbf{r})f)$ with $\oplus_{n>b+c+2} \operatorname{Ext}_{\Lambda^*}^n(\Lambda^*/\mathbf{r}^*, \Lambda^*/\mathbf{r}^*)$ and denoting this ring by Δ, $\oplus_{n>b+c+2} \operatorname{Ext}_\Lambda^n(\Phi(X), (\Lambda/\mathbf{r})f)$ and $\oplus_{n>b+c+2} \operatorname{Ext}_{\Lambda^*}^n(\Phi(F(X)), \Lambda^*/\mathbf{r}^*)$ are isomorphic as graded Δ-modules.

Proof. Let X be a graded Λ-module and let

$$\mathcal{P}^\bullet\colon \cdots \to P^2 \xrightarrow{\delta^2} P^1 \xrightarrow{\delta^1} P^0 \xrightarrow{\delta^0} X \to 0$$

be a minimal graded projective Λ-resolution of the graded module X. By our assumption that $\operatorname{id}_\Lambda((\Lambda/\mathbf{r})e) = b$, for $n > b$, P^n belongs to f. Hence, applying the functor F and Corollary 2.10, we see that

$$F(\mathcal{P}^{>b})\colon \cdots \to F(P^{b+2}) \xrightarrow{F(\delta^{b+2})} F(P^{b+1}) \xrightarrow{F(\delta^{b+1})} F(\Omega^{b+1}(X)) \to 0$$

is a minimal graded projective Λ^*-resolution of $F(\Omega^{b+1}(X))$.

Let S be a simple graded Λ-module belonging to f and let $S^* = F(S)$. First we show that, using the above isomorphisms, if $n > b + c + 2$, then F induces a monomorphism

$$\operatorname{Ext}_\Lambda^n(\Phi(X), \Phi(S)) \to \operatorname{Ext}_{\Lambda^*}^n(\Phi(F(X)), \Phi(S^*)).$$

We view this morphism as the composition

$$\begin{aligned}\operatorname{Ext}_\Lambda^n(\Phi(X), \Phi(S)) &\to \operatorname{Ext}_{\Lambda^*}^{n-b-1}(\Phi(F(\Omega^{b+1}(X))), \Phi(S^*))\\ &\xrightarrow{\cong} \operatorname{Ext}_{\Lambda^*}^{n-b-1}(\Phi(\Omega^{b+1}(F(X))), \Phi(S^*))\\ &\xrightarrow{\cong} \operatorname{Ext}_{\Lambda^*}^n(\Phi(F(X)), \Phi(S^*)).\end{aligned}$$

The last map is an isomorphism by dimension shift and the commutativity of Φ and Ω. Since $n > b + c + 2$, we have $n - b - 1 > c + 1$, and the second map is an isomorphism by Theorem 2.7. We now describe the first map. We recall that

$$\operatorname{Ext}^n_\Lambda(\Phi(X), \Phi(S)) \cong \operatorname{Ext}^n_{\mathbf{Gr}(\Lambda)}(X, \oplus_{g\in G} S[g])$$

and that

$$\operatorname{Ext}^{n-b-1}_{\Lambda^*}(\Phi(F(\Omega^{b+1}(X))), \Phi(S^*)) \cong \operatorname{Ext}^{n-b-1}_{\mathbf{Gr}(\Lambda^*)}(F(\Omega^{b+1}(X)), \oplus_{g\in G} S^*[g]).$$

Suppose $\alpha\colon P^n \to S[g]$ represents an element in $\operatorname{Ext}^n_{\mathbf{Gr}(\Lambda)}(X, S[g])$. Since $F(\mathcal{P}^{>b})$ is a projective resolution of $F(\Omega^{b+1}(X))$, the map $F(\alpha)\colon F(P^n) \to S^*[g]$ represents an element in $\operatorname{Ext}^{n-b-1}_{\mathbf{Gr}(\Lambda^*)}(F(\Omega^{b+1}(X)), S^*[g])$. Since both P^n and S belong to f and $F(\mathcal{P}^{>b})$ is minimal, if α is non-zero, then $F(\alpha)$ is non-zero in $\operatorname{Ext}^{n-b-1}_{\mathbf{Gr}(\Lambda^*)}(F(\Omega^{b+1}(X)), S^*[g])$. In this way F induces a monomorphism

$$\operatorname{Ext}^n_\Lambda(\Phi(X), \Phi(S)) \to \operatorname{Ext}^{n-b-1}_{\Lambda^*}(\Phi(F(\Omega^{b+1}(X))), \Phi(S^*))$$

and hence a monomorphism

$$\operatorname{Ext}^n_\Lambda(\Phi(X), \Phi(S)) \to \operatorname{Ext}^n_{\Lambda^*}(\Phi(F(X)), \Phi(S^*)).$$

Having shown that if $n > b + c + 2$, then F induces a monomorphism

$$\operatorname{Ext}^n_\Lambda(\Phi(X), \Phi(S)) \to \operatorname{Ext}^n_{\Lambda^*}(\Phi(F(X)), \Phi(S^*)),$$

we now show that F induces an epimorphism. Since

$$\operatorname{Ext}^n_\Lambda(\Phi(X), \Phi(S)) \cong \operatorname{Hom}_\Lambda(\Phi(P^n), \Phi(S)) \cong \operatorname{Hom}_{\Lambda_0}(\Phi(P^n/\mathbf{r}P^n), \Phi(S))$$

and

$$\begin{aligned}\operatorname{Ext}^n_{\Lambda^*}(\Phi(F(X)), \Phi(S^*)) &\cong \operatorname{Hom}_{\Lambda^*}(\Phi(F(P^n)), \Phi(S^*))\\ &\cong \operatorname{Hom}_{\Lambda^*_0}(\Phi(F(P^n)/\mathbf{r}^*F(P^n)), \Phi(S^*)),\end{aligned}$$

we conclude that the lengths of $\operatorname{Ext}^n_\Lambda(\Phi(X), \Phi(S))$ and $\operatorname{Ext}^n_{\Lambda^*}(\Phi(F(X)), \Phi(S^*))$ are equal as modules over the commutative Artin ring C, over which both $\Lambda/\mathbf{r}$ and $\Lambda^*/\mathbf{r}^*$ are finite length modules. Since F induces a monomorphism, we conclude that F is an isomorphism.

