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Spectral triples and wavelets for higher-rank graphs

Carla Farsi, Elizabeth Gillaspy, Antoine Julien, Sooran Kang, and Judith Packer

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Abstract

In this paper, we present a new way to associate a finitely summable spectral triple to a higher-rank graph Λ, via the infinite path space Λ^∞ of Λ. Moreover, we prove that this spectral triple has a close connection to the wavelet decomposition of Λ^∞ which was introduced by Farsi, Gillaspy, Kang, and Packer in 2015. We first introduce the concept of stationary k-Bratteli diagrams, in order to associate a family of ultrametric Cantor sets, and their associated Pearson-Bellissard spectral triples, to a finite, strongly connected higher-rank graph Λ. We then study the zeta function, abscissa of convergence, and Dixmier trace associated to the Pearson-Bellissard spectral triples of these Cantor sets, and show these spectral triples are ζ-regular in the sense of Pearson and Bellissard. We obtain an integral formula for the Dixmier trace given by integration against a measure μ, and show that μ is a rescaled version of the measure M on Λ^∞ which was introduced by an Huef, Laca, Raeburn, and Sims. Finally, we investigate the eigenspaces of a family of Laplace-Beltrami operators associated to the Dirichlet forms of the spectral triples. We show that these eigenspaces refine the wavelet decomposition of $L^2(\Lambda^\infty, M)$ which was constructed by Farsi et al.

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Key words and phrases: Finitely summable spectral triple, wavelets, higher-rank graph, ζ-function, Laplace-Beltrami operator, Dixmier trace, k-Bratteli diagram, ultrametric Cantor set.

Contents

1 Introduction

Both spectral triples and wavelets are algebraic structures which encode geometrical information. In this paper, we expand the correspondence established in [27] between wavelets and spectral triples for the infinite path space of the Cuntz algebras $\mathcal{O}_N$ to the setting of higher-rank graphs. To be precise, we associate a family of Pearson-Bellissard spectral triples [59] to the infinite path space of a higher-rank graph (or k-graph) Λ, and relate these spectral triples with the representation of the higher-rank graph C^*-algebra $C^*(\Lambda)$ on the infinite path space, and the associated wavelet decomposition, which were introduced in [28]. We also investigate the geometry of ultrametric Cantor sets associated to Λ by studying the ζ-functions and Dixmier traces associated to these spectral triples.

Spectral triples were introduced by Connes in [19] as a noncommutative generalization of a compact Riemannian manifold. A spectral triple consists of a representation of a pre-C^*-algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, together with a Dirac-type operator D on $\mathcal{H}$, which satisfy certain commutation relations. In the case when $\mathcal{A} = C^\infty(X)$ is the algebra of smooth functions on a compact spin manifold X, Connes showed [20] that the algebraic structure of the associated spectral triple suffices to reconstruct the Riemannian metric on X. Moreover, Connes established in [19] that the spectral dimension and Dixmier trace of this spectral triple recover the Riemannian volume form on X. To be precise, the dimension δ of the manifold X agrees with the spectral dimension of $(C^\infty(X), D, \mathcal{H})$. Furthermore, for any $f \in C^\infty(X)$, the Dixmier trace $\mathrm{Tr}_\omega(f|D|^{-\delta})$ is independent of the choice of generalized limit ω, and gives a rescaled version of $\int_X f\, d\nu$, where ν denotes the volume form associated to the Riemannian metric. For more general spectral triples, the ζ-function and Dixmier trace associated to a spectral triple also play important roles in the applications of spectral triples to physics, from the standard model [21] to classical field theory [44].

In addition to spin manifolds, Connes studied spectral triples for the triadic Cantor set and Julia set in [19, 22]. Shortly thereafter, Lapidus [53] suggested studying spectral triples $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is a commutative algebra of functions on a fractal space X, and investigating which aspects of the geometry of X are recovered from the spectral triple. Of the many authors (cf. [15, 35, 59]) who have pursued Lapidus' program, we focus here on the spectral triples introduced by Pearson and Bellissard in [59].

Motivated by a desire to apply the tools of noncommutative geometry to the study of transversals of aperiodic Delone sets [3], Pearson and Bellissard constructed in [59] spectral triples for ultrametric Cantor sets associated to Michon trees. They also showed how to recover geometric information about the Cantor set $\mathcal{C}$ from their spectral triple: using the ζ-function and the Dixmier trace, Pearson and Bellissard reconstructed the ultrametric and the upper box dimension of $\mathcal{C}$. Moreover, they constructed a family of Laplace-Beltrami operators Δ_s, $s \in \mathbb{R}$, on $L^2(\mathcal{C}, \mu)$, where the measure μ arises from the Dixmier trace. Julien and Savinien subsequently applied the Pearson-Bellissard spectral triples to the study of substitution tilings in [42], by sharpening many of the results from [59] and reinterpreting them using stationary Bratteli diagrams.

In this paper, we extend the Pearson-Bellissard spectral triples to the setting of higher-rank graphs. A k-dimensional generalization of directed graphs, higher-rank graphs (also called k-graphs) were introduced by Kumjian and Pask in [51]. The combinatorial character of k-graph C^*-algebras has facilitated the analysis of their structural properties, such as simplicity and ideal structure [60, 62, 24, 45, 12], quasidiagonality [18] and KMS states [40, 39, 38]. In particular, results such as [64, 9, 8, 58] show that higher-rank graphs often provide concrete examples of C^*-algebras which are relevant to Elliott's classification program for simple separable nuclear C^*-algebras.

By associating Pearson-Bellissard spectral triples to k-graphs, this paper establishes a link between k-graphs and their C^*-algebras, and the extensive literature on the spectral geometry of fractal and Cantor sets (cf. [13, 15, 16, 35, 46, 47, 52] and the references therein). In these cases, as is the case in the present paper, the pre-C^*-algebra of the spectral triple is abelian. Since the C^*-algebra of a graph or k-graph is rarely abelian, other researchers (cf. [11, 31, 32]) have studied non-abelian spectral triples for graph C^*-algebras and related objects; the research in this paper offers a complementary perspective on the noncommutative geometry of higher-rank graph C^*-algebras, and in particular on the connection between wavelets and spectral triples.

In order to associate Pearson-Bellissard spectral triples to k-graphs, we introduce a new class of Bratteli diagrams: namely, the stationary k-Bratteli diagrams. Where a stationary Bratteli diagram is completely determined by a single square matrix A, the stationary k-Bratteli diagrams are determined by k matrices $A_1, \ldots, A_k$; see Definition 2.5 below. The space of infinite paths $X_{\mathcal{B}}$ of a stationary k-Bratteli diagram $\mathcal{B}$ is often a Cantor set, enabling us to study its associated Pearson-Bellissard spectral triple. Indeed, if the matrices $A_1, \ldots, A_k$ are the adjacency matrices for a k-graph Λ, then the space of infinite paths in Λ is homeomorphic to the Cantor set $X_{\mathcal{B}}$ (also called $\partial\mathcal{B}$). In other words, the Pearson-Bellissard spectral triples for stationary k-Bratteli diagrams can also be viewed as spectral triples for higher-rank graphs.

We then proceed to study, in Section 3, the geometrical information encoded by these spectral triples. Theorem 3.14 establishes that the Pearson-Bellissard spectral triple associated to $(X_{\mathcal{B}_\Lambda}, d_\delta)$ is finitely summable, with dimension $\delta \in (0, 1)$. Section 3.3 focuses on the Dixmier traces of the spectral triples, and establishes both an integral formula for the Dixmier trace (Theorems 3.23 and 3.28) and a concrete expression for the measure induced by the Dixmier trace (Theorem 3.26). These computations also reveal that the ultrametric Cantor sets $(X_{\mathcal{B}_\Lambda}, d_\delta)$ are ζ-regular in the sense of [59, Definition 11]. Other settings in the literature in which spectral triples on Cantor sets admit an integral formula for the Dixmier trace include [13, 47, 17, 14].

