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The impact of variable fluid properties on hydromagnetic boundary layer and heat transfer flows over an exponentially stretching sheet

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Abstract

This paper put forward an analysis of variable fluid properties and their impact on hydromagnetic boundary and thermal layers in a quiescent fluid which is developed due to the exponentially stretching sheet. The viscous incompressible fluid has been set into motion due to aforementioned sheet. We assume that the viscosity and the thermal conductivity of the Newtonian fluid are temperature dependent. The governing boundary layer equations containing continuity, momentum and energy equations are coupled and nonlinear in nature, thereby, cannot be solvable easily by using analytical methods. Since the general boundary layer equations admits a similarity solutions, thus a generalized Howarth-Dorodnitsyn transformations have been exploited for the set-up of a coupled nonlinear ODEs. These transformed ODEs are solved numerically by a shooting method and is verified from MATLAB built-in collocation solver *bvp4c* for different parameters appearing in the work. We show results graphically and in a tabulated form for a constant and a variable fluid properties. We find that the temperature dependent variable viscosity and a thermal conductivity influence a velocity and a temperature profiles. We show that the thermal boundary layer decreases for constant variable fluid properties and increases for variable fluid properties

Nomenclature

(u, v)	the velocity components
(x, y)	Cartesian coordinates
L	characteristic length
U_w	sheet velocity
T_w	sheet temperature
μ	the coefficient of viscosity
ρ	the density of fluid
M	magnetic parameter
T	fluid temperature
k	the thermal conductivity of the fluid
Pr_0	ambient Prandtl number
C_p	the specific heat at constant pressure
T_0	the ambient fluid temperature
C_f	local skin friction coefficient

Nu_x local Nusselt number

1. Introduction

The principles of heat transfer in manufacturing industry is a chief theory behind the design and production of many household appliances and commercially used devices. The examples of heat transfer can be found in air conditioning system, refrigerators, the TV and the DVD player, to name a few. Even heat transfer flows are more important due to stretching sheet which has abundance of applications in industries, engineering, metallurgy, paper production, drawing of plastic films, hot rolling wires, elongation bubbles, extrusion processes in which the deformed materiel is pass out from die for final product, geological stretching of the tectonic plates during earthquake etc.

A Blasius type moving flow due to a stretching sheet issuing steadily from the slit has been investigated by Sakiadis [1]. The numerical and integral methods have been carried out to obtain the solution of the underlying study. He indicated that the boundary layer behavior on such surface is different than the surface of finite length. Owing to the need of definitive experiment for the boundary layer of continuous surface, the combination of experimental and analytical verifications have been considered in Tsou *et al* [2]. A three page article by Crane [3] extended the work of Sakiadis [1] in that he took the boundary layer flow over a stretching sheet where velocity varies linearly from the slit. The work on unsteady viscous flow has been only assumed adjacent to stagnation point by Rott [4] but far away from the plate the flow is taken as steady. The plate performed harmonic motion in its own plane i.e. along x-direction and he has shown that this problem is solvable exactly. Danberg and Fansle [5] enhanced this idea further for non-similar stretching wall where velocity is proportional to the distance x. Chakrabarti and Gupta[6] has extended the specialized case of Danberg and Fansle [5] and considered an electrically conducting fluid with a uniform transverse magnetic field. The motion in the fluid is caused by a stretching of the wall. Soundalgekar and Murty [7] tackled a heat transfer problem past a continuous semi-infinite flat plate in which temperature varies nonlinearly i.e. Ax^n, where A is a constant and n is never o or 1. They observed that the Nusselt number increases with increasing the exponent n. Wang [8], on the other hand, moved one step further and presented analysis for the three dimensional flow caused by two lateral directions where wall velocities varies linearly. The list of available literature on boundary layer flows for different fluids and flows over a stretching sheet with different aspects is long. For detail the reader is referred to Dutta *et al* [9], Grubka and Bobba [10–21], and forthcoming cited literature in next paragraphs.

