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Spectral triples for higher-rank graph C^*-algebras

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Abstract

In this note, we present a new way to associate a spectral triple to the noncommutative C^*-algebra $C^*(\Lambda)$ of a strongly connected finite higher-rank graph Λ. Our spectral triple builds on an approach used by Consani and Marcolli to construct spectral triples for Cuntz-Krieger algebras. We prove that our spectral triples are intimately connected to the wavelet decomposition of the infinite path space of Λ which was introduced by Farsi, Gillaspy, Kang, and Packer in 2015. In particular, we prove that the wavelet decomposition of Farsi et al. describes the eigenspaces of the Dirac operator of our spectral triple. The paper concludes by discussing other properties of the spectral triple, namely, θ-summability and Brownian motion.

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1 Introduction

Connes' spectral triples [4] are a valuable tool for transferring geometric questions to an algebraic context. Indeed, even within purely geometric settings, spectral triples often shed new light. For example, the spectral triples associated to fractal spaces (cf. [2, 10, 27, 22, 18, 16, 24]) often reveal the dimension of the fractal space as well as other aspects of the fractal geometry. The goal of this paper is to increase our geometric understanding of higher-rank graph C^*-algebras $C^*(\Lambda)$, by constructing a new spectral triple for $C^*(\Lambda)$ when Λ is finite and strongly connected. Our spectral triple was inspired by work of Consani and Marcolli [5] on spectral triples for certain Cuntz–Krieger algebras. They used these spectral triples and other C^*-algebraic techniques to study Arakelov geometry and Archimedean cohomology.

In this note, in addition to presenting the second known construction of a spectral triple for higher-rank graph C^*-algebras $C^*(\Lambda)$, we also establish the compatibility of these spectral triples with the representations and wavelets for higher-rank graphs which were developed in [8]. Indeed, both spectral triples and wavelets are algebraic structures which encode geometrical information, so it is natural to ask about the relationship between wavelets and spectral triples.

Higher-rank graphs (also called k-graphs) were introduced by Kumjian and Pask in [19] to provide a combinatorial model to the higher-dimensional Cuntz-Krieger algebras given by Robertson and Steger in [30]. The C^*-algebras $C^*(\Lambda)$ of k-graphs Λ have been studied by many authors and provided concrete, computable examples of many classifiable C^*-algebras. The graphical character of k-graphs has also facilitated the analysis of structural properties of $C^*(\Lambda)$, such as simplicity and ideal structure [28, 29, 6, 17, 1], quasidiagonality [3] and KMS states [14, 13, 12].

However, the analysis of the noncommutative geometry of $C^*(\Lambda)$ is in its infancy. Although Pask, Rennie, and Sims establish in [26] that higher-rank graph C^*-algebras often provide tractable examples of noncommutative manifolds, the current literature contains only one class of (semifinite) spectral triples for $C^*(\Lambda)$, namely those studied in [25]. (One can also associate Pearson–Bellissard spectral triples $(\mathcal{A}, \mathcal{H}, D)$ to higher-rank graphs, cf. [7]. However, the algebra $\mathcal{A} = C_{\text{Lip}}(\Lambda^\infty)$ used in these spectral triples is a commutative subalgebra of $C^*(\Lambda)$ and does not capture the noncommutative geometry of $C^*(\Lambda)$.) Moreover, the spectral triples described in this paper can be constructed for many higher-rank graphs to which the techniques of [25] do not apply (see Remark 2.6 below). Thus, the spectral triples for the noncommutative C^*-algebra $C^*(\Lambda)$, which we construct in Theorem 3.4 below, constitute an important step forward in our understanding of the noncommutative geometry of $C^*(\Lambda)$, in particular because of the link we establish between these spectral triples and wavelet theory for $C^*(\Lambda)$.

Wavelets for higher-rank graphs Λ were introduced by four of the authors of the current paper in [8], building on work of Marcolli and Paolucci [23] for Cuntz–Krieger algebras, which in turn was inspired by the wavelets for fractal spaces developed by Jonsson [15] and Strichartz [31]. In all of these settings, the wavelets give an orthogonal decomposition of $L^2(X, \mu)$ for a fractal space X, which arises from applying dilation and translation operators to a finite family of "mother wavelets" $f_i \in L^2(X, \mu)$. The dilation and translation operators are determined by the underlying geometry. In Jonsson and Strichartz' work, the self-similar structure of the fractal space X dictates the dilation and translation operators, while in the higher-rank graph case, the dilation and translation operators arise from the graph structure. (See Section 2.2 for more details.)

To further our understanding of the noncommutative geometry of $C^*(\Lambda)$, we construct in Theorem 3.4 a spectral triple $(\mathcal{A}_\Lambda, L^2(\Lambda^\infty, M), D)$, where $\mathcal{A}_\Lambda$ is a dense (noncommutative) subalgebra of $C^*(\Lambda)$. This spectral triple was inspired by the spectral triples for Cuntz–Krieger algebras constructed in [5], and offers a different perspective on the noncommutative geometry of $C^*(\Lambda)$ than the spectral triples of [25]. Theorem 3.5 then establishes our link between spectral triples and wavelets for higher-rank graphs by showing that the eigenspaces of the Dirac operator D of this spectral triple agree with the wavelet decomposition of [8]. Finally, we conclude the paper in Section 4 by analyzing the θ-summability and Brownian motion of our spectral triple.

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2 Background material

We begin by detailing some foundational material needed for our results, and in particular reviewing the definition of a higher-rank graph Λ, the definition of its C^*-algebra $C^*(\Lambda)$, and associated wavelets.

2.1 Higher-rank graphs and their C^*-algebras

Throughout this paper, we will view $\mathbb{N} := \{0, 1, 2, \dots\}$ as a monoid under addition, or as a category. In this interpretation, the natural numbers are the morphisms in $\mathbb{N}$. Thus, for consistency with the standard notation $n \in \mathbb{N}$, we will write

$$\lambda \in \Lambda$$

to indicate that λ is a morphism in the category Λ.

Definition 2.1. A *higher-rank graph* or *k-graph* by definition is a countable small category Λ with a degree functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *factorization property*: for any morphism $\lambda \in \Lambda$ and any $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n \in \mathbb{N}^k$, there exist unique morphisms $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$ and $d(\mu) = m$, $d(\nu) = n$.