By taking direct sums over simple modules belonging to f, the isomorphisms $\operatorname{Ext}^n_\Lambda(\Phi(X), \Phi(S)) \to \operatorname{Ext}^n_{\Lambda^*}(\Phi(F(X)), \Phi(S^*))$ induced by F extend to an isomorphism

$$\operatorname{Ext}^n_\Lambda(\Phi(X), (\Lambda/\mathbf{r})f) \cong \operatorname{Ext}^n_{\Lambda^*}(\Phi(F(X)), \Lambda^*/\mathbf{r}^*),$$

Taking $X = (\Lambda/\mathbf{r})f$ we obtain the isomorphism

$$\operatorname{Ext}^n_\Lambda((\Lambda/\mathbf{r})f, (\Lambda/\mathbf{r})f) \cong \operatorname{Ext}^n_{\Lambda^*}(\Lambda^*/\mathbf{r}^*, \Lambda^*/\mathbf{r}^*),$$

Since F is an exact functor, the induced isomorphism

$$\bigoplus_{n>b+c+2} \operatorname{Ext}^n_\Lambda((\Lambda/\mathbf{r})f, (\Lambda/\mathbf{r})f) \cong \bigoplus_{n>b+c+2} \operatorname{Ext}^n_{\Lambda^*}(\Lambda^*/\mathbf{r}^*, \Lambda^*/\mathbf{r}^*),$$

is an isomorphism of $\mathbb{Z} \times G$-graded rings (without identity). The assertion about $\oplus_{n>b+c+2} \operatorname{Ext}^n_\Lambda(\Phi(X), (\Lambda/\mathbf{r})f) \cong \oplus_{n>b+c+2} \operatorname{Ext}^n_{\Lambda^*}(\Phi(F(X)), \Lambda^*/\mathbf{r}^*)$ being a graded module isomorphism follows. $\square$

We have the following consequence of the above proof.

Proposition 2.12. *Keeping the notation and hypothesis of Theorem 2.11, let X be a graded Λ-module having a finitely generated minimal graded projective resolution. Let $n > b+c+2$. Then $\mathrm{pd}_\Lambda(\Phi(X)) \le n-1$ if and only if $\mathrm{pd}_{\Lambda^*}(\Phi(F(X))) \le n-1$. In particular, if every graded simple Λ-module belonging to e has a finitely generated minimal graded projective resolution, then $\mathrm{pd}_\Lambda((\Lambda/\mathbf{r})e) \le b+c+2$.*

Proof. From the proof of Theorem 2.11, we see that for every graded simple Λ-module S belonging to f,

$$\mathrm{Ext}^n_\Lambda(\Phi(X), \Phi(S)) \cong \mathrm{Ext}^n_{\Lambda^*}(\Phi(F(X)), \Phi(F(S))),$$

for $n > b+c+2$. We have $\mathrm{Ext}^n_{\Lambda^*}(\Phi(F(X)), -) = 0$ if and only if $\mathrm{pd}_{\Lambda^*}(\Phi(F(X))) \le n-1$. Since any simple Λ^*-module is isomorphic to a module of the form $F(S)$ with S a simple Λ-module belonging to f, and $F(\Omega^{b+1}(X))$ has a finitely generated minimal graded projective resolution, it follows that

$$\mathrm{Ext}^n_{\Lambda^*}(\Phi(F(X)), -) \cong \mathrm{Ext}^{n-b-1}_{\Lambda^*}(\Phi(F(\Omega^{b+1}(X))), -) = 0$$

if and only if

$$\mathrm{Ext}^n_{\Lambda^*}(\Phi(F(X)), \Phi(F(S))) \cong \mathrm{Ext}^{n-b-1}_{\Lambda^*}(\Phi(F(\Omega^{b+1}(X))), \Phi(F(S))) = 0$$

for every graded simple Λ-module S belonging to f. By our assumption on finitely generated resolutions, and that $\mathrm{id}_\Lambda((\Lambda/\mathbf{r})e) = b$, we see that $\mathrm{Ext}^n_\Lambda(\Phi(X), -) = 0$ if and only if $\mathrm{Ext}^n_\Lambda(\Phi(X), \Phi(S)) = 0$ for all graded simple modules S belonging to f. Finally, $\mathrm{Ext}^n_\Lambda(\Phi(X), -) = 0$ if and only if $\mathrm{pd}_\Lambda(\Phi(X)) \le n-1$.

If every graded simple Λ-module belonging to e has a finitely generated minimal graded projective resolution, then so has $(\Lambda/\mathbf{r})e$. Since $f\Lambda e \subseteq \mathbf{r}$, we get $F((\Lambda/\mathbf{r})e) = f\Lambda \otimes_\Lambda (\Lambda/\mathbf{r})e = 0$. The last statement follows. □

By combining Proposition 2.8 and Theorem 2.11, we obtain the desired result.