In boundary layer flow, if a temperature difference is strong then the assumption of fluid properties are constant may lead to different results and hence wrong interpretation of the post processing. The dynamic viscosity is highly dependent on a temperature and is weakly dependent on thermodynamic pressure. Takhar *et al* [22] was the first who has discussed variable fluid properties. Pantokratoras [23] have discussed results of variable viscosity on the flow due to a continuous moving flat plate. He assumed that the Prandtl number is variable across a boundary layer. His assumption is based on the definition of Prandtl number which depends on viscosity i.e. if viscosity is variable so do the Prandtl number. This assumption is not correct as discussed in Andersson and Aarsaeth [24] . A compact analysis on variable fluid properties for Sakiadis problem have been presented by Andersson and Aarsaeth [24]. They clarify some of the misconceptions prevalent in scientific community over a variable fluid properties. Lai and Kulacki [25] investigated variable fluid properties for convective heat transfer in a saturated porous medium since previous studies mostly dealt with constant fluid properties for water. Their work is also concerned on heat transfer analysis for gases too. Kameswaran *et al* [26] studied the effect of radiation on the MHD Newtonian fluid flow due to an exponentially stretching sheet when considering the effects of viscous dissipation and frictional heating on the heat transport. Hayat *et al* [27] have deliberated axisymmetric hydromagnetic flow of a third grade fluid. The idea was to observe characteristics of flow over a stretching cylinder. They reported that the velocity and momentum boundary layer thickness is dependent on the curvature parameter. They also mentioned that velocity profile is higher for third grade fluid than the Newtonian and second grade fluid with and without MHD. Very recently Babu *et al* [28] discussed MHD dissipative flow across slendering stretching sheet with temperature dependent variable viscosity. Study of viscoelastic boundary layer flow and heat transfer over an exponentially stretching sheet was examined by Khan and Sanjayanand [29]. Pop *et al* [30] have examined the influence of variable viscosity on laminar boundary layer flow. They assumed the fluid viscosity varies inversely with temperature. Ali [31] considered heat transfer characteristics over a nonlinearly stretching sheet. Prasad *et al* [32] similar to Ali [31] have studied the effect of variable viscosity and thermal conductivity over a nonlinearly stretching sheet. Magyari and Keller [33] considered mass and heat transfer in the boundary layers on acontinuous surface which is stretched exponentially. The flow of a viscoelastic fluid over a stretching sheet with transverse magnetic field is assumed by Andersson [34]. He showed that the MHD has the same effect on the flow as viscoelasticity. In a similar work, a power-law fluid over a stretching sheet was investigated by Andersson *et al* [35]. They have shown that the

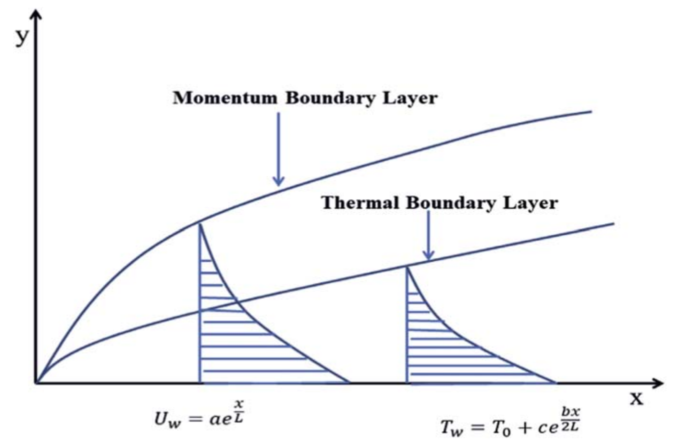

Figure 1. Geometry of the problem.

magnetic field make the boundary layer thinner for the underlying case. Nadeem *et al* [36] analyzed the heat transfer characteristic while presenting two cases, Prescribed exponential order surface temperature (PEST) and prescribed exponential order heat flux (PEHF). They studied Jeffrey fluid over an exponentially stretching surface. Although, viscous dissipation is a key term appearing in energy equation but considered by very few scientists. Pavithra *et al* [37] took this task to include viscous dissipation in dusty fluid over an exponentially stretching sheet and also discussed two cases for heat transfer analysis: Prescribed exponential order surface temperature (PEST) and prescribed exponential order heat flux (PEHF). Mabood *et al* [38] did analysis on viscous incompressible flow along with radiation effect while taking exponentially stretching sheet. They obtained the solution by using homotopy analysis method (HAM). Mukhopadhyay [39] studied MHD boundary layer flow and heat transfer towards an exponentially stretching sheet embedded in a thermally stratified permeable medium. Singh and Agarwal [40] investigated the effects of variable fluid properties of Maxwell fluid over an exponentially stretching sheet. They applied Keller-Box method to find a numerical solution. A variable thermal conductivity has been accounted with Cattaneo—Christov heat flux formulation in Hayat *et al* [13].

All studies of the past have considered variable fluid properties with many different fluids over a different type of stretching sheets. Not much work has been done on variable fluid properties, specifically temperature dependent viscosity and thermal conductivity, over an exponentially stretching sheet with MHD effect. We fill these gaps and present some interesting results on this topic.

The present paper has been organized as follows. In section 2, we present a mathematical model for the flow and heat transfer analysis. The three distinct cases have been discussed in section 3. The computational procedure has been explained in section 4. In section 5, we present the graphs, tables and their discussion. The conclusion has been drawn in section 6.

2. Problem formulation

Consider a steady, two dimensional, incompressible flow of an electrically conducting fluid over a sheet that has been stretched exponentially. The x-axis is taken along the sheet and y-axis is normal to it. B_o is the strength of uniform magnetic field which is applied normal to the sheet. The induced magnetic field is neglected because the value of a magnetic Reynolds number is less than unity in an electrically conducting fluids. T_w is a temperature of the sheet and T_o is the temperature of the ambient fluid. The geometrical configuration of the problem can be seen in the figure 1 for better understanding and visualization. The governing equations with these assumptions are given by Andersson and Aarseth [4]

$$\partial_x(\rho u) + \partial_y(\rho v) = 0, \tag{1a}$$

$$\rho(uu_x + vu_y) = \partial_y(\mu u_y) - \sigma B_0^2 u, \tag{1b}$$

$$\rho C_p(uT_x + vT_y) = \partial_y(kT_y), \tag{1c}$$

with boundary conditions

$$u(x,\,0)=U_w(x)=ae^{x/L},\qquad v(x,\,0)=0,\qquad T(x,\,0)=T_w(x)=T_0+ce^{bx/2L}$$
$$u\to 0,\qquad T\to T_0,\qquad as\ y\to\infty \tag{2}$$

where u is a $x-$ component and v is a $y-$ component of a fluid's velocity. Fluid density is represented by ρ, B_0 is the strength of an applied magnetic field, μ is the dynamic viscosity, specific heat is denoted by C_p, fluid's temperature is symbolized by T and the factor k appearing in energy equation is commonly known as a thermal conductivity. U_w represents the velocity of the sheet, wall temperature is denoted by $T_w(x)$.