We often think of k-graphs as a generalization of directed graphs, so we call objects $v \in \Lambda^0$ "vertices" and morphisms $\lambda \in \Lambda$ are called "paths." We write $r, s : \Lambda \to \Lambda^0$ for the range and source maps and $v\Lambda w = \{\lambda \in \Lambda : r(\lambda) = v, s(\lambda) = w\}$. Similarly, for any $n \in \mathbb{N}^k$, we write $v\Lambda^n = \{\lambda \in \Lambda : r(\lambda) = v, d(\lambda) = n\}$.

For $m, n \in \mathbb{N}^k$, we denote by $m \vee n$ the coordinatewise maximum of m and n. Given $\lambda, \eta \in \Lambda$, we write

$$\Lambda^{\min}(\lambda, \eta) := \{(\alpha, \beta) \in \Lambda \times \Lambda \,:\, \lambda\alpha = \eta\beta,\ d(\lambda\alpha) = d(\lambda) \vee d(\eta)\}.$$

We say that a k-graph Λ is *finite* if Λ^n is a finite set for all $n \in \mathbb{N}^k$ and say that Λ *has no sources* or *is source-free* if $v\Lambda^n \neq \emptyset$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. It is well known that this is equivalent to the condition that $v\Lambda^{e_i} \neq \emptyset$ for all $v \in \Lambda$ and all basis vectors e_i of $\mathbb{N}^k$. Also we say that a k-graph is *strongly connected* if, for all $v, w \in \Lambda^0$, $v\Lambda w \neq \emptyset$. If Λ is strongly connected then it is source-free by [14, Lemma 2.1].

Definition 2.2. [19] If Λ is a finite k-graph with no sources, write $C^*(\Lambda)$ for the universal C^*-algebra generated by partial isometries $\{s_\lambda\}_{\lambda \in \Lambda}$ satisfying the Cuntz–Krieger conditions:

(CK1) $\{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections;

(CK2) Whenever $s(\lambda) = r(\eta)$ we have $s_\lambda s_\eta = s_{\lambda\eta}$;

(CK3) For any $\lambda \in \Lambda$, $s_\lambda^* s_\lambda = s_{s(\lambda)}$;

(CK4) For all $v \in \Lambda^0$ and all $n \in \mathbb{N}^k$, $\sum_{\lambda \in v\Lambda^n} s_\lambda s_\lambda^* = s_v$.

Condition (CK4) implies that for any $\lambda, \eta \in \Lambda$ we have

$$s_\lambda^* s_\eta = \sum_{(\alpha,\beta) \in \Lambda^{\min}(\lambda,\eta)} s_\alpha s_\beta^*,$$

where we interpret empty sums as zero. Consequently, $C^*(\Lambda) = \overline{\operatorname{span}}\{s_\lambda s_\eta^* : \lambda, \eta \in \Lambda\}$.

Definition 2.3. Let $\mathcal{A}_\Lambda$ denote the dense $*$-subalgebra of $C^*(\Lambda)$ spanned by $\{s_\lambda s_\eta^*\}_{\lambda,\eta \in \Lambda}$.

An important example of a k-graph is the category Ω_k, where

$$\operatorname{Obj}(\Omega_k) = \mathbb{N}^k, \qquad \operatorname{Mor}(\Omega_k) = \{(p, q) \in \mathbb{N}^k : p \leq q\}.$$

The range and source maps r, s in Ω_k are given by $r(p, q) = p$, $s(p, q) = q$, and the degree map $d : \Omega_k \to \mathbb{N}^k$ is given by

$$d(p, q) = q - p.$$

Definition 2.4. An *infinite path* in a k-graph Λ is a degree preserving functor $x : \Omega_k \to \Lambda$. We write Λ^∞ for the set of infinite paths in Λ.

Given $\lambda \in \Lambda$, we define the *cylinder set* $[\lambda] \subseteq \Lambda^\infty$ by

$$[\lambda] := \{x \in \Lambda^\infty : x(0, d(\lambda)) = \lambda\}$$

to be the infinite paths with initial segment λ. It is well-known (cf. [19]) that the collection of cylinder sets $\{[\lambda]\}_{\lambda\in\Lambda}$ forms a compact open basis for a locally compact Hausdorff topology on Λ^∞. If a k-graph Λ is finite, then Λ^∞ is compact in this topology.

For each $m \in \mathbb{N}^k$, we have a shift map σ^m on Λ^∞ given by

$$\sigma^m(x)(p, q) = x(p + m, q + m). \tag{1}$$

for $x \in \Lambda^\infty$ and $(p, q) \in \Omega_k$. In duality to the shift map σ^m, for each $\lambda \in \Lambda$ we also have a prefixing map $\sigma_\lambda : [s(\lambda)] \to [\lambda]$ given by

$$\sigma_\lambda(x) = \lambda x = \left[(p, q) \mapsto \begin{cases} \lambda(p, q), & q \le d(\lambda) \\ x(p - d(\lambda), q - d(\lambda)), & p \ge d(\lambda) \\ \lambda(p, d(\lambda))\, x(0, q - d(\lambda)), & p < d(\lambda) < q \end{cases} \right] \tag{2}$$

According to [14, Proposition 8.1], for any finite and strongly connected k-graph Λ, there is a unique self-similar Borel probability measure M on Λ^∞. To describe M, we require more definitions.

Definition 2.5. For a finite k-graph Λ and $1 \le i \le k$, the *vertex matrix* $A_i \in M_{\Lambda^0}(\mathbb{N})$ is

$$A_i(v, w) = \#(v\Lambda^{e_i}w).$$

Lemma 3.1 of [14] establishes that if Λ is finite and strongly connected, then there exists a unique vector $\kappa^\Lambda \in (0, \infty)^{\Lambda^0}$, called the *Perron–Frobenius eigenvector* of Λ, such that

$$\sum_{v\in\Lambda^0} \kappa^\Lambda_v = 1 \qquad \text{and} \qquad A_i\kappa^\Lambda = \rho_i\kappa^\Lambda \quad \forall\, 1 \le i \le k.$$

The unique self-similar Borel probability measure M of [14] is given on cylinder sets by

$$M([\lambda]) = (\rho(\Lambda))^{-d(\lambda)}\kappa^\Lambda_{s(\lambda)} \quad \text{for } \lambda \in \Lambda.$$

Here $\rho(\Lambda) = (\rho_1, \dots \rho_k)$, where ρ_i denotes the spectral radius of the vertex matrix $A_i \in M_{\Lambda^0}(\mathbb{N})$, and $(\rho(\Lambda))^n := \rho_1^{n_1} \dots \rho_k^{n_k}$ for $n = (n_1, \dots n_k) \in \mathbb{R}^k$. We call the measure M the *Perron–Frobenius measure* on Λ^∞.