Theorem 2.13. *Let G be a group and $\Lambda = \oplus_{g\in G}\Lambda_g$ be a properly G-graded ring in which graded idempotents lift. Assume that every graded simple Λ-module has a finitely generated minimal graded projective Λ-resolution. Let $\mathbf{r}$ denote the graded Jacobson radical of Λ. Suppose that (e, f) is a suitable idempotent pair in $\Lambda_\mathfrak{e}$ and set Λ^* to be the ring $f\Lambda f$ and $\mathbf{r}^* = f\mathbf{r}f$. Assume that $\mathrm{pd}_{\Lambda^*}(f\Lambda e) = c < \infty$, and that $\mathrm{id}_\Lambda((\Lambda/\mathbf{r})e) = b < \infty$. Then, for $n > b+c+2$, there are isomorphisms $\mathrm{Ext}^n_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r}) \cong \mathrm{Ext}^n_{\Lambda^*}(\Lambda^*/\mathbf{r}^*, \Lambda^*/\mathbf{r}^*)$ such that the induced isomorphism*

$$\bigoplus_{n>b+c+2} \mathrm{Ext}^n_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r}) \cong \bigoplus_{n>b+c+2} \mathrm{Ext}^n_{\Lambda^*}(\Lambda^*/\mathbf{r}^*, \Lambda^*/\mathbf{r}^*)$$

is an isomorphism of $\mathbb{Z}\times G$-graded rings without identity.

Letting $\Delta = \oplus_{n>b+c+2}\mathrm{Ext}^n_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r})$, if X is a graded Λ-module having a finitely generated projective resolution, then

$$\bigoplus_{n>b+c+2} \mathrm{Ext}^n_\Lambda(\Phi(X), \Lambda/\mathbf{r}) \text{ and } \bigoplus_{n>b+c+2} \mathrm{Ext}^n_{\Lambda^*}(\Phi(F(X)), \Lambda^*/\mathbf{r}^*)$$

are isomorphic as graded Δ-modules.

Proof. Since every graded simple Λ-module has a finitely generated minimal graded projective Λ-resolution, Proposition 2.12 applies, and $\mathrm{pd}_\Lambda((\Lambda/\mathbf{r})e) \le b+c+2$. The rest follows from Proposition 2.8 and Theorem 2.11. □

3. Applications

We begin this section with a well-known result whose proof we include for completeness.

Proposition 3.1. *Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ be a finitely generated positively $\mathbb{Z}$-graded C-algebra where C is a commutative ring. Let N be a fixed positive integer. Then there is a positive integer D with $N < D$ such that the following holds.*

> *If $j \geq D$ and $r \in R_j$, then $r = \sum_i c_i u_{i,1} u_{i,2} \cdots u_{i,m_i}$, where $c_i \in C$ and each $u_{i,k}$ is homogeneous with $N \leq \deg(u_{i,k}) < D$.*

Proof. Assume that R can be generated over C by homogeneous elements $z_1, \ldots, z_m$ of degree 0 and $x_1, \ldots, x_n$ with each x_i having degree at least 1 and set $L = \max\{\deg x_i \mid 1 \leq i \leq n\}$. Set $D = 2LN$ and suppose $r \in R_j$ with $j \geq D$. Then, by finite generation, $r = \sum_i c_i y_{i,1} \cdots y_{i,t_i}$ where, for all i and k, $c_i \in C$, $y_{i,k}$ is of the form $w_{i,k} x_l w'_{i,k}$ or $x_l w'_{i,k}$ or $w_{i,k} x_l$ or x_l, where $w_{i,k}$ and $w'_{i,k}$ are products of z_s's and $\sum_{k=1}^{t_i} \deg(y_{i,k}) = j$, for each i. Fix i and write y_k instead of $y_{i,k}$ and set $t = t_i$. We see that

$$D = 2NL \leq j = \sum_{k=1}^{t} \deg(y_k) \leq Lt.$$

Hence $2N \leq t$. Write $t = AN + S$, where $0 \leq S < N$. For $i = 1, \ldots, A-1$, set $u_i = y_{(i-1)N+1} y_{(i-1)N+2} \cdots y_{iN}$ and $u_A = y_{(A-1)N+1} \cdots y_{AN} \cdot y_{AN+1} \cdots y_t$. It is immediate that for $1 \leq i \leq A$, $N \leq \deg(u_i) < 2NL = D$. This completes the proof. $\square$

We have some immediate consequences.

Corollary 3.2. *Let $R = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$ be a positively $\mathbb{Z}$-graded ring such that, R_0 is an Artin algebra over a commutative Artin ring C, and, for $i \geq 0$, R_i has finite length over R_0. Let N be a fixed positive integer. Then R is finitely generated as a ring over C if and only if $T = \oplus_{i \geq N} R_i$ is finitely generated as a ring (without identity) over C.*

Proof. Note that $R_0 \oplus R_1 \oplus \cdots \oplus R_{N-1}$ is of finite length over C. If T is finitely generated over C, adding a finite numbers generators of $R_0 \oplus R_1 \oplus \cdots \oplus R_{N-1}$ over C to a set of generators T yields a finite generating set for R.