Since governing equations are written in general set-up, we cannot apply usual similarity transformation. But we take the following Howarth-Dorodnitsyn transformations Dorodnitsyn [41], Howarth [42]:

$$\eta=\sqrt{\frac{a}{2\nu_0 L}}\,e^{x/2L}\int_0^y\frac{\rho}{\rho_0}dy,\qquad \psi=\rho_0\sqrt{2a\nu_0 L}\,e^{x/2L}f(\eta),\qquad \theta(\eta)=\frac{T-T_0}{T_w-T_0} \tag{3}$$

here the stream function is denoted by ψ and its relation with u and v have been given as Andersson and Aarseth [4].

$$\rho u=\frac{\partial\psi}{\partial y}\qquad \rho v=-\frac{\partial\psi}{\partial x} \tag{4}$$

Using equation (4) the x and y components of velocity can be written as

$$u=ae^{x/L}f'(\eta),\qquad v=-e^{x/2L}\sqrt{\frac{a\nu_0}{2L}}(\eta f'+f) \tag{5}$$

Inserting equations (3), (4) and (5) into (1a), (1b) and (1c), we get a system of nonlinear ODEs

$$\left(\frac{\rho\mu}{\rho_0\mu_0}f''\right)'-2Mf'-2(f')^2+ff''=0 \tag{6a}$$

$$\left(\frac{\rho k}{\rho_0 k_0}\theta'\right)'+\frac{C_p}{C_{p0}}Pr_0(f\theta'-bf'\theta)=0 \tag{6b}$$

where Pr_o, M are Prandtl number and magnetic parameter, respectively. These parameters are defined as follows

$$Pr_o=\frac{\mu_o C_{po}}{k_o}\qquad M=\frac{\sigma B_0^2 L}{\rho_0 ae^{x/L}}$$

The connected transformed boundary conditions of the ODEs (2) are :

$$f(0)=0,\qquad f'(0)=1,\qquad \theta(0)=1,$$
$$f'(\eta)=0,\qquad \theta(\eta)=0\quad as\quad \eta\to\infty \tag{7}$$

where f' denotes dimensionless velocity and θ denotes dimensionless temperature.

The skin friction coefficient C_f and local Nusselt number Nu_x are defined as follows:

$$C_f=\frac{2\tau_w}{\rho U_w^2},\qquad Nu_x=\frac{xq_w}{k(T_w-T_0)}, \tag{8}$$

where τ_w is a shear stress and q_w regarded as a heat flux, and these are defined as:

$$\tau_w=\mu a\sqrt{\frac{a}{2\nu_0 L}}\,e^{3x/2L}f''(0),\qquad q_w=-kce^{bx/2L}\sqrt{\frac{a}{2\nu_0 L}}\,e^{x/2L}\theta'(0), \tag{9}$$

Using above equations (8) and (9)we get

$$C_f Re^{1/2}=(2X)^{\frac{1}{2}}f''(0),\qquad Nu_x Re^{-1/2}=-\left(\frac{X}{2}\right)^{\frac{1}{2}}\theta'(0),$$

where Re denotes local Reynolds number.

It is important to note that all the fluid properties considered here are constant except the viscosity and thermal conductivity which are temperature dependent.

3. Special cases

3.1. Case A: constant fluid properties

For this case, we assume all the fluid properties as constant. By this assumption the momentum equation (6a) and energy equation (6b) becomes

$$f''' + ff'' - 2f'^2 - 2Mf' = 0, \tag{10}$$

$$\theta'' + Pr_0(f\theta' - bf'\theta) = 0, \tag{11}$$

The boundary conditions given in equation (7) remains the same.

3.2. Case B: variable fluid properties

For this case, we assume viscosity and thermal conductivity as variable that depends on a temperature when keeping the other physical properties as constant. For this case the momentum boundary layer equation equation (6a) becomes

$$\left(f''\frac{\mu}{\mu_0}\right)' + ff'' - 2f'^2 - 2Mf' = 0. \tag{12}$$

Energy equation (6b) reads as

$$\left(\frac{k}{k_0}\theta'\right)' + Pr_0(f\theta' - bf'\theta) = 0. \tag{13}$$

Lai and Kulachi [5], Ling and Dybbs [31] and Pop et al [10] suggested the following relation between viscosity and temperature:

$$\mu(T) = \frac{\mu_{ref}}{[1 + \delta(T - T_{ref})]},$$

where δ is property of the fluid that depends on the reference temperature T_{ref}.