Remark 2.6. The spectral triples for $C^*(\Lambda)$ studied by Pask, Rennie, and Sims [25] are constructed using a $\mathbb{Z}^k$ grading on a Hilbert space that arises from the gauge action of $\mathbb{T}^k$ on the C^*-algebra $C^*(\Lambda)$ and a *faithful k-graph trace* (see [25, Definition 3.5]) on the k-graph Λ. For source-free k-graphs (in particular for strongly connected k-graphs), the set $v\Lambda^{\le n}$ of [25] coincides with $v\Lambda^n$. It follows by taking $n = e_i$ in [25, Definition 3.5] that in this case, a faithful k-graph trace is a common eigenvector g for the adjacency matrices $A_1, \dots, A_k$, such that $A_i(g) = g$ for all i and all entries of g are positive. Thus, [14, Proposition 3.1 and Lemma 4.1] imply that for a strongly connected k-graph Λ with $\rho(\Lambda) \ne 1$, $C^*(\Lambda)$ does not admit a Pask-Rennie-Sims spectral triple. There are many examples of strongly connected k-graphs Λ for which $\rho(\Lambda) \ne 1$; cf. [21, Example 7.7] or [8, Example 3.7]. For these k-graphs Λ, Theorem 3.4 below establishes the first spectral triple for $C^*(\Lambda)$.

2.2 Wavelets on higher-rank graphs

According to Proposition 3.4 and Theorem 3.5 of [8], there is a separable representation π of $C^*(\Lambda)$ on $L^2(\Lambda^\infty, M)$ when Λ is a finite, strongly connected k-graph. Theorem 3.4 below identifies a Dirac operator D for which this representation gives a spectral triple $(\mathcal{A}_\Lambda, L^2(\Lambda^\infty, M), D)$.

Before stating Theorem 3.4, we review the definition of the representation π and the associated wavelet decomposition of $L^2(\Lambda^\infty, M)$. For $p \in \mathbb{N}^k$ and $\lambda \in \Lambda$, let σ^p and σ_λ be the shift and prefixing maps on Λ^∞ given in (1) and (2). If we let $S_\lambda := \pi(s_\lambda)$, the image of the standard generator s_λ of $C^*(\Lambda)$ under the representation π, then [8, Theorem 3.5] tells us that S_λ is given on characteristic functions of cylinder sets by

$$\begin{aligned} S_\lambda \chi_{[\eta]}(x) = \chi_{[\lambda]}(x)\rho(\Lambda)^{d(\lambda)/2}\chi_{[\eta]}(\sigma^{d(\lambda)}(x)) &= \begin{cases} \rho(\Lambda)^{d(\lambda)/2} & \text{if } x = \lambda\eta y \text{ for some } y \in \Lambda^\infty \\ 0 & \text{otherwise} \end{cases} \\ &= \rho(\Lambda)^{d(\lambda)/2}\chi_{[\lambda\eta]}(x). \end{aligned} \tag{3}$$

Moreover, the adjoint S_λ^* of S_λ is given on characteristic functions of cylinder sets by

$$\begin{aligned} S_\lambda^* \chi_{[\eta]}(x) = \chi_{[s(\lambda)]}(x)\rho(\Lambda)^{-d(\lambda)/2}\chi_{[\eta]}(\sigma_\lambda(x)) &= \begin{cases} \rho(\Lambda)^{-d(\lambda)/2} & \text{if } \lambda x = \eta y \text{ for some } y \in \Lambda^\infty \\ 0 & \text{otherwise} \end{cases} \\ &= \rho(\Lambda)^{-d(\lambda)/2} \sum_{(\zeta,\xi)\in\Lambda^{\min}(\lambda,\eta)} \chi_{[\zeta]}(x). \end{aligned} \tag{4}$$

We can think of the operators S_λ as combined "scaling and translation" operators, since they change both the size and the range of a cylinder set $[\eta]$, and are intimately tied to the geometry of the k-graph Λ.

This perspective enabled four of the authors of the current paper to use the representation π to construct a wavelet decomposition of $L^2(\Lambda^\infty, M)$; we recall the details from [8, Section 4]. For each vertex v in Λ, let

$$D_v = v\Lambda^{(1,\dots,1)}.$$

One can show (cf. [14, Lemma 2.1(a)]) that D_v is always nonempty when Λ is strongly connected.

Enumerate the elements of D_v as $D_v = \{\lambda_0, \dots, \lambda_{\#(D_v)-1}\}$. Observe that if $D_v = \{\lambda\}$ is a 1-element set, then $[v] = [\lambda]$. If $\#(D_v) > 1$, then for each $1 \le i \le \#(D_v) - 1$, we define

$$f^{i,v} = \frac{1}{M[\lambda_0]}\chi_{[\lambda_0]} - \frac{1}{M[\lambda_i]}\chi_{[\lambda_i]}. \tag{5}$$

One easily checks that in $L^2(\Lambda^\infty, M)$, $\langle f^{i,v}, \chi_{[w]}\rangle = 0$ for all i and all vertices v, w, and that

$$\{f^{i,v} : v \in \Lambda^0, 1 \le i \le \#(D_v) - 1\}$$

is an orthogonal set. Therefore, the functions $\{f^{i,v}\}_{i,v}$ span the subspace $\mathcal{W}_{0,\Lambda} \subseteq L^2(\Lambda^\infty, M)$ from [8, Theorem 4.2], which we will henceforth call $\mathcal{W}_0$.

The following theorem, which was proved in [8], justifies our labeling of the orthogonal decomposition (6) as a wavelet decomposition: the subspaces $\mathcal{W}_n$ are given by applying "scaling and translation" operators S_λ to the finite family of "mother functions" $\{f^{i,v}\}_{i,v}$.