In full generality, Dixmier traces are defined on the Dixmier-Macaev (also called Lorentz) ideal $\mathcal{M}_{1,\infty} \subseteq \mathcal{K}(\mathcal{H})$ inside the compact operators and are computed using a generalized limit ω (roughly speaking, a linear functional that lies between lim sup and lim inf). Although the theory of Dixmier traces can be quite intricate, many of the computations simplify substantially in our setting, and so our treatment of the general theory will be brief; we refer the interested reader to the extensive literature on Dixmier traces and other singular traces (cf. [19, 55, 54, 10, 47, 34, 56]). For each such generalized limit ω, there is an ω-Dixmier trace $\mathcal{T}_\omega$ defined on $\mathcal{M}_{1,\infty}$; however, if $T \in \mathcal{M}_{1,\infty}$ is measurable in the sense of Connes, then the value of $\mathcal{T}_\omega(T)$ is independent of ω, and in many cases can be computed via residue formulas. Indeed this is the case for $T = |D|^{-\delta}$, see Corollary 3.19, if D is the Dirac operator of the Pearson-Bellissard spectral triple associated to the ultrametric Cantor set $(X_{\mathcal{B}_\Lambda}, d_\delta)$. The calculation of the Dixmier trace of $|D|^{-\delta}$ is one of the most technical results of the paper, since it relies on the explicit computation of a residue formula, and was inspired by a related result (Theorem 3.9 of [42]) for the case of stationary Bratteli diagrams with primitive adjacency matrices. Theorem 3.18 underlies the major results mentioned in the previous paragraph.

The complexity of stationary k-Bratteli diagrams, as compared to the stationary Bratteli diagrams studied in [42], complicates the analysis of the ζ-function and Dixmier trace of our spectral triples. However, a side benefit of our approach is that, when restricted to the setting of stationary Bratteli diagrams, the theorems in Section 3 below hold for an irreducible matrix A. Thus, even for stationary Bratteli diagrams, the results in this paper are new: the authors of [59, 42] imposed on A the stronger requirement of primitivity.

As mentioned earlier, one of our motivations for studying Pearson-Bellissard spectral triples for k-graphs was to understand their relationship with the wavelets and representations for k-graphs introduced in [28]. Wavelet analysis has many applications in various areas of mathematics, physics and engineering. For example, it has been used to study p-adic spectral analysis [50], pseudodifferential operators and dynamics on ultrametric spaces [48, 49], and the theory of quantum gravity [26, 2].

Although wavelets were introduced as orthonormal bases or frames for $L^2(\mathbb{R}^n)$ which behaved well under compression algorithms, wavelet decompositions for $L^2(X)$, where X is a fractal space, were defined by Jonsson [41] and Strichartz [65] shortly thereafter. In this fractal setting, the wavelet orthonormal bases reflect the self-similar structure of X. A few years later, Jonsson and Strichartz' fractal wavelets inspired Marcolli and Paolucci [57] to construct a wavelet decomposition of $L^2(\Lambda_A, \mu)$ for the Cuntz-Krieger algebra $\mathcal{O}_A$, where A is an $N \times N$ matrix, Λ_A denotes the limit set of infinite sequences in an alphabet on N letters, and μ is a Hausdorff measure on Λ_A. Similar wavelets were developed in the higher-rank graph setting by four of the authors of the current paper [28], using a separable representation π of the k-graph C^*-algebra $C^*(\Lambda)$. In particular, this representation gave us a wavelet decomposition of $L^2(\Lambda^\infty, M)$, where Λ^∞ denotes the space of infinite paths in the k-graph Λ, and the measure M was introduced by an Huef et al. in [40]. This wavelet decomposition is given by

$$L^2(\Lambda^\infty, M) = \mathscr{V}_0 \oplus \bigoplus_{n \geq 0} \mathcal{W}_n. \tag{1}$$

Each subspace[1] $\mathcal{W}_n = \{S_\lambda f : f \in \mathcal{W}_0, \lambda \in \Lambda^{(n,\dots,n)}\}$ is constructed from $\mathcal{W}_0$ by means of limit"scaling and translation" operators $S_\lambda := \pi(s_\lambda)$ which reflect the (higher-rank) graph structure of Λ. (See Theorem 4.2 of [28] or Section 4 below.)

One of the main results of this paper, Theorem 4.6, proves that the spectral triples of Pearson and Bellissard [59] are intimately tied to the wavelets of [28]. Recall that a Pearson-Bellissard spectral triple for an ultrametric Cantor set $\mathcal{C}$ gives rise to a family of Laplace-Beltrami operators Δ_s, $s \in \mathbb{R}$, on $L^2(\mathcal{C}, \mu)$ associated to the spectral triple's Dirichlet form as in Equation (28) below. Julien and Savinien established in [42] that in the Bratteli diagram setting the eigenspaces of Δ_s are parametrized by the finite paths γ in the Bratteli diagram. Theorem 4.6 establishes that when $(\mathcal{C}, \mu) = (\Lambda^\infty, M)$, the eigenspaces E_γ of the Laplace-Beltrami operators refine the wavelet decomposition of (1).

This paper is organized as follows. In Section 2, we recall the basic facts about higher-rank graphs (or k-graphs) and we develop the machinery of stationary k-Bratteli diagrams (Definition 2.5). This enables us to construct a family of ultrametrics $\{d_\delta : \delta \in (0,1)\}$ on the infinite path space Λ^∞ of a k-graph Λ, identified as the boundary of the associated stationary k-Bratteli diagram $\mathcal{B}_\Lambda$. In many situations, $\Lambda^\infty \cong X_{\mathcal{B}_\Lambda}$ is a Cantor set (see Proposition 2.4); Section 3 studies the fine structure of the Pearson-Bellissard spectral triples associated to the ultrametric Cantor sets $\{X_{\mathcal{B}_\Lambda}, d_\delta\}_{\delta \in (0,1)}$. We begin by allowing δ to range over the interval $(0,1)$ because there is no *a priori* preferred value of δ in this range; later, we see in Corollary 3.15 that the Pearson-Bellissard spectral triple of $(X_{\mathcal{B}_\Lambda}, d_\delta)$ has dimension δ. However, other properties of the spectral triple (cf. Theorem 3.26) are independent of the choice of $\delta \in (0,1)$.

The major technical achievements of this paper are Theorems 3.14 and 3.18. These results underpin Theorems 3.26 and 3.28, which offer less computationally intensive perspectives on the Dixmier trace. Theorem 3.14 establishes that the ζ-function of the spectral triple associated to the ultrametric Cantor set $(X_{\mathcal{B}_\Lambda}, d_\delta)$ has abscissa of convergence δ, while Theorem 3.18 enables the computation of the Dixmier trace integral formula in Theorems 3.23 and 3.28, which in turn reveals the ζ-regularity of $(X_{\mathcal{B}_\Lambda}, d_\delta)$. Theorem 3.26 then shows that under mild additional hypotheses, the measures μ_δ which appear in the Dixmier trace integral formula are simply a rescaling of the measure M on the infinite path space $X_{\mathcal{B}_\Lambda}$ that was introduced in Proposition 8.1 of [40] and which we used in [28] to construct a wavelet decomposition of $L^2(\Lambda^\infty, M)$.

Finally, Section 4 presents the promised connection between the Pearson-Bellissard spectral triples and the wavelet decomposition of $L^2(\Lambda^\infty, M)$ from [28]. Under appropriate hypotheses we show in Theorem 4.6 that the eigenspaces E_γ of the Laplace-Beltrami operator Δ_s refine the wavelet decomposition of (1): namely, for all $n \in \mathbb{N}$,

$$\mathcal{W}_n = \bigoplus_{nk \leq |\gamma| < (n+1)k} E_\gamma.$$

[1]The subspaces denoted in this paper by $\mathcal{W}_n$ were labeled $\mathcal{W}_{j,\Lambda}$ for $j \in \mathbb{N}$ in Theorem 4.2 of [28].

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2 Higher-rank graphs and ultrametric Cantor sets

In this section, we review the basic definitions and results that we will need about directed graphs, higher-rank graphs, (weighted/stationary) Bratteli diagrams, infinite path spaces, and (ultrametric) Cantor sets. Throughout this article, $\mathbb{N}$ will denote the non-negative integers.