If R is finitely generated as a ring over C, the proof of Proposition 3.1 implies that T is generated as a ring over C by $R_N \oplus R_{N+1} \oplus R_{N+2} \oplus \cdots \oplus R_{2NL-1}$. Since $R_N \oplus R_{N+1} \oplus R_{N+2} \oplus \cdots \oplus R_{2NL-1}$ is of finite length over C, there exists a finite set of generators for T as a ring over C. $\square$

Before stating the main theorem of the section, we consider low terms in resolutions of simple Λ- and Λ^*-modules. More precisely, suppose that G is a group and that Λ is a properly G-graded ring in which graded idempotents lift. Let (e, f) be a suitable idempotent pair in Λ and let $\Lambda^* = f\Lambda f$, $\mathbf{r}$ and $\mathbf{r}^*$ the graded Jacobson radicals of Λ and Λ^* respectively. Assume all the conditions of Theorem 2.13. Let S be a graded simple Λ-module and $S^* = f\Lambda \otimes_\Lambda S$, viewed as a graded Λ^*-module. Example 4.1 shows that even if S has a finitely generated graded projective Λ-resolution, S^* need not have a finitely generated graded projective Λ^*-resolution. To remedy this situation, we have the following result and its corollary.

Proposition 3.3. *Let G be a group and $R = \oplus_{g\in G} R_g$ be a G-graded ring. Let $\cdots \to X^2 \xrightarrow{d^2} X^1 \xrightarrow{d^1} X^0 \xrightarrow{d^0} M \to 0$ be an exact sequence of graded R-modules. If, for all $n \geq 0$, X^n has a finitely generated graded projective R-resolution, then M has a finitely generated graded projective R-resolution.*

Proof. For $j \geq 0$, let $X^{0,j} = X^j$ and $Y^{0,j} = \mathrm{Im}(d^j)$. Note that $Y^{0,0} = M$. For each $j \geq 0$, let

$$\cdots \to P^{2,j} \xrightarrow{\delta^{2,j}} P^{1,j} \xrightarrow{\delta^{1,j}} P^{0,j} \xrightarrow{\delta^{0,j}} X^{0,j} \to 0$$

be a finitely generated graded projective R-resolution of $X^{0,j}$. For $i \geq 0$, define $X^{i,j} = \mathrm{Im}(\delta^{i,j})$. Thus, for each $i \geq 0$, we have short exact sequences

$$0 \to X^{i+1,j} \to P^{i,j} \to X^{i,j} \to 0.$$

We inductively construct graded R-modules $Y^{i,j}$ and finitely generated graded projective R-modules $Q^{i,j}$ such that

(1) for each $i, j \geq 0$, there is a short exact sequence $0 \to Y^{i+1,j} \to Q^{i,j} \to Y^{i,j} \to 0$,
(2) for $i \geq 0$ and $j \geq 1$, there is a short exact sequence $0 \to X^{i+1,j-1} \to Y^{i+1,j-1} \to Y^{i,j} \to 0$ and,
(3) for $i \geq 1$ and $j \geq 0$, $Q^{i,j} = Q^{i-1,j+1} \oplus P^{i,j}$.

Once this is accomplished, splicing together the short exact sequences $0 \to Y^{i+1,0} \to Q^{i,0} \to Y^{i,0} \to 0$ we obtain a long exact sequence

$$\cdots \to Q^{2,0} \to Q^{1,0} \to Q^{0,0} \to Y^{0,0} \to 0.$$

But $Y^{0,0} = M$ and the result follows.

We have defined $Y^{0,s}$ and $P^{0,s}$ for all $s \geq 0$. Set $Q^{0,i} = P^{0,i}$, for all $i \geq 0$. We have exact sequences

$$0 \to Y^{0,s+1} \to X^{0,s} \to Y^{0,s} \to 0,$$

for all $s \geq$. We also have exact sequences $0 \to X^{1,s} \to P^{0,s} \to X^{0,s} \to 0$ for all $s \geq 0$. From these exact sequences we obtain the following commutative diagram.

$$\begin{tikzcd}
 & & 0 & 0 & & \\
0 \arrow[r] & Y^{0,s+1} \arrow[r] & X^{0,s} \arrow[r] \arrow[u] & Y^{0,s} \arrow[r] \arrow[u] & 0 & \\
 & 0 \arrow[r] & P^{0,s} \arrow[r, equal] \arrow[u] & P^{0,s} \arrow[r] \arrow[u] & 0 & \\
 & 0 \arrow[r] & X^{1,s} \arrow[r] \arrow[u] & Y^{1,s} \arrow[r] \arrow[u] & Y^{0,s+1} \arrow[r] & 0 \\
 & & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

where $Y^{1,s}$ is defined to be the kernel of the surjection $P^{0,s}$ to $Y^{0,s}$. Thus, we have defined $Y^{1,j}$ such that (1) holds for all $i = 0$ and $j \geq 0$ and (2) holds for all $j \geq 1$ and $i = 0$. For $i = 0$, (3) vacuously holds.

Now consider $0 \to X^{1,s} \to Y^{1,s} \to Y^{0,s+1} \to 0$ Using the exact sequences $0 \to Y^{2,s} \to Q^{1,s} \to Y^{1,s} \to 0$ and $0 \to X^{1,s+1} \to P^{0,s+1} \to X^{0,s+1} \to 0$ and using the Horseshoe Lemma, we obtain the following commutative diagram

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & X^{1,s} \arrow[r] \arrow[u] & Y^{1,s} \arrow[r] \arrow[u] & Y^{0,s+1} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & P^{1,s} \arrow[r] \arrow[u] & P^{1,s} \oplus Q^{0,s+1} \arrow[r] \arrow[u] & Q^{0,s+1} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & X^{2,s} \arrow[r] \arrow[u] & Y^{2,s} \arrow[r] \arrow[u] & Y^{1,s+1} \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

where $Y^{2,s}$ is the kernel of $P^{1,s} \oplus Q^{0,s+1} \to Y^{1,s}$. Let $Q^{1,s} = P^{1,s} \oplus Q^{0,s+1}$. It is immediate that (1) holds for all $j \geq 0$, (2) holds for all $j \geq 1$ and $i = 1$, and (3) holds for $i = 1$ and all $j \geq 0$.