Theorem 2.7. *[8, Theorem 4.2] Let Λ be a finite, strongly connected k-graph and define $\mathscr{V}_0 := span\{\chi_{[v]} : v \in \Lambda^0\}$. Let $\mathscr{V}_0 := span\{\chi_{[v]} : v \in \Lambda^0\}$, and set*

$$\mathcal{W}_n = span\{S_\lambda f^{i,s(\lambda)} : d(\lambda) = (n, \ldots, n), 1 \le i \le \#(D_{s(\lambda)}) - 1\}$$

for each $n \in \mathbb{N}$. Then $\{S_\lambda f^{i,s(\lambda)} : d(\lambda) = (n, \ldots, n), 1 \le i \le \#(D_{s(\lambda)}) - 1\}$ is a basis for $\mathcal{W}_n$ and

$$L^2(\Lambda^\infty, M) \cong \mathscr{V}_0 \oplus \bigoplus_{n=0}^{\infty} \mathcal{W}_n. \tag{6}$$

3 Spectral triples of Consani-Marcolli type for strongly connected finite higher-rank graphs

In Section 6 of [5], Consani and Marcolli construct a spectral triple for the Cuntz-Krieger algebra $\mathcal{O}_A$ associated to a matrix $A \in M_n(\mathbb{N})$. Recall from [20] that if E is the 1-graph with adjacency matrix A, then $\mathcal{O}_A \cong C^*(E)$.

In this section, we generalize the construction of Consani and Marcolli to build spectral triples for higher-rank graph C^*-algebras $C^*(\Lambda)$. For these spectral triples (described in Theorem 3.4 below), it is shown in Theorem 3.5 that the eigenspaces of the Dirac operator agree with the wavelet decomposition from [8]. We also discuss in Remark 3.6 at the end of the section how to modify the construction of the spectral triple to make the eigenspaces of the Dirac operator compatible with the J-shape wavelets of [9].

Definition 3.1. Let Λ be a finite, strongly connected k-graph. Define $\mathcal{R}_{-1} \subset L^2(\Lambda^\infty, M)$ to be the linear subspace of constant functions on Λ^∞. For $s \in \mathbb{N}$, define $\mathcal{R}_s \subset L^2(\Lambda^\infty, M)$ by

$$\mathcal{R}_s = \text{span}\left\{\chi_{[\eta]} :\ \eta \in \Lambda,\ \sup\{d(\eta)_i : 1 \le i \le k\} \le s\right\},$$

where $d(\eta) = (d(\eta)_1, \ldots, d(\eta)_k) \in \mathbb{N}^k$.

Let Ξ_s be the orthogonal projection in $L^2(\Lambda^\infty, M)$ onto the subspace $\mathcal{R}_s$. For a pair $(s, r) \in \mathbb{N} \times (\mathbb{N} \cup \{-1\})$ with $s > r$, let

$$\widehat{\Xi}_{s,r} = \Xi_s - \Xi_r.$$

Since $\mathcal{R}_r \subset \mathcal{R}_s$, $\widehat{\Xi}_{s,r}$ is the orthogonal projection onto the subspace $\mathcal{R}_s \cap (\mathcal{R}_r)^\perp$.

Given an increasing sequence $\alpha = \{\alpha_q\}_{q\in\mathbb{N}}$ of positive real numbers with $\lim_{q\to\infty} \alpha_q = \infty$, we define an operator D on $L^2(\Lambda^\infty, M)$ by

$$D := \sum_{q\in\mathbb{N}} \alpha_q\, \widehat{\Xi}_{q,q-1}. \tag{7}$$

Note first that the operator D has eigenvalues α_q with eigenspaces $\mathcal{R}_q \cap \mathcal{R}^\perp_{(q-1)}$ by construction. Also note that when Λ has one vertex, $\mathcal{R}_{-1} = \mathcal{R}_0$ and the orthogonal projection $\widehat{\Xi}_{0,-1}$ is the zero projection.

Proposition 3.2. *The operator D on $L^2(\Lambda^\infty, M)$ of Equation (7) is unbounded and self-adjoint.*

Proof. The fact that D is unbounded follows from the hypothesis that $\lim_{q\to\infty} \alpha_q = \infty$. Thus, to see that D is self-adjoint we must first check that it is densely defined, and then show that D and D^* have the same domain. For the first assertion, recall from Lemma 4.1 of [8] that

$$\{[\eta] : d(\eta) = (n, \ldots, n) \text{ for some } n \in \mathbb{N}\}$$

generates the topology on Λ^∞, and hence $\operatorname{span}\{\chi_{[\eta]} : d(\eta) = (n, n, \ldots, n),\ n \in \mathbb{N}\}$ is dense in $L^2(\Lambda^\infty, M)$. Given such a "square" cylinder set $[\eta]$ with $d(\eta) = (s, \ldots, s)$, since $\chi_{[\eta]} \in \mathcal{R}_s$, we can write $\chi_{[\eta]} = \sum_{r\leq s} \widehat{\Xi}_{r,r-1}(\chi_{[\eta]})$. Then,

$$D(\chi_{[\eta]}) = \sum_{r\leq s} \alpha_r \widehat{\Xi}_{r,r-1}(\chi_{[\eta]}),$$

which is a finite linear combination of vectors with finite L^2-norm, and hence is in $L^2(\Lambda^\infty, M)$. In other words, for any finite linear combination ξ of characteristic functions of square cylinder sets, $D\xi$ is in $L^2(\Lambda^\infty, M)$. Thus D is defined on (at least) the finite linear combinations of square cylinder sets, which form a dense subspace of $L^2(\Lambda^\infty, M)$.

Moreover, our definition of D as a diagonal operator on $L^2(\Lambda^\infty, M)$ with real eigenvalues implies that $D = D^*$ formally; since the operators D and D^* are given by the same diagonal formula, their domains also agree, and hence we do indeed have $D = D^*$ as unbounded operators. □

Proposition 3.3. *Let D be the operator on $L^2(\Lambda^\infty, M)$ given in* (7)*. For all complex numbers $\lambda \notin \{\alpha_n\}_{n\in\mathbb{N}}$, the resolvent $R_\lambda(D) := (D - \lambda)^{-1}$ is a compact operator on $L^2(\Lambda^\infty, M)$.*

Proof. By definition, D is given by multiplication by α_q on $\mathcal{R}_q \cap \mathcal{R}^\perp_{(q-1)}$. Consequently, for all $q \in \mathbb{N}$, $(D - \lambda)^{-1}$ is given by multiplication by $\frac{1}{\alpha_q-\lambda}$ on $\mathcal{R}_q \cap \mathcal{R}^\perp_{(q-1)}$.