If $T_{ref} \approx T_0$, the above formula becomes

$$\mu = \frac{\mu_0}{1 - \frac{T - T_0}{\theta_{ref}(T_w - T_0)}} = \frac{\mu_0}{1 - \frac{\theta(\eta)}{\theta_{ref}}}, \tag{14}$$

here $\theta_{ref} \equiv \frac{-1}{(T_w - T_0)\gamma}$ and $\Delta T = (T_w - T_0)$.

By using equation (14) in equation (12), we get the following momentum equation

$$f''' + \frac{\theta'}{\theta_{ref} - \theta}f'' + \left(\frac{\theta_{ref} - \theta}{\theta_{ref}}\right)(ff'' - 2Mf' - 2f'^2) = 0. \tag{15}$$

The thermal conductivity is defined as Subhas et al [36]

$$k(T) = k_0(1 + \epsilon\theta),$$

$$\frac{k}{k_0} = 1 + \epsilon\theta, \tag{16}$$

using the above relation (16) in equation (13) we get the following energy equation.

$$(1 + \epsilon\theta)\theta'' + \epsilon\theta'^2 + Pr_0(f\theta' - bf'\theta) = 0. \tag{17}$$

3.3. Case C: exponential temperature dependency

Similar to Case B, viscosity is again taken as variable but its variation depends exponentially on temperature White [43]

$$\ln\left(\frac{\mu}{\mu_{ref}}\right) = -\left(2.10 + 4.45\frac{T_{ref}}{T} - 6.55\left(\frac{T_{ref}}{T}\right)^2\right). \tag{18}$$

Substituting the above equation (18) in equation (12) the equation results into:

$$f''' = -f''\theta'\Delta T\left(4.45\frac{T_{ref}}{T^2} - 13.1\frac{T_{ref}^2}{T^3}\right) + \frac{\mu_0}{\mu}(2f'^2 - ff'' + 2Mf'). \tag{19}$$

while energy equation remains the same as shown in equation (17).

4. Numerical procedure

Here we find the numerical solution of nonlinear (ODEs) for each Cases A, B and C with the boundary conditions as given in equation (12). We apply shooting technique to obtain numerical results. The basic idea behind the shooting technique is to transform BVP into an IVP. Then find the roots by using Newton-Raphson technique and Runge-Kutta technique of fifth order on the resultant IVP. Results obtained from shooting

Table 1. Values of skin friction and wall temperature gradient for different physical parameters for Case A.

Pr	M	*bvp4c* $-f''(0)$	$-\theta'(0)$	Shooting method $-f''(0)$	$-\theta'(0)$	cpu time(bvp4c)
7	0	1.281 830 9	3.013 197 6	1.281 808 6	3.013 278 3	1.702 288 s
—	0.1	1.358 984	2.299 339 66	1.358 956 9	2.993 482	0.772 894 s
—	0.2	1.431 606	2.974 728 9	1.431 573 7	2.297 481 7	0.722 582 s
—	0.3	1.500 470 9	2.957 044 9	1.500 464 3	2.957 069 9	1.685 556 s
—	0.4	1.566 199 1	2.940 072 7	1.566 191 6	2.940 097 4	1.402 203 s
3	0.1	1.358 981 4	1.848 470 2	1.358 957 1	1.848 469 8	0.722 257 s
5	—	1.358 980 1	2.480 004 5	1.358 956 9	2.480 048	0.729 234 s
7	—	1.358 961 7	2.993 455 7	1.358 956 9	2.993 482	0.747 933 s
10	—	1.358 961 5	3.640 761 6	1.358 956 9	3.640 832 3	0.749 508 s

technique are verified with *bvp4c*, a built-in solver in MATLAB. For numerical solutions of different cases, we adopted the strategy as explained below:

(a) Case A: system of equations for momentum and energy becomes

$$y_3' = -y_1y_3 + 2y_2^2 + 2My_2, \tag{20}$$

$$y_5' = Pr_0(by_2y_4 - y_1y_5). \tag{21}$$

(b) Case B: momentum equation becomes,

$$y_3' = \frac{y_3y_5}{0.25 + y_4} + \frac{0.25 + y_4}{0.25}(2y_2^2 + 2My_2 - y_1y_3). \tag{22}$$

Energy equation takes the form

$$y_5' = -\frac{1}{1 + \epsilon y_4}(\epsilon y_5^2 + Pr_0(y_1y_5 - by_2y_4)). \tag{23}$$

(c) Case C: momentum equation becomes,

$$y_3' = -y_3y_5\Delta T\left(4.45\frac{T_{ref}}{T^2} - 13.1\frac{T_{ref}^2}{T_3}\right) + \frac{\mu_0}{\mu}(2y_2^2 - y_1y_3 + 2My_2). \tag{24}$$

here White [43]

$$\frac{\mu}{\mu_0} = \frac{\mu_{ref}}{\mu_0}e^{-\left(2.10+4.45\left(\frac{T_{ref}}{T}\right)-6.65\left(\frac{T_{ref}}{T}\right)^2\right)},$$

here $\mu_{ref} = 0.001\ 792kg/ms$, $\mu_0 = 0.001\ 520kg/ms$, $T_{ref} = 273$ K and $T_0 = 278$ K while energy equation remains same as shown in equation (23).