2.1 Bratteli diagrams

A *directed graph* is given by a quadruple $E = (E^0, E^1, r, s)$, where E^0 is the set of vertices of the graph, E^1 is the set of edges, and $r, s : E^1 \to E^0$ denote the range and source of each edge. A vertex v in a directed graph E is a *sink* if $s^{-1}(v) = \emptyset$; we say v is a *source* if $r^{-1}(v) = \emptyset$.

Definition 2.1. [6] A *Bratteli diagram* $\mathcal{B} = (\mathcal{V}, \mathcal{E})$ is a directed graph with vertex set $\mathcal{V} = \bigsqcup_{n\in\mathbb{N}} \mathcal{V}_n$, and edge set $\mathcal{E} = \bigsqcup_{n\geq 1} \mathcal{E}_n$, where $\mathcal{E}_n$ consists of edges whose source vertex lies in $\mathcal{V}_n$ and whose range vertex lies in $\mathcal{V}_{n-1}$, and $\mathcal{V}_n$ and $\mathcal{E}_n$ are finite sets for all n.

For a Bratteli diagram $\mathcal{B} = (\mathcal{V}, \mathcal{E})$, define a sequence of adjacency matrices $A_n = (f^n(v, w))_{v,w}$ of $\mathcal{B}$ for $n \geq 1$, where

$$f^n(v, w) = \#\Big(\{e \in \mathcal{E}_n : r(e) = v \in \mathcal{V}_{n-1},\ s(e) = w \in \mathcal{V}_n\}\Big),$$

where by $\#(Q)$ we denote the cardinality of the set Q. A Bratteli diagram is *stationary* if $A_n = A_1 =: A$ are the same for all $n \geq 1$. We say that η is a *finite* path of $\mathcal{B}$ if there exists $m \in \mathbb{N}$ such that $\eta = \eta_1 \dots \eta_m$ for $\eta_i \in \mathcal{E}_i$, and in that case the *length* of η, denoted by $|\eta|$, is m.

Remark 2.2. In the literature, Bratteli diagrams traditionally have $s(\mathcal{E}_n) = \mathcal{V}_n$ and $r(\mathcal{E}_n) = \mathcal{V}_{n+1}$; our edges point the other direction for consistency with the standard conventions for higher-rank graphs and their C^*-algebras.

It is also common in the literature to require $|\mathcal{V}_0| = 1$ and to call this vertex the *root* of the Bratteli diagram; we will NOT invoke this hypothesis in this paper.

Definition 2.3. Given a Bratteli diagram $\mathcal{B} = (\mathcal{V}, \mathcal{E})$, denote by $X_{\mathcal{B}}$ the set of all of its infinite paths:

$$X_{\mathcal{B}} = \{(x_n)_{n\geq 1} : x_n \in \mathcal{E}_n \text{ and } s(x_n) = r(x_{n+1}) \ \text{ for } n \geq 1\}.$$

For each finite path $\lambda = \lambda_1\lambda_2 \cdots \lambda_\ell$ in $\mathcal{B}$ with $r(\lambda) \in \mathcal{V}_0$, $\lambda_i \in \mathcal{E}_i$, define the *cylinder set* $[\lambda]$ by

$$[\lambda] = \{x = (x_n)_{n\geq 1} \in X_{\mathcal{B}} : x_i = \lambda_i \ \text{ for } \ 1 \leq i \leq \ell\}.$$

The collection $\mathcal{T}$ of all cylinder sets forms a compact open sub-basis for a locally compact Hausdorff topology on $X_{\mathcal{B}}$ and cylinder sets are clopen; we will always consider $X_{\mathcal{B}}$ with this topology.

The following proposition will tell us when $X_{\mathcal{B}}$ is a *Cantor set*; that is, a totally disconnected, compact, perfect topological space.

Proposition 2.4. *(Lemma 6.4. of [1]) Let $\mathcal{B} = (\mathcal{V}, \mathcal{E})$ be a Bratteli diagram such that $\mathcal{B}$ has no sinks outside of $\mathcal{V}_0$, and no sources. Then $X_{\mathcal{B}}$ is a totally disconnected compact Haudorff space, and the following statements are equivalent:*

1. *The infinite path space $X_{\mathcal{B}}$ of $\mathcal{B}$ is a Cantor set;*
2. *For each infinite path $x = (x_1, x_2, ...)$ in $X_{\mathcal{B}}$ and each $n \geq 1$ there is an infinite path $y = (y_1, y_2, ...)$ with*
$$x \neq y \text{ and } x_k = y_k \text{ for } 1 \leq k \leq n;$$
3. *For each $n \in \mathbb{N}$ and each $v \in \mathcal{V}_n$ there is $m \geq n$ and $w \in \mathcal{V}_m$ such that there is a path from w to v and*
$$\#(r^{-1}(\{w\})) \geq 2.$$

2.2 Higher-rank graphs and stationary k-Bratteli diagrams

Definition 2.5. Let $A_1, A_2, \cdots, A_k$ be $N \times N$ matrices with non-negative integer entries. The *stationary k-Bratteli diagram* associated to the matrices $A_1, \ldots, A_k$, which we will call $\mathcal{B}_{(A_j)_{j=1,\ldots,k}}$, is the Bratteli diagram given by a set of vertices $\mathcal{V} = \bigsqcup_{n\in\mathbb{N}} \mathcal{V}_n$ and a set of edges $\mathcal{E} = \bigsqcup_{n\geq 1} \mathcal{E}_n$, where the edges in $\mathcal{E}_n$ go from $\mathcal{V}_n$ to $\mathcal{V}_{n-1}$, such that:

(a) For each $n \in \mathbb{N}$, $\mathcal{V}_n$ consists of N vertices, which we will label $1, 2, \ldots, N$.

(b) When $n \equiv i \pmod{k}$, there are $A_i(p, q)$ edges whose range is the vertex p of $\mathcal{V}_{n-1}$ and whose source is the vertex q of $\mathcal{V}_n$.

In other words, the matrix A_1 determines the edges with source in $\mathcal{V}_1$ and range in $\mathcal{V}_0$; then the matrix A_2 determines the edges with source in $\mathcal{V}_2$ and range in $\mathcal{V}_1$; etc. The matrix A_k determines the edges with source in $\mathcal{V}_k$ and range in $\mathcal{V}_{k-1}$, and the matrix A_1 determines the edges with range in $\mathcal{V}_k$ and source in $\mathcal{V}_{k+1}$.

Note that a stationary 1-Bratteli diagram is often called a *stationary Bratteli diagram* in the literature (cf. [6, 42]).

Just as a directed graph has an associated adjacency matrix A which also describes a stationary Bratteli diagram $\mathcal{B}_A$, the higher-dimensional generalizations of directed graphs known as *higher-rank graphs* or k-graphs give us k commuting matrices $A_1, \ldots, A_k$ and hence a stationary k-Bratteli diagram.

We use the standard terminology and notation for higher-rank graphs, which we review below for the reader's convenience.

Definition 2.6. [51] A *k-graph* is a countable small category Λ equipped with a degree functor[2] $d : \Lambda \to \mathbb{N}^k$ satisfying the *factorization property*: whenever λ is a morphism in Λ such that $d(\lambda) = m + n$, there are unique morphisms $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$.

We use the arrows-only picture of category theory; thus, $\lambda \in \Lambda$ means that λ is a morphism in Λ. For $n \in \mathbb{N}^k$, we write
$$\Lambda^n := \{\lambda \in \Lambda : d(\lambda) = n\}.$$
When $n = 0$, Λ^0 is the set of objects of Λ, which we also refer to as the *vertices* of Λ.

[2]We view $\mathbb{N}^k$ as a category with one object, namely 0, and with composition of morphisms given by addition.