Since $\lambda \notin \{\alpha_n\}_{n\in\mathbb{N}}$ and $\lim_{n\to\infty} \alpha_n = \infty$, given $\epsilon > 0$, we can choose N so that for all $n \geq N$, $\frac{1}{|\alpha_n-\lambda|} < \epsilon$. Fix $s \in \mathbb{N}$; then for any $f \in \mathcal{R}_s \cap \mathcal{R}^\perp_{s-1}$ of norm 1,

$$\begin{aligned}\left\| \left(\sum_{q=1}^{N} \frac{1}{\alpha_q - \lambda} \widehat{\Xi}_{q,q-1}(f) \right) - (D-\lambda)^{-1}(f) \right\| &= \left\| \sum_{q>N} \frac{1}{\alpha_q - \lambda} \widehat{\Xi}_{q,q-1}(f) \right\| \\ &= \begin{cases} \left|\frac{1}{\alpha_s-\lambda}\right| \|f\| & \text{if } s > N \\ 0 & \text{if } s \leq N \end{cases} \\ &< \epsilon,\end{aligned}$$

since $\|f\| = 1$ by hypothesis. Since the subspaces $\{\mathcal{R}_s \cap \mathcal{R}^\perp_{s-1} : s \in \mathbb{N}\}$ span $L^2(\Lambda^\infty, M)$, it follows that $(D - \lambda)^{-1}$ is the norm limit of finite rank operators and hence is compact. □

Theorem 3.4. *Let Λ be a finite, strongly connected k-graph, and denote by π the representation of $C^*(\Lambda)$ on $L^2(\Lambda^\infty, M)$ given in Equations* (3) *and* (4)*. Let $\mathcal{A}_\Lambda$ be the dense $*$-subalgebra of $C^*(\Lambda)$ given in Definition 2.3 and let D be the operator given in* (7)*. If there exists a constant $C \geq 0$ such that the sequence $\alpha = \{\alpha_q\}_{q\in\mathbb{N}}$ satisfies*

$$|\alpha_{q+1} - \alpha_q| \leq C,\ \forall q \in \mathbb{N},$$

then the commutator $[D, \pi(a)]$ is a bounded operator on $L^2(\Lambda^\infty, M)$ for any $a \in \mathcal{A}_\Lambda$.

Combined with the above results, this implies that the data $(\mathcal{A}_\Lambda, L^2(\Lambda^\infty, M), D)$ gives a spectral triple for $C^*(\Lambda)$.

Proof. To prove that $(\mathcal{A}_\Lambda, L^2(\Lambda^\infty, M), D)$ is a spectral triple we need to show that D is self-adjoint, $(D^2+i)^{-1}$ is compact and $[D, \pi(a)]$ is bounded for all $a \in \mathcal{A}_\Lambda$. The first statement is the content of Proposition 3.2, and the second follows from Proposition 3.3, thanks to the fact that $\pm i \notin \{\alpha_n\}_{n\in\mathbb{N}}$ and hence $(D \pm i)^{-1}$ is compact. Thus, to complete the proof of the Theorem, we will now show that $[D, \pi(a)]$ is bounded for all finite linear combinations $a = \sum_{i\in F} c_i s_{\lambda_i} s^*_{\eta_i} \in \mathcal{A}_\Lambda$, where $c_i \in \mathbb{C}$.

Given $\lambda \in \Lambda$, write $\max_\lambda = \max_j\{d(\lambda)_j\}$ and $\min_\lambda = \min_j\{d(\lambda)_j\}$. Then the formula (3) implies immediately that, for any fixed $s \in \mathbb{N}$, the operator S_λ on $L^2(\Lambda^\infty, M)$ takes $\mathcal{R}_s$ to $\mathcal{R}_{s+\max_\lambda}$.

Moreover, Equation (4) implies that the operator S^*_λ on $L^2(\Lambda^\infty, M)$ takes $\mathcal{R}_s$ to $\mathcal{R}_{s-\min_\lambda}$ if $\min_\lambda \leq s$, and to $\mathcal{R}_0$ otherwise. To see this, suppose $\chi_{[\eta]} \in \mathcal{R}_s$ and $d(\eta) = (n_1, \ldots, n_k)$. Then $S^*_\lambda \chi_{[\eta]}$ is a linear combination of cylinder sets $\chi_{[\zeta]}$ with

$$d(\zeta)_i = \begin{cases} 0, & d(\lambda)_i \geq d(\eta)_i \\ d(\eta)_i - d(\lambda)_i, & d(\lambda)_i < d(\eta)_i \end{cases}$$

Consequently, we see that (as desired)

$$\max\{d(\zeta)_i\} = \max\{0, n_i - d(\lambda)_i : 1 \leq i \leq k\} \leq s - \min_\lambda.$$

If $s < \min_\lambda$, then $n_i - d(\lambda)_i \leq 0$ for all i, so $S^*_\lambda \chi_{[\eta]} \in \mathcal{R}_0$ for all $\chi_{[\eta]} \in \mathcal{R}_s$.

Similarly, if $f \in \mathcal{R}_s^\perp$, then $S_\lambda f \in \mathcal{R}^\perp_{s+\min_\lambda}$. Namely, if $\langle f, h\rangle = 0$ for all $h \in \mathcal{R}_s$, then our description of S^*_λ above yields

$$\langle f, S^*_\lambda g\rangle = 0 \ \forall\ g \in \mathcal{R}_{s+\min_\lambda}.$$

An analogous argument shows that S^*_λ takes $\mathcal{R}^\perp_s$ to $\mathcal{R}^\perp_{s-\max_\lambda}$ if $s \geq \max_\lambda$.

Now fix $q \in \mathbb{N}$, $f \in \mathcal{R}_q \cap \mathcal{R}^\perp_{q-1}$, and fix $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$. We use the reasoning of the previous paragraphs to identify the subspaces $\mathcal{R}_s, \mathcal{R}_t^\perp$ which contain $S_\lambda S^*_\mu f$.

If $\max_\mu \geq q$, then we cannot guarantee that $S^*_\mu f$ is orthogonal to any $\mathcal{R}_t$ with $t \geq 0$; in order to do so, we must have $\langle S^*_\mu f, \xi\rangle = \langle f, S_\mu \xi\rangle = 0$ for all $\xi \in \mathcal{R}_t$. In other words, we must have $S_\mu \xi \in \mathcal{R}_{q-1}$ for all $\xi \in \mathcal{R}_t$. However, S_μ takes R_t into $\mathcal{R}_{t+\max_\mu} \supsetneqq \mathcal{R}_{q-1}$ if $\max_\mu \geq q$ and $t \geq 0$.