Continuing in this fashion, we define the $X^{i,j}$ and $P^{i,j}$ for all $i, j \geq 0$ satisfying (1), (2), and (3). $\square$

Corollary 3.4. *Let G be a group and $\Lambda = \oplus_{g\in G}\Lambda_g$ be a properly G-graded ring in which graded idempotents lift. Suppose that (e, f) is suitable idempotent pair and set Λ^* to be the ring $f\Lambda f$. Assume that, as a left Λ^*-module, $f\Lambda e$ has a finitely generated graded projective resolution. Let M be a graded Λ-module having a finitely generated graded projective Λ-resolution. Then $F(M)$ has a finitely generated graded projective Λ^*-resolution.*

Proof. Let M be a graded Λ-module and let $\mathcal{P} : \cdots \to P^1 \to P^0 \to M \to 0$ be a finitely generated graded projective Λ-resolution of M. Applying the exact functor F, we get an exact sequence graded Λ^*-modules $F(\mathcal{P}) : \cdots \to F(P^1) \to F(P^0) \to F(M) \to 0$. The result will follow if we show that each $F(P^n)$ has a finitely generated graded projective Λ^*-resolution. For each $n \geq 0$, set $P^n = P^n_e \oplus P^n_f$, where P^n_e belongs to e and P^n_f belongs to f. By our hypothesis, $F(P^n_e)$ has a finitely generated graded projective Λ^*-resolution. Since $F(P^n_f)$ is a graded projective Λ^*-module and since $F(P^n) = F(P^n_e) \oplus F(P^n_f)$ we are done. $\square$

We can state the main theorem of this section. If Λ is a ring, then let $\mathrm{GKdim}(\Lambda)$ denote the Gelfand-Kirillov dimension of Λ (see [4] for an introduction to the subject) and $\mathrm{gl.dim}(\Lambda)$ denote the (left) global dimension of Λ. To simplify notation, we write $E(\Lambda)$ for the cohomology ring $\oplus_{n\geq 0} \mathrm{Ext}^n_\Lambda(\Lambda/\mathbf{r}, \Lambda/\mathbf{r})$.

Theorem 3.5. *Let G be a group and $\Lambda = \oplus_{g\in G}\Lambda_g$ be a properly G-graded ring in which graded idempotents lift. Assume that every graded simple Λ-module has a finitely generated minimal graded projective Λ-resolution. Let $\mathbf{r}$ denote the graded Jacobson radical of Λ. Suppose that (e, f) is a suitable idempotent pair and set Λ^* to be the ring $f\Lambda f$ and $\mathbf{r}^* = f\mathbf{r}f$. Suppose that $\mathrm{pd}_{\Lambda^*}(f\Lambda e) < \infty$, and that $\mathrm{id}_\Lambda((\Lambda/\mathbf{r})e) < \infty$. Then the following hold.*

(1) *Assume that* $f\Lambda e$ *has a finitely generated minimal graded projective* Λ^**-resolution. The cohomology ring* $E(\Lambda)$ *is finitely generated over* $\operatorname{Ext}^0_\Lambda(\Lambda/\mathbf{r},\Lambda/\mathbf{r}) \cong Hom_\Lambda(\Lambda/\mathbf{r},\Lambda/\mathbf{r}) \cong (\Lambda/\mathbf{r})^{op}$ *if and only if the cohomology ring* $E(\Lambda^*)$ *is finitely generated as a* $(\Lambda^*/\mathbf{r}^*)^{op}$*-algebra.*

(2) *Assume that* Λ *is* K*-algebra, where* K *is a field and that* $\Lambda/\mathbf{r}$ *is a finite dimensional* K*-algebra. Assume further that both* $E(\Lambda)$ *and* $E(\Lambda^*)$ *are finitely generated* K*-algebras. Then* $\operatorname{GKdim}(E(\Lambda)) = \operatorname{GKdim}(E(\Lambda^*))$*.*

(3) *We have that* $\operatorname{pd}_\Lambda(\Phi(S)) < \infty$*, for all graded simple* Λ*-modules* S *if and only if* $\operatorname{pd}_{\Lambda^*}(\Phi(S^*)) < \infty$*, for all graded simple* Λ^**-modules* S^**.*

Proof. Suppose C is a commutative Artin algebra over which $\Lambda/\mathbf{r}$ has finite length. Note that if S^* is a graded simple Λ^*-module, then there exists a graded simple Λ-module S such that $S^* \cong F(S)$. By Corollary 3.4 and our assumptions, it follows that every graded simple Λ^*-module has a finitely generated graded projective Λ^*-resolution. In particular, for $n \geq 0$, $\operatorname{Ext}^n_{\Lambda^*}(\Lambda^*/\mathbf{r}^*,\Lambda^*/\mathbf{r}^*)$ has finite length over C. Part (1) follows from Theorems 2.11 and 2.13, Proposition 3.1, and Corollary 3.2. Part (2) follows from the definition of Gelfand-Kirillov dimension and Theorem 2.13. Part (3) follows from Proposition 2.12. $\square$

Applying these results to the Artin algebra case we get the following corollary.