Let $r, s : \Lambda \to \Lambda^0$ identify the range and source of each morphism, respectively. For $v \in \Lambda^0$ a vertex, we define

$$v\Lambda^n := \{\lambda \in \Lambda^n : r(\lambda) = v\} \text{ and } \Lambda^n w := \{\lambda \in \Lambda^n : s(\lambda) = w\}.$$

We say that Λ is *finite* if $\#(\Lambda^n) < \infty$ for all $n \in \mathbb{N}^k$, and we say Λ is *source-free* or *has no sources* if $\#(v\Lambda^n) > 0$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

For $1 \leq i \leq k$, write e_i for the ith standard basis vector of $\mathbb{N}^k$, and define a matrix $A_i \in M_{\Lambda^0}(\mathbb{N})$ by

$$A_i(v, w) = \#(v\Lambda^{e_i} w).$$

We call A_i the *ith adjacency matrix* of Λ. Note that the factorization property implies that the matrices A_i commute.

Despite their formal definition as a category, it is often useful to think of k-graphs as k-dimensional generalizations of directed graphs. In this interpretation, Λ^{e_i} is the set of "edges of color i" in Λ. The factorization property implies that each $\lambda \in \Lambda$ can be written as a concatenation of edges in the following sense: A morphism $\lambda \in \Lambda$ with $d(\lambda) = (n_1, n_2, \ldots, n_k)$ can be thought of as a k-dimensional hyper-rectangle of dimension $n_1 \times n_2 \times \cdots \times n_k$. Any minimal-length lattice path in $\mathbb{N}^k$ through the rectangle lying between 0 and $(n_1, \ldots, n_k)$ corresponds to a choice of how to order the edges making up λ, and hence to a unique decomposition or "factorization" of λ. For example, the lattice path given by walking in straight lines from 0 to $(n_1, 0, \ldots, 0)$ to $(n_1, n_2, 0, \ldots, 0)$ to $(n_1, n_2, n_3, 0, \ldots, 0)$, and so on, corresponds to the factorization of λ into edges of color 1, then edges of color 2, then edges of color 3, etc.

For any directed graph E, the category of its finite paths Λ_E is a 1-graph; the degree functor $d : \Lambda_E \to \mathbb{N}$ takes a finite path λ to its length $|\lambda|$. Example 2.7 below gives a less trivial example of a k-graph. The k-graphs Ω_k of Example 2.7 are also fundamental to the definition of the space of infinite paths in a k-graph.

Example 2.7. For $k \geq 1$, let Ω_k be the small category with

$$\mathrm{Obj}\,(\Omega_k) = \mathbb{N}^k,\ \mathrm{Mor}\,(\Omega_k) = \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}, \quad r(m, n) = m,\ s(m, n) = n.$$

If we define $d : \Omega_k \to \mathbb{N}^k$ by $d(m, n) = n - m$, then Ω_k is a k-graph with degree functor d.

Definition 2.8. Let Λ be a k-graph. An *infinite path* of Λ is a k-graph morphism

$$x : \Omega_k \to \Lambda;$$

we write Λ^∞ for the set of infinite paths in Λ. For each $p \in \mathbb{N}^k$, we have a map $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ given by

$$\sigma^p(x)(m, n) = x(m + p, n + p)$$

for $x \in \Lambda^\infty$ and $(m, n) \in \Omega_k$.

Remark 2.9. (a) Given $x \in \Lambda^\infty$, we often write $r(x) := x(0) = x(0, 0)$ for the terminal vertex of x. This convention means that an infinite path has a range but not a source.

We equip Λ^∞ with the topology generated by the sub-basis $\{[\lambda] : \lambda \in \Lambda\}$ of compact open sets, where

$$[\lambda] = \{x \in \Lambda^\infty : x(0, d(\lambda)) = \lambda\}.$$

Remark 2.5 of [51] establishes that, with this topology, Λ^∞ is a locally compact Hausdorff space.

Note that we use the same notation for a cylinder set of Λ^∞ and a cylinder set of $X_{\mathcal{B}}$ in Definition 2.3 since we will prove in Proposition 2.10 and Remark 2.11 (a) that Λ^∞ is homeomorphic and Borel isomorphic to $X_{\mathcal{B}_\Lambda}$ for a finite, source-free k-graph Λ.

(b) For any $\lambda \in \Lambda$ and any $x \in \Lambda^\infty$ with $r(x) = s(\lambda)$, we write λx for the unique infinite path $y \in \Lambda^\infty$ such that $y(0, d(\lambda)) = \lambda$ and $\sigma^{d(\lambda)}(y) = x$. If $d(\lambda) = p$, the maps σ^p and $\sigma_\lambda := x \mapsto \lambda x$ are local homeomorphisms which are mutually inverse:

$$\sigma^p \circ \sigma_\lambda = id_{[s(\lambda)]}, \quad \sigma_\lambda \circ \sigma^p = id_{[\lambda]},$$

although the domain of σ^p is $\Lambda^\infty \supsetneq [\lambda]$.

Informally, one should think of σ^p as "chopping off" the initial segment of length p, and the map $x \mapsto \lambda x$ as "gluing λ on" to the front of x. By "front" and "initial segment" we mean the range of x, since an infinite path has no source.

We can now state precisely the connection between k-graphs and stationary k-Bratteli diagrams.

Proposition 2.10. *Let Λ be a finite, source-free k-graph with adjacency matrices $A_1, \ldots, A_k$. Denote by $\mathcal{B}_\Lambda$ the stationary k-Bratteli diagram associated to the matrices $\{A_i\}_{i=1}^k$. Then $X_{\mathcal{B}_\Lambda}$ is homeomorphic to Λ^∞.*

Proof. Fix $x \in \Lambda^\infty$ and write $\mathbf{1} := (1, 1, \ldots, 1) \in \mathbb{N}^k$. Then the factorization property for Λ^∞ implies that there is a unique sequence

$$(\lambda_i)_i \in \prod_{i=1}^{\infty} \Lambda^{\mathbf{1}}$$

such that $x = \lambda_1 \lambda_2 \lambda_3 \cdots$ with $\lambda_i = x((i-1)\mathbf{1}, i\mathbf{1})$. (See the details in Remark 2.2 and Proposition 2.3 of [51]). Since there is a unique way to write $\lambda_i = f_1^i f_2^i \cdots f_k^i$ as a composable sequence of edges with $d(f_j^i) = e_j$, we have

$$x = f_1^1 f_2^1 \cdots f_k^1 f_1^2 f_2^2 \cdots f_k^2 f_1^3 \cdots ,$$

where the $nk + j$th edge has color j. Thus, for each i, f_j^i corresponds to an entry in A_j, and hence

$$f_1^1 f_2^1 \cdots f_k^1 f_1^2 f_2^2 \cdots f_k^2 f_1^3 \cdots \in X_{\mathcal{B}_\Lambda}.$$

Conversely, given $y = (g_\ell)_\ell \in X_{\mathcal{B}_\Lambda}$, we construct an associated k-graph infinite path $\tilde{y} \in \Lambda^\infty$ as follows. To $y = (g_\ell)_\ell$ we associate a sequence $(\eta_n)_{n \geq 1}$ of finite paths in Λ, where

$$\eta_n = g_1 \cdots g_{nk}$$

is the unique morphism in Λ of degree $(n, \ldots, n)$ represented by the sequence of composable edges $g_1 \cdots g_{nk}$. Recall from [51] Remark 2.2 that a morphism $\tilde{y} : \Omega_k \to \Lambda$ is uniquely determined by $\{\tilde{y}(0, n\mathbf{1})\}_{n \in \mathbb{N}}$. Thus, the sequence $(\eta_n)_n$ determines $\tilde{y}$:

$$\tilde{y}(0,0) = r(y) = r(g_1), \qquad \tilde{y}(0, n\mathbf{1}) := \eta_n \ \forall\, n \geq 1.$$

The map $y \mapsto \tilde{y}$ is easily checked to be a bijection which is inverse to the map $x \mapsto f_1^1 f_2^1 \cdots f_k^1 f_1^2 f_2^2 \cdots f_k^2 f_1^3 \cdots$.