Moreover, if $q < \min_\mu$, then $S^*_\mu f \in \mathcal{R}_0$. Thus,

$$q < \min_\mu \Rightarrow S_\lambda S^*_\mu f \in \mathcal{R}_{\max_\lambda}; \quad \min_\mu \leq q \leq \max_\mu \Rightarrow S_\lambda S^*_\mu f \in \mathcal{R}_{q+\max_\lambda - \min_\mu};$$

$$q > \max_\mu \Rightarrow S_\lambda S^*_\mu f \in \mathcal{R}_{q+\max_\lambda - \min_\mu} \cap \mathcal{R}^\perp_{(q-1)+\min_\lambda - \max_\mu}.$$

For now, assume $q > \max_\mu$. Writing $g = S_\lambda S^*_\mu f$, we have

$$g = \Big(\Xi_{q+\max_\lambda - \min_\mu} - \Xi_{(q-1)+\min_\lambda - \max_\mu}\Big) g = \sum_{w=q+\min_\lambda - \max_\mu}^{q+\max_\lambda - \min_\mu} \Big(\Xi_w - \Xi_{w-1}\Big) g$$

and consequently

$$D(S_\lambda S^*_\mu f) =: Dg = \sum_{w=q+\min_\lambda - \max_\mu}^{q+\max_\lambda - \min_\mu} D\Big(\Big(\Xi_w - \Xi_{w-1}\Big) g\Big) = \sum_{w=q+\min_\lambda - \max_\mu}^{q+\max_\lambda - \min_\mu} \alpha_w \Big(\Big(\Xi_w - \Xi_{w-1}\Big) g\Big).$$

It now follows that, if $f \in \mathcal{R}_q \cap \mathcal{R}^\perp_{q-1}$ for $q > \max_\mu$,

$$[D, S_\lambda S^*_\mu] f = D S_\lambda S^*_\mu f - S_\lambda S^*_\mu D f = \sum_{w=q+\min_\lambda - \max_\mu}^{q+\max_\lambda - \min_\mu} (\alpha_w - \alpha_q)\Big(\Big(\Xi_w - \Xi_{w-1}\Big) S_\lambda S^*_\mu f\Big).$$

Consequently, since $|\alpha_w - \alpha_{w-1}| \le C$ for all w,

$$\begin{aligned} \|[D, S_\lambda S_\mu^*] f\| &\le \sum_{w=q+\min_\lambda - \max_\mu}^{q+\max_\lambda - \min_\mu} |\alpha_w - \alpha_q| \, \|S_\lambda S_\mu^* f\| \\ &\le \|S_\lambda S_\mu^* f\| \sum_{w=q+\min_\lambda - \max_\mu}^{q+\max_\lambda - \min_\mu} C|w-q| = \|S_\lambda S_\mu^* f\| C \sum_{t=\min_\lambda - \max_\mu}^{\max_\lambda - \min_\mu} |t|. \end{aligned}$$

Since $S_\lambda S_\mu^*$ is a partial isometry and hence norm-preserving, whenever $f \in \mathcal{R}_q \cap \mathcal{R}_{q-1}^\perp$ for $q > \max_\mu$, $\|[D, S_\lambda S_\mu^*] f\|$ is bounded above by a constant which depends only on λ and μ.

If we have $\min_\mu \le q \le \max_\mu$, since we no longer know that $S_\lambda S_\mu^* f \in \mathcal{R}_t^\perp$ for any t, in calculating $\|[D, S_\lambda S_\mu^* f \,\|$ we have to begin our summation over w at zero, rather than at $q + \min_\lambda - \max_\mu$. In this case, the final (in)equality above becomes

$$\|[D, S_\lambda S_\mu^*] f\| \le \sum_{t=1}^{\max_\lambda - \min_\mu} Ct \|S_\lambda S_\mu^* f\| + \sum_{t=1}^{q} Ct \|S_\lambda S_\mu^* f\|.$$

In this case, $q \le \max_\mu$, so we obtain the norm bound

$$\|[D, S_\lambda S_\mu^*] f\| \le \|S_\lambda S_\mu^* f\| C \left(\frac{(\max_\lambda - \min_\mu)(\max_\lambda - \min_\mu + 1)}{2} + \frac{\max_\mu (\max_\mu + 1)}{2} \right).$$

In other words, $\|[D, S_\lambda S_\mu^*] f\|$ is again bounded by a constant which only depends on λ and μ. A similar argument shows that if $q < \min_\mu$, $\|[D, S_\lambda S_\mu^*] f\|$ is bounded by a constant which only depends on λ and μ. Since $\{\mathcal{R}_q \cap \mathcal{R}_{q-1}\}_{q \in \mathbb{N}}$ densely spans $L^2(\Lambda^\infty, M)$, it follows that $[D, S_\lambda S_\mu^*]$ is a bounded operator for all $(\lambda, \mu) \in \Lambda \times \Lambda$ with $s(\lambda) = s(\mu)$.

By linearity, it follows that $[D, \pi(a)]$ is bounded for all finite linear combinations $a = \sum_{i \in F} c_i s_{\lambda_i} s_{\eta_i}^*$ of the generators $s_\lambda s_\eta^*$ of $C^*(\Lambda)$. Since every element of the dense $*$-subalgebra $\mathcal{A}_\Lambda$ of $C^*(\Lambda)$ is given by such a finite linear combination, it follows that $(\mathcal{A}_\Lambda, L^2(\Lambda^\infty, M), D)$ is a spectral triple, as claimed. □

Theorem 3.5. *Let $(\mathcal{A}_\Lambda, L^2(\Lambda^\infty, M), D)$ be the spectral triple described in Theorem 3.4. The eigenspaces of the Dirac operator D given in* (7) *agree with the wavelet decomposition*

$$L^2(\Lambda^\infty, M) = \mathscr{V}_0 \oplus \bigoplus_{q=0}^{\infty} \mathcal{W}_q$$

of Theorem 2.7 above (also see [8, Theorem 4.2]). In particular,

$$\mathscr{V}_0 = \mathcal{R}_0 \supseteq \mathcal{R}_{-1} \quad \textit{and} \quad \mathcal{W}_q = \mathcal{R}_{q+1} \cap \mathcal{R}_q^\perp, \ q \ge 0.$$

Proof. By definition, $\mathcal{R}_{-1} \subseteq \mathcal{R}_0 = \mathscr{V}_0 = \operatorname{span}\{\chi_{[v]} : v \in \Lambda^0\}$. For the second assertion, recall that $\mathcal{W}_n = \operatorname{span}\{S_\lambda f : f \in \mathcal{W}_0, \ d(\lambda) = (q, q, \ldots, q)\}$. Since $\max_\lambda = \min_\lambda = q$ for all such λ, each such S_λ takes $\mathcal{R}_s \cap \mathcal{R}_{s-1}^\perp$ to $\mathcal{R}_{s+q} \cap \mathcal{R}_{s+q-1}^\perp$. Thus, it suffices to see that $\mathcal{W}_0 \subseteq \mathcal{R}_1 \cap \mathcal{R}_0^\perp$, and that $\mathcal{W}_q$ and $\mathcal{R}_q \cap \mathcal{R}_{q-1}^\perp$ have the same dimension for all $q \in \mathbb{N}$.