5. Results and discussions

In this part, numerical results of velocity and temperature gradients are discussed. Results are shown in tabular and graphical form. Numerical solutions for $-f''(0)$ (coefficient of skin friction) and $-\theta'(0)$ (temperature gradient) for various values of physical parameters that are Prandtl number, magnetic parameter and the parameter ϵ numerical results of have been shown from numerical results of tables 1 to 4. From tables 1–3 one can observe that the skin friction coefficient increases whereas a reduction in wall temperature have been seen as magnetic parameter arises. The Prandtl number increases wall temperature for all the three cases but skin friction changes slightly. It can also be seen that the parameter ϵ reduces both the skin friction coefficient and wall temperature for the Cases B and C. In table 4 numerical results for skin friction coefficient and heat transfer rate are computed for all the cases by increasing the Prandtl number. The value of skin friction coefficient increases for two Cases B and C but for case A it shows a decreasing behaviour. The wall temperature shows increasing behavior for all the three cases. In table 5 we compare our results with the previously published data.

Table 2. Values of skin friction and wall temperature gradient for different physical parameters for Case B.

Pr	M	ϵ	*bvp4c* $-f''(0)$	$-\theta'(0)$	Shooting method $-f''(0)$	$-\theta'(0)$
7	0	0.1	3.315 254 1	2.480 971 7	3.315 144 1	2.480 938 2
—	0.1	—	3.492 423 9	2.436 261 7	3.492 291	2.436 224 3
—	0.2	—	3.654 614 7	2.395 552 6	3.654 457 1	2.395 511 1
—	0.3	—	3.805 688 1	2.357 806	3.805 504 8	2.357 760 2
—	0.4	—	3.947 960 7	2.322 362 3	3.947 845 7	2.322 337
3	0.1	0.1	3.277 733 5	1.402 271 2	3.277 679 5	1.402 261 8
5	—	—	3.394 529 1	1.972 303 6	3.394 461 8	1.972 289 6
7	—	—	3.492 364 1	2.436 242 8	3.492 291	2.436 224 3
10	—	—	3.615 561 8	3.022 006 2	3.615 481 5	3.021 974 7
7	0.1	0	3.518 218 6	2.612 649 6	3.518 138 7	2.612 625 4
—	—	0.1	3.492 364 1	2.436 242 8	3.492 291	2.436 224 3
—	—	0.2	3.469 090 9	2.286 594 5	3.469 020 1	2.286 579 4

Table 3. Values of skin friction and wall temperature gradient for different physical parameters for Case C.

Pr	M	ϵ	*bvp4c* $-f''(0)$	$-\theta'(0)$	Shooting method $-f''(0)$	$-\theta'(0)$
7	0	0.1	3.268 118 3	2.509 089 3	3.268 09	2.509 08
—	0.1	—	3.441 183 6	2.466 886 7	3.441 15	2.466 88
—	0.2	—	3.599 361 1	2.428 220 6	3.599 32	2.428 21
—	0.3	—	3.746 234 7	2.392 252 9	3.746 19	2.392 24
—	0.4	—	3.884 138 7	2.358 452 9	3.884 08	2.358 44
3	0.1	0.1	3.199 274 3	1.432 527 8	3.199 24	1.432 52
5	—	—	3.333 254 9	2.002 535 6	3.333 21	2.002 53
7	—	—	3.441 183 6	2.466 886 7	3.441 15	2.466 88
10	—	—	3.572 496	3.053 781	3.572 47	3.053 77
7	0.1	0	3.469 863 5	2.644 825	3.469 83	2.644 81
—	—	0.1	3.441 183 6	2.466 886 7	3.441 15	2.466 88
—	—	0.2	3.415 273 4	2.315 989 5	3.415 23	2.315 98

Table 4. Values of skin friction and wall temperature gradient with $M = 0.1$ and $\epsilon = 0.1$.

Cases	M	Pr	*bvp4c* $-f''(0)$	$-\theta'(0)$	Shooting method $-f''(0)$	$-\theta'(0)$
	0.1	3				
CaseA			1.358 981 4	1.848 470 2	1.358 957 1	1.848 469 8
CaseB			3.277 733 5	1.402 271 2	3.277 679 5	1.402 261 8
CaseC			3.199 274 3	1.432 527 8	3.199 24	1.432 52
	0.1	5				
CaseA			1.358 980 1	2.480 004 5	1.358 956 9	2.480 048
CaseB			3.394 529 1	1.972 303 6	3.394 461 8	1.972 289 6
CaseC			3.333 254 9	2.002 535 6	3.333 21	2.002 53
	0.1	7				
CaseA			1.358 961 7	2.993 455 7	1.358 956 9	2.993 482
CaseB			3.492 364 1	2.436 242 8	3.492 291	2.436 224 3
CaseC			3.441 183 6	2.466 886 7	3.441 15	2.466 88

The effect of viscosity and thermal conductivity for all the three cases have been studied. Temperature of ambient fluid is $T_0 = 278$ K while temperature of surface is taken as $T_w = 358$ K. In figures 2–3 velocity and temperature profiles are presented for all Cases A, B and C. In comparison with Case A and C velocity profile for

Table 5. Comparison of $\theta'(0)$ for $M = 0$ and for various Prandtl numbers to previous data.