Moreover, for any $i \in \mathbb{N}$, $0 \leq j \leq k-1$, and any $\lambda = f_1^1 f_2^1 \cdots f_k^1 f_1^2 f_2^2 \cdots f_k^2 f_1^3 \cdots f_j^i$ with $d(\lambda) = (i-1)\mathbf{1} + (\overbrace{1, \ldots, 1}^{j}, 0, \ldots, 0)$, both of these bijections preserve the cylinder set $[\lambda]$. In particular, these bijections preserve the "square" cylinder sets $[\lambda]$ associated to paths λ with $d(\lambda) = i\mathbf{1}$ for some $i \in \mathbb{N}$. (If $i = 0$ then we interpret $d(\lambda) = 0 \cdot \mathbf{1}$ as meaning that λ is a vertex in $\mathcal{V}_0 \cong \Lambda^0$.) From the proof of Lemma 4.1 of [28], any cylinder set can be written as a disjoint union of square cylinder sets, and therefore the square cylinder sets generate the topology on Λ^∞. We deduce that Λ^∞ and $X_{\mathcal{B}_\Lambda}$ are homeomorphic, as claimed. □

Remark 2.11. (a) Thanks to Proposition 2.10, we will usually identify the infinite path spaces $X_{\mathcal{B}_\Lambda}$ and Λ^∞, denoting this space by the symbol which is most appropriate for the context. In particular, the Borel structures on $X_{\mathcal{B}_\Lambda}$ and Λ^∞ are isomorphic, and so any Borel measure on Λ^∞ induces a unique Borel measure on $X_{\mathcal{B}_\Lambda}$ and vice versa.

(b) The bijection of Proposition 2.10 between infinite paths in the k-graph Λ and in the associated Bratteli diagram $\mathcal{B}_\Lambda$ does not extend to finite paths. While any finite path in the Bratteli diagram determines a finite path, or morphism, in Λ, not all morphisms in Λ have a representation in the Bratteli diagram. For example, if e_1 is a morphism of degree $(1, 0, \dots, 0) \in \mathbb{N}^k$ in a k-graph ($k > 1$) with $r(e_1) = s(e_1)$, the composition $e_1 e_1$ is a morphism in the k-graph which cannot be represented as a path on the Bratteli diagram. However, the proof of Proposition 2.10 above establishes that "rainbow" paths in Λ – morphisms of degree $(\overbrace{q+1, \dots, q+1}^{j}, q, \dots, q)$ for some $q \in \mathbb{N}$ and $1 \leq j \leq k$ – can be represented uniquely as paths of length $kq + j$ in the Bratteli diagram.

2.3 Ultrametrics on $X_\mathcal{B}$

Although the Cantor set is unique up to homeomorphism, different metrics on it can induce quite different geometric structures. In this section, we will focus on Bratteli diagrams $\mathcal{B}$ for which the infinite path space $X_\mathcal{B}$ is a Cantor set. In this setting, we construct ultrametrics on $X_\mathcal{B}$ by using weights on $\mathcal{B}$. To do so, we first need to introduce some definitions and notation.

Definition 2.12. A metric d on a Cantor set C is called an *ultrametric* if d induces the Cantor set topology and satisfies the so-called *strong triangle inequality*

$$d(x, y) \leq \max\{d(x, z), d(y, z)\} \quad \text{for all } x, y, z \in C. \tag{2}$$

Definition 2.13. Let $\mathcal{B}$ be a Bratteli diagram. Denote by $F\mathcal{B}$ the set of finite paths in $\mathcal{B}$ with range in $\mathcal{V}_0$. For any $n \in \mathbb{N}$, we write

$$F^n\mathcal{B} = \{\lambda \in F\mathcal{B} : |\lambda| = n\}.$$

Given two (finite or infinite) paths λ, η in $\mathcal{B}$, we say η is a *sub-path* of λ if there is a sequence γ of edges, with $r(\gamma) = s(\eta)$, such that $\lambda = \eta\gamma$.

For any two infinite paths $x, y \in X_\mathcal{B}$, we define $x \wedge y$ to be the longest path $\lambda \in F\mathcal{B}$ such that λ is a sub-path of x and y. We write $x \wedge y = \emptyset$ when no such path λ exists.

Definition 2.14. (cf. [59]) A *weight* on a Bratteli diagram $\mathcal{B}$ is a function $w : F\mathcal{B} \to \mathbb{R}^+$ such that

- If $\mathcal{V}_0$ denotes the set of vertices at level 0, then $\sum_{v \in \mathcal{V}_0} w(v) \leq 1$.
- $\lim_{n\to\infty} \sup\{w(\lambda) : \lambda \in F^n\mathcal{B}\} = 0$.
- If η is a sub-path of λ, then $w(\lambda) < w(\eta)$.

A Bratteli diagram with a weight is often called a weighted Bratteli diagram and denoted by $(\mathcal{B}, w)$.

Observe that the third condition implies that for any path $x = (x_n)_n \in \mathcal{B}$ (finite or infinite),

$$w\left(x_1 x_2 \dots x_n\right) > w\left(x_1 x_2 \cdots x_{n+1}\right) \quad \text{for all } n.$$

The concept above of a weight was inspired by Definition 2.9 of [42] which was in turn inspired by the work of [59]; indeed, if one denotes a weight in the sense of [42] Definition 2.9 by w', and defines $w(\lambda) := w'(s(\lambda))$, then w is a weight on $\mathcal{B}$ in the sense of Definition 2.14 above.

Proposition 2.15. *Let* $(\mathcal{B}, w)$ *be a weighted Bratteli diagram such that* $X_{\mathcal{B}}$ *is a Cantor set. The function* $d_w : X_{\mathcal{B}} \times X_{\mathcal{B}} \to \mathbb{R}^+$ *given by*

$$d_w(x, y) = \begin{cases} 1 & \textit{if } x \wedge y = \emptyset, \\ 0 & \textit{if } x = y, \\ w(x \wedge y) & \textit{else.} \end{cases}$$

is an ultrametric on $X_{\mathcal{B}}$. *Moreover* d_w *metrizes the cylinder set topology on* $X_{\mathcal{B}}$.

Proof. It is evident from the defining conditions of a weight that d_w is symmetric and satisfies $d_w(x, y) = 0 \Leftrightarrow x = y$. Since the inequality (2) is stronger than the triangle inequality, once we show that d_w satisfies the ultrametric condition (2) it will follow that d_w is indeed a metric.

To that end, first suppose that $d_w(x, y) = 1$; in other words, x and y have no common sub-path. This implies that for any $z \in X_{\mathcal{B}}$, at least one of $d_w(x, z)$ and $d_w(y, z)$ must be 1, so

$$d_w(x, y) \leq \max\{d_w(x, z), d_w(y, z)\},$$

as desired. Now, suppose that $d_w(x, y) = w(x \wedge y) < 1$. If $d_w(x, z) \geq d_w(x, y)$ for all $z \in X_{\mathcal{B}}$ then we are done. On the other hand, if there exists $z \in X_{\mathcal{B}}$ such that $d_w(x, z) < d_w(x, y)$, then the maximal common sub-path of x and z must be longer than that of x and y. This implies that

$$d_w(y, z) := w(y \wedge z) = w(y \wedge x) = d_w(x, y);$$

consequently, in this case as well we have $d_w(x, y) \leq \max\{d(x, z), d_w(y, z)\}$.

Finally, we observe that the metric topology induced by d_w agrees with the cylinder set topology. This fact may be known, but because we did not find the proof in the literature, we include it here. Let $B[x, r]$ be the closed ball of center x and radius $r > 0$. We will show first that $B[x, r] \subset [x_1 \cdots x_n]$ for some $n \in \mathbb{N}$. To obtain an easy upper bound on the diameter of $B[x, r]$, choose $y, z \in B[x, r]$ and observe that

$$d_w(y, z) \leq \max\{d_w(x, y), d_w(x, z)\} \leq r.$$

Taking supremums reveals that $\operatorname{diam} B[x, r] \leq r$.