For the first statement, recall that $\mathcal{W}_0$ was constructed precisely to be the span of a family $\{f^{i,v}\}$ of functions (see Equation (5)) which were orthogonal to $\mathscr{V}_0 = \mathcal{R}_0$. Moreover, every function

$f^{i,v}$ is a linear combination of characteristic functions χ_η with $d(\eta) = (1, \ldots, 1)$, and therefore lies in $\mathcal{R}_1 \cap \mathcal{R}_0^\perp$.

From the fact that $\{S_\lambda f^{i,s(\lambda)} : d(\lambda) = (q, q, \ldots, q),\ 1 \leq i \leq \#(D_{s(\lambda)}) - 1\}$ is a basis for $\mathcal{W}_q$, and the factorization rule in Λ, it follows that $\mathcal{W}_q$ has dimension

$$\sum_{v \in \Lambda^0} \#(\Lambda^{(q,\ldots,q)} v) \cdot \big(\#(v\Lambda^{(1,\ldots,1)}) - 1\big) = \#(\Lambda^{(q+1,\ldots,q+1)}) - \#(\Lambda^{(q,\ldots,q)}).$$

Moreover, we know from [8, Lemma 4.1] that "square" cylinder sets generate the topology on Λ^∞; it follows that $\mathcal{R}_s$ is spanned by $\{\chi_{[\lambda]} : d(\lambda) = (s, \ldots, s)\}$. Indeed, this set forms a basis for $\mathcal{R}_s$: if $d(\lambda) = d(\mu) = (s, \ldots, s)$, then the factorization rule implies that

$$\langle \chi_{[\lambda]}, \chi_{[\mu]} \rangle = \int_{\Lambda^\infty} \chi_{[\lambda]} \chi_{[\mu]} \, dM = \delta_{\lambda,\mu} M([\lambda]).$$

Consequently, $\mathcal{R}_{q+1} \cap \mathcal{R}_q^\perp$ also has dimension $\#(\Lambda^{(q+1,\ldots,q+1)}) - \#(\Lambda^{(q,\ldots,q)})$. Hence, $\mathcal{W}_q = \mathcal{R}_{q+1} \cap \mathcal{R}_q^\perp$ for all $q \in \mathbb{N}$, as desired. □

Remark 3.6. Fix $J \in \mathbb{N}^k$ with $J_i > 0$ for all i. We described in Section 5 of [9] how to construct wavelets with "fundamental domain" J – the original construction in Section 4 of [8] used $J = (1, \ldots, 1)$. By defining

$$\widetilde{\mathcal{R}}_s = \operatorname{span}\{\chi_{[\eta]} : d(\eta) \leq sJ\}$$

we can construct a Dirac operator $\widetilde{D}$ on $L^2(\Lambda^\infty, M)$ which gives rise to a spectral triple $(\mathcal{A}_\Lambda, L^2(\Lambda^\infty, M), \widetilde{D})$ whose eigenspaces agree with the wavelet decomposition given in Theorem 5.2 of [9]. We omit the details here as they are completely analogous to the proofs of Theorems 3.4 and 3.5 above.

4 θ-summability and Brownian motion

In this section we identify conditions under which the spectral triple of Theorem 3.4 is θ-summable. We also show that the Dirac operator for this spectral triple generates a Markovian semigroup, which (as discussed in [27]) can be viewed as Brownian motion associated to this representation of $C^*(\Lambda)$ on $L^2(\Lambda^\infty, M)$.

Recall that a spectral triple $(\mathcal{A}, \mathcal{H}, \not{D})$ is *θ-summable* if the operator $e^{-t\not{D}^2}$ is trace class for all $t > 0$. If there exists $s > 0$ such that $(1 + \not{D}^2)^{-s}$ is trace class then we say $(\mathcal{A}, \mathcal{H}, \not{D})$ is *finitely summable.*

Recall that the jth adjacency matrix $A_j \in M_{\Lambda^0}(\mathbb{N})$ of Λ is given by

$$A_j(v, w) = \# v\Lambda^{e_j} w;$$

the factorization rule implies that the matrices A_j pairwise commute. We will write ρ for the spectral radius of $A := A_1 \cdots A_k$. Observe that if Λ is source-free and there exists a vertex v such that $|v\Lambda^{e_j}| \geq 2$, then $\rho > 1$. Finally, we write $f(n) = \Theta(g(n))$ if there exist positive constants c_1, c_2 such that for large n, $c_1 g(n) \leq f(n) \leq c_2 g(n)$.

Proposition 4.1. *If $\rho > 1$ then the spectral triple $(\mathcal{A}_\Lambda, L^2(\Lambda^\infty, M), D)$ is never finitely summable.*

Proof. The proof of Theorem 3.5 establishes that the nth eigenspace of D has dimension

$$\#\Lambda^{(n+1,n+1,\ldots,n+1)} - \#\Lambda^{(n,\ldots,n)} = \sum_{v,w\in\Lambda^0} \left(A^{n+1}(v,w) - A^n(v,w)\right), \text{ where } A = A_1 \ldots A_k.$$

First notice that, since A has a (unique) positive normalized right eigenvector of eigenvalue ρ (namely $\kappa^\Lambda = (\kappa^\Lambda_v)_{v\in\Lambda^0}$), Corollary 8.1.33 of [11] implies that (see also [7, Proof of theorem 3.8])

$$\frac{A^q(v,w)}{\rho^q} \le \frac{\max\{\kappa^\Lambda_b\}_{b\in\Lambda^0}}{\min\{\kappa^\Lambda_b\}_{b\in\Lambda^0}},\ \forall v,w\in\Lambda^0, \forall\, q\in\mathbb{N}\backslash\{0\},$$

$$\text{and therefore} \quad \sum_{v,w}\frac{A^q(v,w)}{\rho^q} \le |\Lambda^0|^2\,\frac{\max\{\kappa^\Lambda_j\}_{j\in\Lambda^0}}{\min\{\kappa^\Lambda_j\}_{j\in\Lambda^0}}, \forall\, q\in\mathbb{N}\backslash\{0\}.$$