Corollary 3.6. *Let* Λ *be an Artin algebra. Let* $\mathbf{r}$ *denote the graded Jacobson radical of* Λ*. Suppose that* (e,f) *is a suitable idempotent pair in* Λ *and set* Λ^* *to be the ring* $f\Lambda f$ *and* $\mathbf{r}^* = f\mathbf{r}f$*. Suppose that* $\operatorname{pd}_{\Lambda^*}(f\Lambda e) < \infty$*, and that* $\operatorname{id}_\Lambda((\Lambda/\mathbf{r})e) < \infty$*. Then the following hold.*

(1) *The cohomology ring* $E(\Lambda)$ *is finitely generated over* $\operatorname{Ext}^0_\Lambda(\Lambda/\mathbf{r},\Lambda/\mathbf{r}) \cong Hom_\Lambda(\Lambda/\mathbf{r},\Lambda/\mathbf{r}) \cong (\Lambda/\mathbf{r})^{op}$ *if and only if the cohomology ring* $E(\Lambda^*)$ *is finitely generated as a* $(\Lambda^*/\mathbf{r}^*)^{op}$*-algebra.*

(2) *Assume that* Λ *is a finite dimensional* K*-algebra, where* K *is a field. Assume further that both* $E(\Lambda)$ *and* $E(\Lambda^*)$ *are finitely generated* K*-algebras. Then* $\operatorname{GKdim}(E(\Lambda)) = \operatorname{GKdim}(E(\Lambda^*))$*.*

(3) *We have that* $\operatorname{gl.dim}(\Lambda)$ *is finite if and only if* $\operatorname{gl.dim}(\Lambda^*)$ *is finite.*

Proof. We take G to be the trivial group and view Λ as a graded algebra. Then the grading is proper and graded idempotents lift. Every (graded) simple Λ-module and (graded) simple Λ^*-module has a finitely generated projective resolution, as does $f\Lambda e$. The result is now a direct consequence of Theorem 3.5. $\square$

4. Concluding remarks and examples

We begin this section with a discussion of the construction of $\Lambda^* = f\Lambda f$ in case Λ is a quotient of a path algebra. We keep the notation of Example 2.1; namely, let K be a field, $\mathcal{Q}$ be a finite quiver, G a group, $W\colon \mathcal{Q}_1 \to G\setminus\{\mathfrak{e}\}$ be a weight function, and I a graded ideal in the path algebra $K\mathcal{Q}$ generated by weight homogeneous elements. We also assume that I is contained in the ideal of $K\mathcal{Q}$ generated by the arrows of $\mathcal{Q}$. Again we assume there is an integer t such that all paths of weight $\mathfrak{e}$ and length greater than t starting and ending at the same vertex belong to I. Setting $\Lambda = K\mathcal{Q}/I$, the G-grading on $K\mathcal{Q}$ obtained from W induces a proper G-grading on Λ such that graded idempotents lift.

To simplify notation, if $x \in K\mathcal{Q}$, then we denote the element $x + I$ of Λ by $\bar{x}$. We wish to describe $\Lambda^* = f\Lambda f$, where $f = \sum_{v\in X} \bar{v}$ and X is a subset of the vertex set $\mathcal{Q}_0$. Set $e = \sum_{v \in \mathcal{Q}_0 \setminus X} \bar{v}$. We keep the notation that $\mathbf{r}$ is the graded Jacobson radical of Λ and $\mathbf{r}^* = f\mathbf{r}f$. Note that $\mathbf{r}$ is generated by all elements of the form $\bar{a}$, for $a \in \mathcal{Q}_1$, Λ^* has a G-grading induced from the G-grading on Λ, and $\mathbf{r}^*$ is the graded Jacobson radical of Λ^*. Furthermore, $\Lambda/\mathbf{r} \cong \prod_{v \in \mathcal{Q}_0} K$ and $\Lambda^*/\mathbf{r}^* \cong \prod_{v \in X} K$.

We define the quiver $\mathcal{Q}^*$ as follows. Let $\mathcal{Q}_0^* = X$. To define the set of arrows $\mathcal{Q}_1^*$, consider the set of paths $\mathcal{M}$ in $\mathcal{Q}$ such that $p \in \mathcal{M}$ if p is a path of length $n \geq 1$ in $\mathcal{Q}$ such that $p = u_1 \xrightarrow{a_1} u_2 \xrightarrow{a_2} u_3 \to \cdots \to u_n \xrightarrow{a_n} u_{n+1}$, with u_i belonging to e for $2 \leq i \leq n$ and u_1 and u_{n+1} belonging to f. Note that a vertex u belongs to f (respectively, to e) just means $u \in X$ (respectively, $u \notin X$). Then $\mathcal{Q}_1^* = \{a_p \mid p \in \mathcal{M}, p \text{ is a path from } u_1 \text{ to } u_{n+1}\}$. A path $p \in \mathcal{M}$ is called a *minimal f-path* and the arrow a_p in $\mathcal{Q}^*$ is called the *arrow in $\mathcal{Q}^*$ associated to minimal f-path p*. We note that if $a\colon u \to v$ is an arrow with u and v belonging to f, then a is a minimal f-path. It is also easy to see that if p is a path in Q from vertex u to vertex v with u and v belonging to f, then p can be uniquely written as a product of paths $p_1 \cdots p_m$, where each p_i is a minimal f-path.

We now turn our attention to relations. Let I^* be the ideal in $K\mathcal{Q}^*$ generated as follows. If $r \in I$ is an element with $r = vru$, where u and v are vertices belonging to f and $r = \sum_i c_i p_i$, where $c_i \in K$ and p_i is a path from u to v, then we set r^* to be $\sum_i c_i p_i^*$ where p_i^* is the path in $\mathcal{Q}^*$ obtained from p_i by replacing each minimal f-subpath p in p_i by a_p. Note that if a minimal f-path is in I, then the associated arrow is in I^*. We also note that although $\mathcal{Q}$ is a finite quiver and $\mathcal{Q}_0^*$ is a finite set, $\mathcal{Q}^*$ may have an infinite number of arrows. The next example demonstrates this and that even if I is an ideal in $K\mathcal{Q}$, finitely generated by homogeneous elements, I^* need not be finitely generated.