b	Pr	Magyari and Kellar [24]	Pal [44]	Present result
0.0	0.5	0.330 493	0.330 49	0.330 496 78
—	1	0.549 643	0.549 64	0.549 650 44
—	3	1.122 188	1.122 09	1.122 091 5
—	5	1.521 243	1.521 24	1.521 232
1.0	0.5	0.594 338	0.594 34	0.594 343 14
—	1	0.954 782	0.954 78	0.954 789 75
—	3	1.869 075	1.869 07	1.869 069 5
—	5	2.500 135	2.500 13	2.500 063 9
3.0	0.5	1.008 405	1.008 41	1.008 416 5
—	1	1.560 294	1.560 30	1.560 305 1
—	3	2.938 535	2.938 54	2.938 552 8
—	5	3.886 555	3.886 56	3.886 566 2

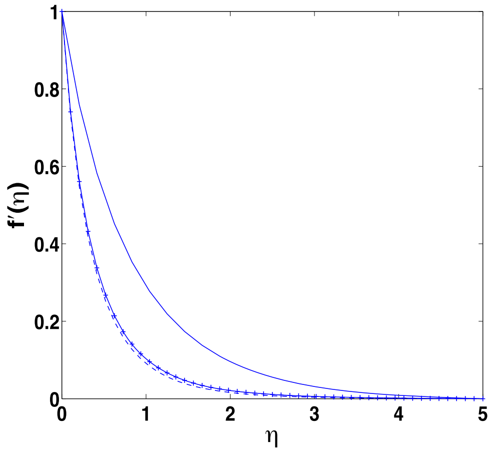

Figure 2. Variation in dimensionless velocity profiles $f'(\eta)$ for different cases at $\mathrm{Pr} = 0.7, \mathrm{M} = 0.1$ and $\epsilon = 0.1$.

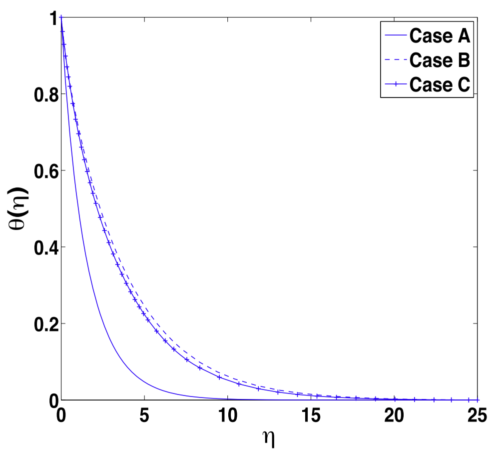

Figure 3. Variation in dimensionless temperature profiles $\theta(\eta)$ for different cases at $\mathrm{Pr} = 0.7, \mathrm{M} = 0.1$ and $\epsilon = 0.1$.

Case B have been reduced adjacent to moving surface as shown in figure 2. The same results have been observed in momentum boundary layer thickness. Comparing with the Case B the temperature profile for both Cases A and C decreases close to moving surface as shown in figure 3. Effect of magnetic parameter M on temperature and velocity profiles have been shown in figures 4–9. Temperature profile increases as we increase M and there is a decreasing effect on momentum boundary layer for all three Cases A, B and C. In figures 10–13 the effect of

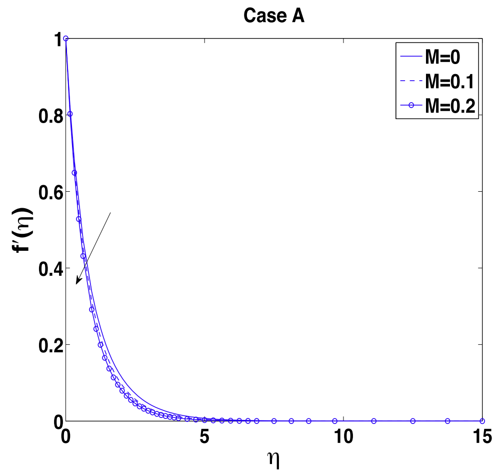

Figure 4. Variation in dimensionless velocity profiles $f'(\eta)$ for different values of M with Pr $= 3$.

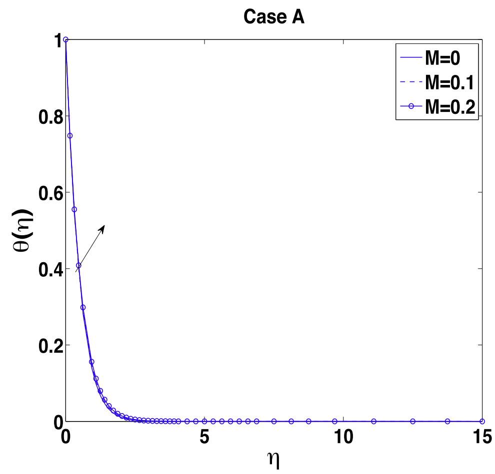

Figure 5. Variation in dimensionless temperature profiles $\theta(\eta)$ for different values of M with Pr $= 3$.