But, since all the entries of the positive normalized Perron–Frobenius eigenvector κ^Λ of Λ are less than 1, we also have

$$\begin{aligned}\sum_{v,w} A^q(v,w) &= \left\langle \begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}, A^q\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}\right\rangle \ge \left\langle \begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}, A^q\kappa^\Lambda\right\rangle \\ &= \left\langle \begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}, \rho^q\kappa^\Lambda\right\rangle = \rho^q\left\langle \begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}, \kappa^\Lambda\right\rangle,\ \forall\, q\in\mathbb{N}\backslash\{0\}.\end{aligned}$$

Therefore,

$$\rho^q\left\langle \begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}, \kappa^\Lambda\right\rangle \ \le\ \sum_{v,w} A^q(v,w)\ \le\ |\Lambda^0|^2\,\frac{\max\{x^\Lambda_b\}_{b\in\mathcal{V}_0}}{\min\{x^\Lambda_b\}_{b\in\mathcal{V}_0}}\,\rho^q,\ \forall\, q\in\mathbb{N}\backslash\{0\}.$$

Consequently $\#\Lambda^{(n+1,n+1,\ldots,n+1)} - \#\Lambda^{(n,\ldots,n)} = \Theta(\rho^n)$. Since $\alpha_n < Cn$ for all n, it follows that

$$\begin{aligned}\mathrm{Tr}((1+D^2)^{-s}) &= \sum_{n\in\mathbb{N}}(1+\alpha_n^2)^{-s}(\dim\mathcal{R}_n - \dim\mathcal{R}_{n-1}) \\ &\sim \sum_{n\in\mathbb{N}}(1+\alpha_n^2)^{-s}(\rho^n) \ge \sum_{n\in\mathbb{N}}\frac{\rho^n}{(1+(Cn)^2)^s},\end{aligned}$$

and this latter series diverges since $\lim_{n\to\infty}\frac{\rho^n}{(1+(Cn)^2)^s} = \infty$ for all $s>0$. □

Proposition 4.2. *If the sequence $(\alpha_n)_{n\in\mathbb{N}}$ satisfies $\alpha_n = \Theta(n^r)$ for $r < 1/2$, then the spectral triple $(\mathcal{A}_\Lambda, L^2(\Lambda^\infty, M), D)$ is θ-summable. If $\alpha_n = \Theta(n^r)$ for $r < 1/2$, then the spectral triple $(\mathcal{A}_\Lambda, L^2(\Lambda^\infty, M), D)$ is not θ-summable.*

Proof. Observe that

$$\mathrm{Tr}(e^{-tD^2}) = \sum_{n\in\mathbb{N}} e^{-t\alpha_n^2} \left(\#\Lambda^{(n+1,n+1,\ldots,n+1)} - \#\Lambda^{(n,\ldots,n)}\right) \sim \sum_{n\in\mathbb{N}} e^{-t\alpha_n^2}\rho^n.$$

If $\alpha_n = \Theta(n^r)$, then there exist constants $B, C > 0$ such that $Bn^r \le \alpha_n \le Cn^r$ for large enough n. Consequently

$$\mathrm{Tr}(e^{-tD^2}) \ge \sum_{n\in\mathbb{N}} \frac{\rho^n}{e^{Ctn^{2r}}} = \sum_{n\in\mathbb{N}} \left(\frac{\rho}{e^{Ctn^{2r-1}}}\right)^n.$$

When $r < 1/2$, the ratio test tells us that this series diverges, so e^{-tD^2} is not trace class. On the other hand,

$$\mathrm{Tr}(e^{-tD^2}) \le \sum_{n\in\mathbb{N}} \frac{\rho^n}{e^{Btn^{2r}}} = \sum_{n\in\mathbb{N}} \left(\frac{\rho}{e^{Btn^{2r-1}}}\right)^n.$$

When $r > 1/2$, this series converges by the ratio test and therefore e^{-tD^2} is trace class. □

We now proceed to show that the Dirac operator D from the spectral triple $(\mathcal{A}_\Lambda, L^2(\Lambda^\infty, M), D)$ also gives rise to a Markovian semigroup, which is an analogue of Brownian motion for noncommutative dynamical sytems. As defined in [27], a *Markovian semigroup* is a family of operators $\{T_t\}_{t\in\mathbb{R}_{>0}} \subseteq B(L^2(X, m))$ for a measure space (X, m), such that

1. $0 \le f \le 1$ a.e. $\Rightarrow$ $\forall t,\ 0 \le T_t f \le 1$ a.e.
2. Each T_t is a self-adjoint contraction.
3. $T_{s+t} = T_s T_t$
4. For all $f \in L^2(X, m)$, we have $\lim_{t\to 0} \|T_t(f) - f\| = 0$.

Proposition 4.3. *For the Dirac operator D, the operators $\{e^{-tD^2}\}_{t>0}$ form a Markovian semigroup.*

Proof. Properties 2 and 3 of the definition of a Markovian semigroup are immediate from the construction. For Property 1, observe that if $0 \le f \le 1$ then we can approximate f a.e. by simple functions which are supported on cylinder sets and have coefficients in $[0, 1]$; for these simple functions ξ, we have $0 \le e^{-tD^2}(\xi) \le 1$, so it follows that $0 \le e^{-tD^2}(f) \le 1$ a.e. as well.

To see Property 4, fix $f \in L^2(\Lambda^\infty, M)$ and $\epsilon > 0$. Choose $n \in \mathbb{N}$ and $\xi \in \mathcal{R}_n$ such that $\|\xi - f\| < \epsilon/3$. Then

$$\begin{aligned}\|e^{-tD^2}(f) - f\| &\le \|e^{-tD^2}(f - \xi)\| + \|e^{-tD^2}(\xi) - \xi\| + \|f - \xi\| \le 2\epsilon/3 + \|e^{-tD^2}(\xi) - \xi\| \\ &\le 2\epsilon/3 + \sum_{k=1}^{n} \|(e^{-t\alpha_k^2} - 1)\widehat{\Xi}_{k,k-1}(\xi)\|.\end{aligned}$$

The fact that for all k we have $\lim_{t\to 0} e^{-t\alpha_k^2} = 1$ implies the existence of $\delta > 0$ such that if $t < \delta$ then $\sum_{k=1}^{n} \|(e^{-t\alpha_k^2} - 1)\widehat{\Xi}_{k,k-1}(\xi)\| < \epsilon/3$. Since $\epsilon > 0$ was arbitrary, it follows that Property 4 holds as well. □

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