Example 4.1. Let $\mathcal{Q}$ be the quiver

$$\begin{tikzcd} \overset{u}{\circ} \arrow[r, "a"'] & \overset{v}{\circ} \arrow[loop above, "b"] \arrow[r, "c"'] & \overset{w}{\circ} \end{tikzcd}$$

Take $e = v$ and $f = u + w$. It is not hard to see that each path of the form $cb^n a$, $n \geq 0$ is a minimal f-path and that these are the only minimal f-paths. Hence, $\mathcal{Q}^*$ is the quiver with two vertices u and w, and a countable number of arrows $a_{ca}, a_{cba}, a_{cb^2a}, \ldots$, each starting at u and ending at w.

Let $W\colon \mathcal{Q}_1 \to \mathbb{Z}_{>0}$ by $W(a) = W(b) = W(c) = 1$ and I be the ideal in $K\mathcal{Q}$ generated by b^2. Set $\Lambda = K\mathcal{Q}/I$ and $\Lambda^* = K\mathcal{Q}^*/I^*$. Then $\Lambda^* = f\Lambda f$ and $I^* = fIf$. We have I^* is generated by $\{a_{cb^n a} \mid n \geq 2\}$. Note that both Λ and Λ^* are Artin algebras. Note that $\mathrm{pd}_\Lambda((\Lambda/\mathbf{r})e) = \mathrm{id}_\Lambda((\Lambda/\mathbf{r})e) = \infty$, where $\mathbf{r}$ is the graded Jacobson radical of Λ. Moreover, $\mathrm{gl.dim}(\Lambda^*) = 1$. This example shows that the finiteness of the injective dimension of $(\Lambda/\mathbf{r})e$ cannot be removed as a condition from Corollary 3.6.

If we take $I = (0) = I^*$ above, then both $\Lambda = K\mathcal{Q}$ and $\Lambda^* = K\mathcal{Q}^*$ are hereditary algebras. Hence Theorem 2.13 holds; in fact, the Ext^n's are 0 for $n \geq 2$. But $\mathrm{Ext}^1_{\Lambda^*}(\Lambda^*/\mathbf{r}^*, \Lambda^*/\mathbf{r}^*)$ is infinite dimensional and hence Theorem 3.5 fails. Note that $f\Lambda e$ does not have a finitely generated graded projective Λ^*-resolution.

We leave the proof of the following result to the reader.

Proposition 4.2. *Keeping the notation above, $\Lambda^* \cong K\mathcal{Q}^*/I^*$.* $\square$

In the quiver case where e is a idempotent associated to a vertex, the next result gives a sufficient condition for exactness of the functor H where $H\colon \mathbf{Gr}(\Lambda^*) \to \mathbf{Gr}(\Lambda)$ given by $H(X) = \mathrm{Hom}_{\Lambda^*}(f\Lambda, X)$ (see Section 2).

Proposition 4.3. *Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional K-algebra where K is field and I is an admissible ideal in the path algebra $K\mathcal{Q}$; that is, for some $n \geq 2$, $J^n \subseteq I \subseteq J^2$ where J is the ideal generated by the arrows of $\mathcal{Q}$. Assume that $K\mathcal{Q}$ is G-graded with the grading coming from a weight function $W\colon \mathcal{Q}_1 \to G$ and that I can be generated by homogeneous elements. Let e be an idempotent element of $K\mathcal{Q}$ associated to a vertex v. If $\mathrm{pd}_\Lambda((\Lambda/\mathbf{r})e) \leq 1$, then $\mathrm{pd}_{\Lambda^*}(f\Lambda e) = 0$ and H is exact.*

Proof. Assume $\mathrm{pd}_\Lambda((\Lambda/\mathbf{r})e) \leq 1$. By the Strong No Loop Theorem [3], there is no loop at v. (Alternatively, a loop at v would imply that Λe is a direct summand of $\mathbf{r}e$, a contradiction.) Let e be the idempotent in $K\mathcal{Q}$ associated to the vertex v and let $f = 1 - e$. Consider the short exact sequence $0 \to \mathbf{r}e \to \Lambda e \to (\Lambda/\mathbf{r})e \to 0$. Applying the functor F, we obtain

$$\begin{tikzcd}
0 \arrow[r] & f\Lambda \otimes_\Lambda \mathbf{r}e \arrow[r] \arrow[d, "\cong"] & f\Lambda \otimes_\Lambda \Lambda e \arrow[r] \arrow[d, "\cong"] & f\Lambda \otimes_\Lambda (\Lambda/\mathbf{r})e \arrow[r] \arrow[d, "\cong"] & 0 \\
 & f\mathbf{r}e & f\Lambda e & 0 &
\end{tikzcd}$$

It follows that $f\Lambda e \cong f\mathbf{r}e$. Since $\mathrm{pd}_\Lambda(\Lambda/\mathbf{r})e \leq 1$, $\mathbf{r}e \cong \oplus \Lambda w$ where the direct sum runs over the arrows $v \to w$ in $\mathcal{Q}$ and w belongs to f, and where Λw is the projective Λ-module associated to the vertex w. Since each w belongs to f, it follows that $f\Lambda w = f\Lambda f w = \Lambda^* w$, which is a projective Λ^*-module. Thus $f\Lambda e$ is a projective Λ^*-module and by the remark after Proposition 2.5, H is exact. $\square$