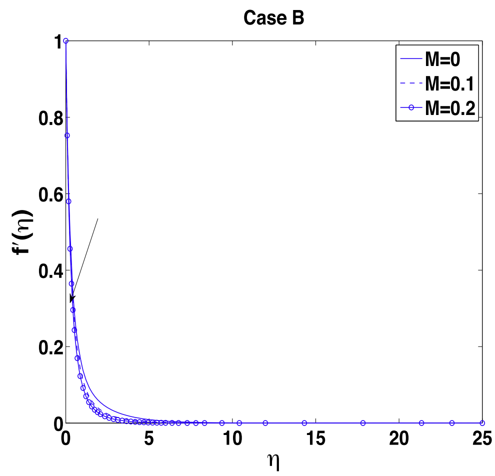

Figure 6. Variation in dimensionless velocity profiles $f'(\eta)$ for different values of M with $\epsilon = 0.1$ and Pr $= 3$.

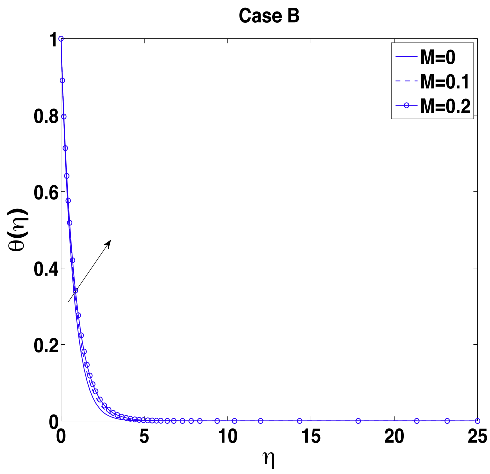

Figure 7. Variation in dimensionless temperature profiles $\theta(\eta)$ for different values of M with $\epsilon = 0.1$ and $\Pr = 3$.

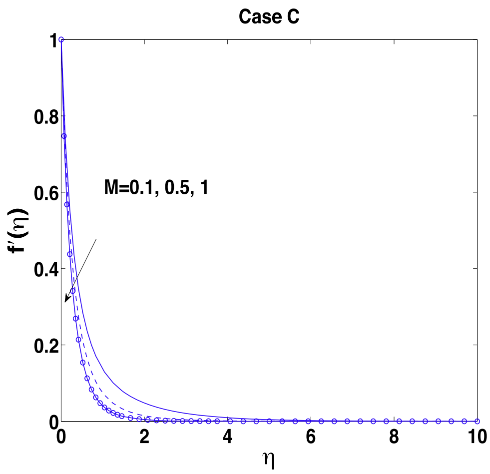

Figure 8. Variation in dimensionless velocity profiles $f'(\eta)$ for different values of M with $\epsilon = 0.1$ and $\Pr = 3$.

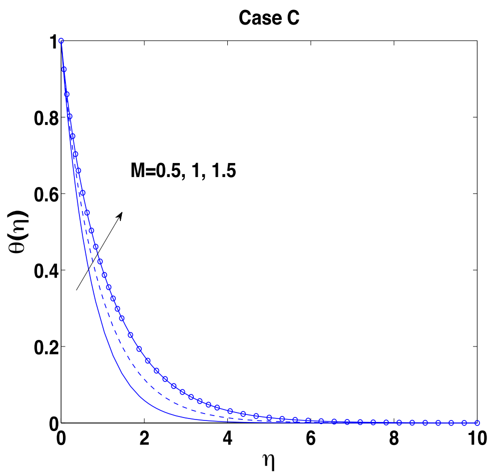

Figure 9. Variation in dimensionless temperature profiles $\theta(\eta)$ for different values of M with $\epsilon = 0.1$ and $\Pr = 3$.

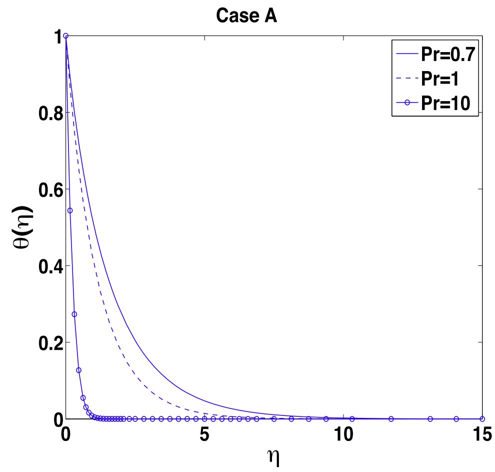

Figure 10. Variation in dimensionless temperature profiles $\theta(\eta)$ for different values of Pr with $\mathrm{M} = 0.1$.

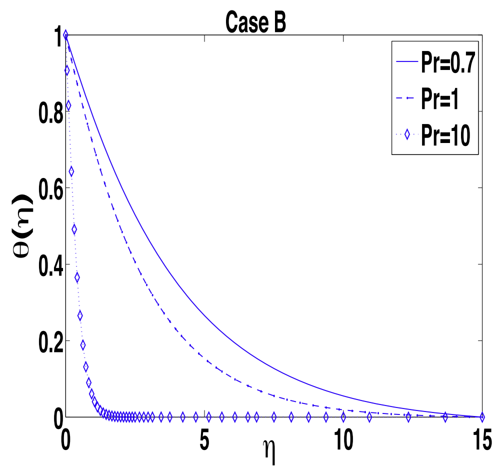

Figure 11. Variation in dimensionless temperature profiles $\theta(\eta)$ for different values of Pr with $\mathrm{M} = 0.1$ and $\epsilon = 0.1$.