We now check that $B[x, r] = [x_1 \cdots x_n]$ for some $n \in \mathbb{N}$. By the definition of the weight w, there is a smallest $n \in \mathbb{N}$ such that

$$w(x_1 \cdots x_n) \leq \operatorname{diam} B[x, r].$$

If $y \in B[x, r]$, then

$$\operatorname{diam} B[x, r] \geq d_w(x, y) = w(x \wedge y) = w(x_1 \cdots x_m)$$

for some $m \geq n \in \mathbb{N}$ by Definition 2.14 and the minimality of n. It follows that $y \in [x_1 \cdots x_n]$, so that $B[x, r] \subset [x_1 \cdots x_n]$. On the other hand, if $z \in [x_1 \cdots x_n]$ then

$$d_w(z, x) = w(z \wedge x) \leq w(x_1 \cdots x_n) \leq \operatorname{diam} B[x, r] \leq r.$$

so $z \in B[x, r]$ by construction, and hence $[x_1 \cdots x_n] \subset B[x, r]$. In other words, $B[x, r] = [x_1 \cdots x_n]$ as claimed, so cylinder sets of $X_{\mathcal{B}}$ and closed balls (which are open in the topology induced by the metric d_w) agree. (If $n = 0$ then we interpret $[x_1 \cdots x_n]$ as $[r(x)]$.) $\square$

2.4 Strongly connected higher-rank graphs

When Λ is a finite k-graph whose adjacency matrices satisfy some additional properties, there is a natural family $\{w_\delta\}_{0<\delta<1}$ of weights on the associated Bratteli diagram $\mathcal{B}_\Lambda$ which induce ultrametrics on the infinite path space $X_{\mathcal{B}_\Lambda}$. We describe these additional properties on Λ and the formula of the weights w_δ below.

Definition 2.16. A k-graph Λ is *strongly connected* if, for all $v, w \in \Lambda^0$, $v\Lambda w \neq \emptyset$.

In Lemma 4.1 of [40], an Huef et al. show that a finite k-graph Λ is strongly connected if and only if the adjacency matrices $A_1, \ldots, A_k$ of Λ form an *irreducible family of matrices*. Also, Proposition 3.1 of [40] implies that if Λ is a finite strongly connected k-graph, then there is a unique positive vector $x^\Lambda \in (0,\infty)^{\Lambda^0}$ such that $\sum_{v\in\Lambda^0} x^\Lambda_v = 1$ and for all $1 \le i \le k$,

$$A_i x^\Lambda = \rho_i x^\Lambda,$$

where ρ_i denotes the spectral radius of A_i. We call x^Λ the *Perron-Frobenius eigenvector* of Λ. Moreover, an Huef et al. constructed a Borel probability measure M on Λ^∞ in Proposition 8.1 of [40] when Λ is finite, strongly connected k-graph. The measure M on Λ^∞ is given by

$$M([\lambda]) = \rho(\Lambda)^{-d(\lambda)} x^\Lambda_{s(\lambda)} \quad \text{for } \lambda \in \Lambda, \tag{3}$$

where x^Λ is the Perron-Frobenius eigenvector of Λ and $\rho(\Lambda) = (\rho_1, \ldots, \rho_k)$, and for $n = (n_1, \ldots, n_k) \in \mathbb{N}^k$,

$$\rho(\Lambda)^n := \rho_1^{n_1} \cdots \rho_k^{n_k}.$$

We know from Remark 2.11 that every finite path $\lambda \in \mathcal{B}_\Lambda$ corresponds to a unique morphism in Λ. Using this correspondence and the homeomorphism $X_{\mathcal{B}_\Lambda} \cong \Lambda^\infty$ of Proposition 2.10, Equation (3) translates into the formula

$$M([\lambda]) = (\rho_1 \cdots \rho_t)^{-(q+1)} (\rho_{t+1} \cdots \rho_k)^{-q} x^\Lambda_{s(\lambda)} \tag{4}$$

for $[\lambda] \subseteq X_{\mathcal{B}_\Lambda}$, where $\lambda \in F\mathcal{B}_\Lambda$ with $|\lambda| = qk + t$ and x^Λ is the Perron-Frobenius eigenvector of Λ.

In the proof that follows, we rely heavily on the identification between Λ^∞ and $X_{\mathcal{B}_\Lambda}$ by Proposition 2.10 and Remark 2.11 (a). We also use the observation from Remark 2.11 that every finite path in $F\mathcal{B}_\Lambda$ corresponds to a unique finite path $\lambda \in \Lambda$.

Proposition 2.17. *Let Λ be a finite, strongly connected k-graph with adjacency matrices A_i. Then the infinite path space Λ^∞ is a Cantor set whenever $\prod_i \rho_i > 1$.*

Proof. We let $A = A_1 \ldots A_k$; it is a matrix whose entries are indexed by $\Lambda^0 \times \Lambda^0$, and its spectral radius is $\prod_i \rho_i$. We assume that Λ^∞ is not a Cantor set, and will prove that the spectral radius of A is at most 1, hence proving the Proposition.

Since Λ^∞ is compact Hausdorff and totally disconnected, but not a Cantor set, it has an isolated point x. We write $\{\gamma_n\}_{n\in\mathbb{N}}$ for the increasing sequence of finite paths in $\mathcal{B}_\Lambda$ which are sub-paths of x. If $n = \ell k + t$, then $|\gamma_n| = n$ and (thinking of γ_n as an element of Λ) $d(\gamma_n) = (\ell+1, \ldots, \ell+1, \ell, \ldots, \ell)$ with t occurrences of $\ell + 1$. Since x is an isolated point, there exists $N \in \mathbb{N}$ such that for all $n \ge N$, $[\gamma_n] = \{x\}$. Without loss of generality, we can assume that $N = dk$ is a multiple of k, so that $d(\gamma_N) = (d, \ldots, d)$. For $n \ge N$, we write $\gamma_n = \gamma_N \eta_n$, with $|\gamma_n| = n$ and $|\eta_n| = n - N = qk + t$, so that $d(\eta_n) = (q+1, \ldots, q+1, q, \ldots, q)$, with t occurrences of $q + 1$.

By Proposition 2.4, our hypothesis that x is an isolated point implies that for all $n \ge N$, η_n is the unique path of degree $d(\eta_n)$ whose range is $s(\gamma_N) = r(\eta_n)$. This, in turn, implies that for all $n \ge N$, we have $A^q A_1 \dots A_t(r(\eta_n), z)$ equal to 1 for a single z, and 0 otherwise. In other words, if we consider the column vector δ_v which is 1 at the vertex v and 0 else, we have that

$$\left(\delta_{r(\eta_n)}\right)^T \cdot A^q A_1 \dots A_t = \left(\delta_{s(\eta_n)}\right)^T.$$

Note that for each $n \geq N$ with $n - N = qk + t$, $s(\eta_{n+1})$ is the label of the only non-zero entry in row $s(\eta_n)$ of the matrix A_t. Since each entry in the sequence $(s(\eta_n))_{n\in\mathbb{N}}$ is completely determined by a finite set of inputs – namely, the previous entry in the sequence, and the entries of the matrices A_t – and the set Λ^0 of vertices is finite, the sequence $(s(\eta_n))_{n\in\mathbb{N}}$ is eventually periodic. Let p be a period for this sequence. Then kp is also a period, so there exists J such that for all $n \geq J$ we have

$$(A^p)^T \delta_{s(\eta_n)} = \delta_{s(\eta_n)}.$$

If we average along one period and define

$$\vec{v} = \frac{1}{kp} \sum_{j=J+1}^{J+kp} \delta_{s(\eta_j)},$$

then we can compute that

$$A^T \vec{v} = \frac{1}{kp} \sum_{j=J+1}^{J+kp} \delta_{s(\eta_j)} = \vec{v},$$

so $\vec{v}$ is an eigenvector of A^T with eigenvalue 1, with non-negative entries.