Let $\Lambda = K\mathcal{Q}/I$ be a finite dimensional K-algebra where K is field and I is an admissible ideal in the path algebra $K\mathcal{Q}$. Assume that $K\mathcal{Q}$ is G-graded with the grading coming from a weight function $W\colon \mathcal{Q}_1 \to G$ and that I can be generated by homogeneous elements. Let e be an idempotent element of $K\mathcal{Q}$ associated to a vertex v. As usual let $f = 1 - e$. It is well known that $\mathrm{pd}_\Lambda((\Lambda/\mathbf{r})e) \leq 1$ if and only if there exists a uniform set ρ of generators of I such that $gv = 0$ for all $g \in \rho$, where an element $r \in K\mathcal{Q}$ is *uniform* if there exist vertices u and w in $\mathcal{Q}$ such that $r = wru$. Thus, if there exists a uniform set ρ of generators of I such that $gv = 0$ for all $g \in \rho$ and if $\mathrm{id}_\Lambda((\Lambda/\mathbf{r})e) < \infty$, then Theorems 2.13 and 3.5 apply, as in the following example.

Example 4.4. Let $\mathcal{Q}$ be the quiver

$$\begin{tikzcd}
\overset{u}{\circ} \arrow[r, "a"] & \overset{v}{\circ} \arrow[r, "b"] & \overset{w}{\circ} \arrow[d, "c"] \\
 & \underset{z}{\circ} \arrow[ul, "h"] & \underset{x}{\circ} \arrow[l, "d"]
\end{tikzcd}$$

Let I be the admissible ideal in $K\mathcal{Q}$ with a uniform set of generators $\rho = \{dcb, bahd\}$, and let $W\colon \mathcal{Q}_1 \to G$ be some weight function. Consider the G-graded finite dimensional K-algebra $\Lambda = K\mathcal{Q}/I$. Let e be the idempotent element associated to the vertex u. Then $gu = 0$ for all $g \in \rho$ and $\mathrm{id}_\Lambda((\Lambda/\mathbf{r})e) = 1 < \infty$, so Theorems 2.13 and 3.5 apply.

In this example $\Lambda^* \cong K\mathcal{Q}^*/I^*$, where $\mathcal{Q}^*$ is the quiver

$$\begin{tikzcd} \overset{v}{\circ} \arrow[r, "b"] & \overset{w}{\circ} \arrow[d, "c"] \\ \overset{z}{\circ} \arrow[u, "a_{ah}"] & \overset{x}{\circ} \arrow[l, "d"] \end{tikzcd},$$

and I^* is generated by $\{dcb, ba_{ah}d\}$.

The next result gives sufficient conditions so that $\mathrm{pd}_{\Lambda^*}(f\Lambda e) < \infty$.

Proposition 4.5. *Let G be a group and $\Lambda = \oplus_{g\in G}\Lambda_g$ be a properly G-graded ring in which graded idempotents lift. Suppose that (e, f) is a suitable idempotent pair and set Λ^* to be the ring $f\Lambda f$. Suppose that $0 \to P^n \to \cdots \to P^1 \to P^0 \to (\Lambda/\mathbf{r})e \to 0$ is a minimal graded projective Λ-resolution of $(\Lambda/\mathbf{r})e$ and that each P^i, for $i \geq 1$, is a direct sum of indecomposable projective Λ-modules of the form Λw with w a vertex belonging to f. Then $\mathrm{pd}_{\Lambda^*}(f\Lambda e) < \infty$.*

Proof. Note that $f\Lambda \otimes_\Lambda (\Lambda/\mathbf{r})e = 0$ and that $P^0 \cong \Lambda e$; so that $f\Lambda \otimes_\Lambda \Lambda e \cong f\Lambda e$. We see that the result follows by tensoring the projective resolution of $(\Lambda/\mathbf{r})e$ with $f\Lambda \otimes_\Lambda -$. $\square$

Example 4.6. We end with a nontrivial class of examples where the hypotheses of main theorems of the paper hold. Let K be a field and Δ and Σ be finite dimensional K-algebras. Suppose that A is finite dimensional K-Σ-bimodule, B is a finite dimensional Δ-K-bimodule, C is a finite dimensional Δ-Σ-bimodule, and $\mu\colon B \otimes_K A \to C$ is a Δ-Σ-bimodule homomorphism. Let

$$\Lambda = \begin{pmatrix} K & A & 0 \\ 0 & \Sigma & 0 \\ B & C & \Delta \end{pmatrix},$$

where the ring operations are given by matrix addition and multiplication. Part of the multiplication involves μ, if $a \in A$ and $b \in B$, then

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \mu(b\otimes a) & 0 \end{pmatrix}.$$

Set $e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Note that $f\Lambda f = \begin{pmatrix} \Sigma & 0 \\ C & \Delta \end{pmatrix}$.

The reader may verify that if $\mathrm{id}_{\Sigma^{op}}(A) < \infty$ and $\mathrm{pd}_\Delta(B) < \infty$, then $\mathrm{id}_\Lambda((\Lambda/\mathbf{r})e) < \infty$ and $\mathrm{pd}_{f\Lambda f}(f\Lambda e) < \infty$. Note that C is an arbitrary finite dimensional bimodule. The bimodule homomorphism μ can also be chosen arbitrarily, it can for instance be chosen to be zero. Thus if $\mathrm{id}_{\Sigma^{op}}(A) < \infty$ and $\mathrm{pd}_\Delta(B) < \infty$, then, taking $G = \{\mathfrak{e}\}$ to be the trivial group, Theorem 2.13 and Corollary 3.6 can be applied.

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