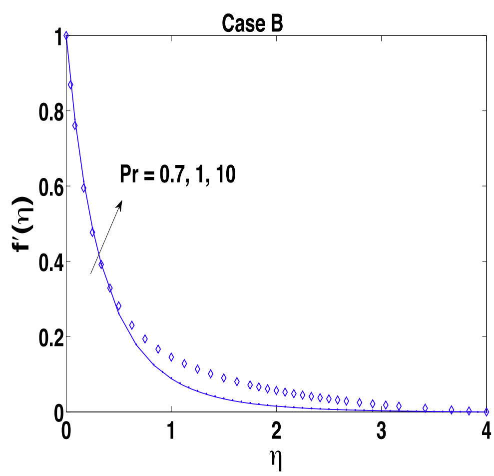

Figure 12. Variation in dimensionless velocity profiles $f'(\eta)$ for different values of Pr with $\mathrm{M} = 0.1$ and $\epsilon = 0.1$.

Prandtl number has been shown. The wall temperature reduces for all the Cases A, B and C whereas the velocity profile increases in Case B. In figures 14–15 the effect of parameter ϵ on temperature profile has been shown. For both the Cases B and C there is an increment in temperature profile.

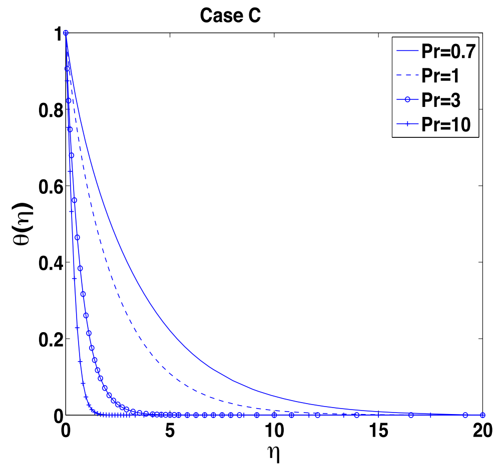

Figure 13. Variation in dimensionless temperature profiles $\theta(\eta)$ for different values of Pr with $M = 0.1$ and $\epsilon = 0.1$.

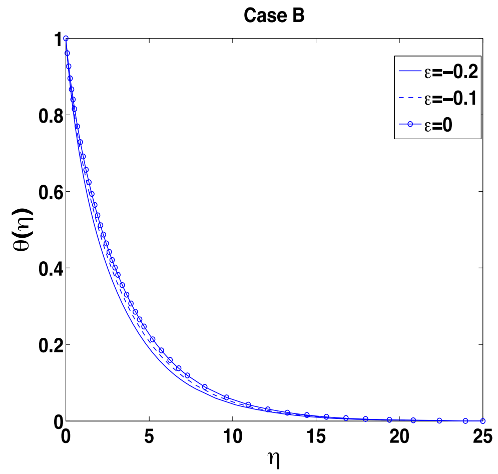

Figure 14. Variation in dimensionless temperature profiles $\theta(\eta)$ for different values of parameter ϵ with $Pr = 0.7$ and $M = 0.1$.

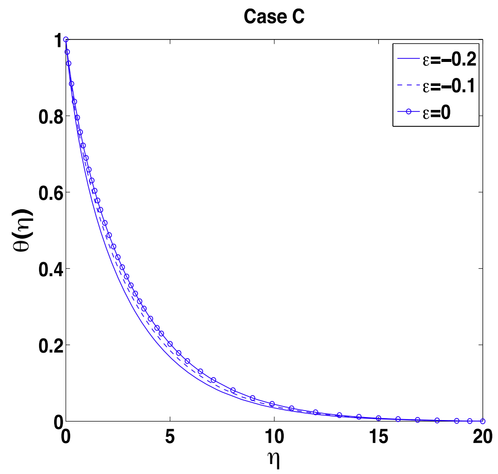

Figure 15. Variation in dimensionless temperature profiles $\theta(\eta)$ for different values of parameter ϵ with $Pr = 0.7$ and $M = 0.1$.

6. Conclusions

In this paper, MHD flow and transfer of heat for viscous fluid with changeable fluid properties over an exponentially stretching surface has been discussed. The problem has following governing parameters: Magnetic parameter M, Prandtl number Pr and parameter ϵ. Their effect on MHD flow and transfer of heat characteristics have been discussed. Main focus of our study has been to describe viscosity and thermal conductivity as functions of temperature. The boundary layer equations together with the boundary conditions have been reduced to nonlinear ordinary differential equations by using similarity variables. The resulting differential equations are then solved numerically by shooting method and verified by *bvp4c* and from the literature.

The results are summarized as follows:

- It is observed that skin friction and thermal boundary layer both increases with increment in magnetic parameter while velocity profile and wall temperature decreases.
- The Prandtl number causes a slight change in momentum boundary layer and skin friction whereas wall temperature and momentum boundary layer thickens for the case of variable viscosity. Thermal boundary layer reduces as Prandtl number rises .
- The parameter ϵ reduces both the skin friction coefficient and Nusselt number whereas it enhances thermal boundary layer thickness.

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