Since Λ is strongly connected by hypothesis, Lemma 4.1 of [40] implies that there exists a matrix A_F which is a finite sum of finite products of the matrices A_i and which has positive entries. This matrix A_F commutes with A, and therefore

$$A^T A_F^T \vec{v} = A_F^T A^T \vec{v} = A_F^T \vec{v},$$

and so $\vec{u} := A_F^T \vec{v}$ is an eigenvector of A^T with eigenvalue 1. Since A_F is positive and $\vec{v}$ is non-negative, $\vec{u}$ is positive. Therefore, we can apply Lemma 3.2 of [40] and conclude that $\prod_i \rho_i = \rho(A) \leq 1$. □

Remark 2.18. The proof of Proposition 2.17 simplifies considerably if we add the hypothesis that each row sum of each adjacency matrix A_i is at least 2. In this case, any finite path γ in the Bratteli diagram has at least two extensions γe and γf. In terms of neighbourhoods, this means that each clopen set $[\gamma]$ contains at least two disjoint non-trivial sets $[\gamma e], [\gamma f]$. It is therefore impossible to have a cylinder set $[\gamma]$ consist of a single point. Therefore, there is no isolated point in $X_{\mathcal{B}_\Lambda}$, and the path space is a Cantor set.

The next Proposition constructs, for any $\delta \in (0, 1)$, a weight w_δ on the stationary k-Bratteli diagram $\mathcal{B}_\Lambda$ of any k-graph Λ which satisfies certain mild hypotheses. In Section 3 below, we will examine the Pearson-Bellissard spectral triples associated to the ultrametric Cantor sets $(X_{\mathcal{B}_\Lambda}, d_{w_\delta})$ and in particular the relationship between the parameter δ and various properties of the spectral triple. For example, Corollary 3.15 establishes that the spectral triple associated to $(X_{\mathcal{B}_\Lambda}, d_{w_\delta})$ has spectral dimension δ, while Theorem 3.26 shows that the measure on $X_{\mathcal{B}_\Lambda}$ induced by the spectral triple is independent of δ.

Proposition 2.19. *Let Λ be a finite, strongly connected k-graph with adjacency matrices A_i. For $\eta \in F\mathcal{B}_\Lambda$ with $|\eta| = n \in \mathbb{N}$, write $n = qk + t$ for some $q, t \in \mathbb{N}$ with $0 \leq t \leq k-1$. For each $\delta \in (0, 1)$, define $w_\delta : F\mathcal{B}_\Lambda \to \mathbb{R}^+$ by*

$$w_\delta(\eta) = \left(\rho_1^{q+1} \cdots \rho_t^{q+1} \rho_{t+1}^q \cdots \rho_k^q\right)^{-1/\delta} x^\Lambda_{s(\eta)}, \tag{5}$$

where x^Λ is the unimodular Perron-Frobenius eigenvector for Λ. If the spectral radius ρ_i of A_i satisfies $\rho_i > 1\ \forall\ i$, then w_δ is a weight on $\mathcal{B}_\Lambda$.

Proof. Recall that $x^\Lambda \in (0,\infty)^{\Lambda^0}$, $\sum_{v\in\Lambda^0} x^\Lambda_v = 1$ and $A_i x^\Lambda = \rho_i x^\Lambda$ for all $1 \le i \le k$; thus,

$$\sum_{v\in\mathcal{V}_0} w_\delta(v) = \sum_{v\in\mathcal{V}_0} x^\Lambda_v = 1,$$

and the first condition of Definition 2.14 is satisfied. Since $\rho_i > 1$ for all i and $0 < \delta < 1$,

$$\lim_{q\to\infty}(\rho_i^q)^{-1/\delta} = \lim_{q\to\infty}\left(\frac{1}{\rho_i^{1/\delta}}\right)^q = 0.$$

Thus the second condition of Definition 2.14 holds. To see the third condition, we observe that it is enough to show that $w_\delta(\lambda) > w_\delta(\lambda f)$ for any edge f with $s(\lambda) = r(f)$. Note that if $|\lambda| = qk + j$ for $q \in \mathbb{N}$ and $0 \le j \le k-1$, so that $s(\lambda) \in \mathcal{V}_{qk+j}$, then

$$\begin{aligned}
\sum_{\substack{f : r(f)=s(\lambda)\\ d(f)=e_{j+1}}} w_\delta(\lambda f) &= \big((\rho_1\cdots\rho_k)^q\rho_1\dots\rho_{j+1}\big)^{-1/\delta}\sum_{v\in\Lambda^0} A_{j+1}(s(\lambda)),v)x^\Lambda_v\\
&= \big((\rho_1\cdots\rho_k)^q\rho_1\dots\rho_j\big)^{-1/\delta}\rho_{j+1}^{-1/\delta}\rho_{j+1}x^\Lambda_{s(\lambda)}\\
&< w_\delta(\lambda).
\end{aligned}$$

Here the second equality follows since x^Λ is an eigenvector for A_{j+1} with eigenvalue ρ_{j+1}, and the final inequality holds because $\rho_{j+1} > 1$ and $1/\delta > 1$, and consequently

$$\rho_{j+1}^{1-1/\delta} = \frac{1}{\rho_{j+1}^{1/\delta-1}} < 1.$$

□

Our primary application for the results of this section is the following.

Corollary 2.20. *Let Λ be a finite, strongly connected k-graph with adjacency matrices A_i and let ρ_i be the spectral radius for A_i, $1 \le i \le k$. Suppose that $\rho_i > 1$ for all $1 \le i \le k$. Let $(\mathcal{B}_\Lambda, w_\delta)$ be the associated weighted stationary k-Bratteli diagram given in Proposition 2.19. Then the infinite path space $X_{\mathcal{B}_\Lambda}$ is an ultrametric Cantor set with the metric d_{w_δ} induced by the weight w_δ.*

Proof. Combine Proposition 2.19, Proposition 2.17, and Proposition 2.15. □

3 Spectral triples for ultrametric higher-rank graph Cantor sets

Proposition 8 of [59] (also see Proposition 3.1 of [42]) gives a recipe for constructing an even spectral triple for any ultrametric Cantor set induced by a weighted tree. We begin this section by explaining how this construction works in the case of the ultrametric Cantor sets which we associated to a finite strongly connected k-graph in the previous section. Section 3.1 recalls basic facts about spectral triples, and Section 3.2 investigates the ζ-function of the spectral triples coming from the ultrametric Cantor sets that arise from k-graphs. Finally, Section 3.3 uses the theory of Dixmier traces to construct measures on $X_{\mathcal{B}_\Lambda}$ from these spectral triples. We also derive an integral formula for the Dixmier trace in this section.

To be precise, consider the Cantor set $\Lambda^\infty \cong X_{\mathcal{B}_\Lambda}$ with the ultrametric induced by the weight w_δ of Equation (5). (Because of Proposition 2.10, we will identify the infinite path spaces of Λ and of $\mathcal{B}_\Lambda$, and use either Λ^∞ or $X_{\mathcal{B}_\Lambda}$ to denote this space, depending on the context.) Under additional (but mild) hypotheses, Theorem 3.14 establishes that the ζ-function of the associated spectral triple has abscissa of convergence δ, and thus is finitely summable with dimension δ. After proving in Proposition 3.22 that the Dixmier trace of the spectral triple induces a well-defined measure μ_δ on $X_{\mathcal{B}_\Lambda}$, Theorem 3.26 establishes that the normalization ν_δ of μ_δ agrees with the measure M introduced in [40] and used in [28] to construct a wavelet decomposition of $L^2(\Lambda^\infty, M)$, and is therefore independent of δ. Finally, Theorems 3.23 and 3.28 establish a Dixmier trace integral formula; the computations underlying these proofs also establish that the ultrametric Cantor set $(X_{\mathcal{B}_\Lambda}, d_\delta)$ is ζ-regular in the sense of [59].

Analogues of Theorem 3.14 and Proposition 3.22 were proved in Section 3 of [42] for stationary Bratteli diagrams (equivalently, directed graphs) with primitive adjacency matrices. However, even for directed graphs our results in this section are stronger than those of [42], since in this setting, our hypotheses are equivalent to saying that the adjacency matrix is merely irreducible.

A crucial hypothesis for the main results in this section is the following Hypothesis 3.1, which will be a standing hypothesis throughout the paper. Lemma 3.2 below identifies conditions under which the weights w_δ of Equation (5) satisfy Hypothesis 3.1. To state this hypothesis, recall that for any Bratteli diagram $(\mathcal{B}, w)$ and $\lambda \in F\mathcal{B}$,

$$\operatorname{diam}[\lambda] = \sup\{d_w(x, y) \mid x, y \in [\lambda]\}. \tag{6}$$

Hypothesis 3.1. The weight w of a weighted Bratteli diagram $(\mathcal{B}, w)$ satisfies

$$w(\lambda) = \operatorname{diam}[\lambda] \quad \text{for all } \lambda \in F\mathcal{B}. \tag{7}$$

Lemma 3.2. *Let $\mathcal{B} = \mathcal{B}_\Lambda$ for a finite, strongly connected k-graph Λ with no sources. Hypothesis 3.1 holds for the weights w_δ of Equation* (5) *if and only if every vertex $a \in \Lambda^0$ receives at least two edges of each color, i.e. $\sum_{b\in\Lambda^0} A_i(a, b) \geq 2$ for all $a \in \Lambda^0$ and $1 \leq i \leq k$.*

Proof. Recall that, by definition of d_{w_δ} and the third condition of Definition 2.14,

$$\operatorname{diam}[\lambda] = \max\{d_{w_\delta}(x, y) : x, y \in [\lambda]\} = \max\{w_\delta(x \wedge y) : x, y \in [\lambda]\} \leq w_\delta(\lambda).$$

Moreover, the hypothesis that Λ be source-free forces each vertex a to receive at least one edge of each color.

Suppose, then, that every vertex $a \in \Lambda^0$ receives at least two edges e_a, f_a of each color. Then for any $\lambda \in F\mathcal{B}_\Lambda$ with $s(\lambda) = a$, there are then two infinite paths $x = \lambda e_a \cdots, y = \lambda f_a \cdots$ in $[\lambda]$ such that $d_{w_\delta}(x, y) = w_\delta(x \wedge y) = w_\delta(\lambda)$. Conversely, if there is a vertex a and a color i such that there is only one edge e of color i and range a, then for any $x, y \in [\lambda]$ we have $x \wedge y = \lambda e$ and hence

$$w_\delta(\lambda) > w_\delta(\lambda e) \geq \operatorname{diam}[\lambda]. \qquad \square$$

Remark 3.3. Recall that the spectral radius of a non-negative matrix is at least the minimum of its row sums. It follows that if $(\mathcal{B}_\Lambda, w_\delta)$ satisfies Hypothesis 3.1, then $\rho_i \geq 2 > 1$ for all $1 \leq i \leq k$, and hence $\rho = \rho_1 \ldots \rho_k > 1$. Therefore, the function w_δ given in Equation (5) is automatically a weight when it satisfies Equation (7) (and hence Hypothesis 3.1). In this setting, w_δ also gives rise to an ultrametric Cantor set $(X_{\mathcal{B}_\Lambda}, d_{w_\delta})$ by Corollary 2.20.

3.1 A review of spectral triples on Cantor sets and and the associated ζ-functions

We begin by recalling the definitions of a pre-C^*-algebra and of a spectral triple we use in our paper; see [19], [33, Chapter 10].

Definition 3.4. ([19, Section IV γ]) A *pre-C^*-algebra* of a C^*-algebra A is a $*$-subalgebra $\mathcal{A}$ of A, which is stable under the holomorphic functional calculus of A.

Pre-C^*-algebras are called local C^*-algebras in [7]. By [59, page 450], the $*$-algebra $C_{\mathrm{Lip}}(X_\mathcal{B}) \subseteq C(X_\mathcal{B})$ of Lipschitz continuous functions on $(X_\mathcal{B}, d_w)$ is a pre-C^*-algebra of the C^*-algebra $C(X_\mathcal{B})$.

Definition 3.5. (cf. [33, Definition 9.16], [59, Definition 9]) A *spectral triple* is a triple $(\mathcal{A}, \mathcal{H}, D)$ consisting of:

- a pre-C^*-algebra $\mathcal{A} \subseteq A$ (with $\mathcal{A}$ and A unital) equipped with a faithful $*$-representation π of $\mathcal{A}$ by bounded operators on a Hilbert space $\mathcal{H}$; and
- a selfadjoint operator D on $\mathcal{H}$, with dense domain $Dom\ D \subseteq \mathcal{H}$, such that

  $$a\,(Dom\ D) \subseteq Dom\ D,\ \forall a \in \mathcal{A};$$

  the operator $[D, a]$, defined initially on $Dom\ D$, extends to a bounded operator on $\mathcal{H}$ for all $a \in \mathcal{A}$; and D has compact resolvent.

A spectral triple is *even* if it has an associated grading operator $\Gamma : \mathcal{H} \to \mathcal{H}$ satisfying:

$$\Gamma^* = \Gamma;\ \ \Gamma^2 = 1;\ \ \Gamma D = -D\Gamma;\ \ \Gamma\pi(a) = \pi(a)\Gamma,\ \forall a \in \mathcal{A}.$$

We now review the construction of the spectral triple associated to an ultrametric Cantor set from [59] (see also Section 3 of [42]).

Definition 3.6. Let $(\mathcal{B}, w)$ be a weighted Bratteli diagram satisfying Hypothesis 3.1 with $X_\mathcal{B}$ a Cantor set. Let $(X_\mathcal{B}, d_w)$ be the associated ultrametric Cantor space. A *choice function* for $(X_\mathcal{B}, d_w)$ is a map $\tau : F\mathcal{B} \to X_\mathcal{B} \times X_\mathcal{B}$ such that $\tau(\gamma) = (\tau_+(\gamma), \tau_-(\gamma)) \in [\gamma] \times [\gamma]$ and $d_w(\tau_+(\gamma), \tau_-(\gamma)) = \operatorname{diam}[\gamma]$. We denote by Υ the set of choice functions for $(X_\mathcal{B}, d_w)$. Note that Υ is nonempty whenever $X_\mathcal{B}$ is a Cantor set, because Condition (3) of Proposition 2.4 implies that for every finite path γ of $\mathcal{B}$ we can find two distinct infinite paths $x, y \in [\gamma]$ with $x \wedge y = \gamma$.

As in [59, 42], let $C_{\mathrm{Lip}}(X_\mathcal{B})$ be the pre-C^*-algebra of Lipschitz continuous functions on $(X_\mathcal{B}, d_w)$ and let $\mathcal{H} = \ell^2(F\mathcal{B}, \mathbb{C}^2)$. For $\tau \in \Upsilon$, we define a faithful $*$-representation π_τ of $C_{\mathrm{Lip}}(X_\mathcal{B})$ on $\mathcal{H}$ by

$$\pi_\tau(f) = \bigoplus_{\gamma \in F\mathcal{B}} \begin{pmatrix} f(\tau_+(\gamma)) & 0 \\ 0 & f(\tau_-(\gamma)) \end{pmatrix}. \tag{8}$$

A Dirac operator D on $\mathcal{H}$ is given by

$$D = \bigoplus_{\gamma \in F\mathcal{B}} \frac{1}{\operatorname{diam}[\gamma]} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and the grading operator Γ is given by

$$\Gamma = 1_{\ell^2(F\mathcal{B})} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The following results were established by Pearson and Bellissard [59].

Proposition 3.7. *[59, Proposition 8] Let $(\mathcal{B}, w)$ be a weighted Bratteli diagram with $X_{\mathcal{B}}$ a Cantor set, satisfying Hypothesis 3.1. Then $(C_{Lip}(X_{\mathcal{B}}), \ell^2(F\mathcal{B}, \mathbb{C}^2), \pi_\tau, D, \Gamma)$ is an even spectral triple for all $\tau \in \Upsilon$.*

Lemma 3.8. *[59, Section 6.1] $|D|$ is invertible. In particular $|D|^{-1}\psi(\gamma) = \mathrm{diam}[\gamma]\,\psi(\gamma)$, for every $\psi \in \ell^2(F\mathcal{B}, \mathbb{C}^2)$ and every finite path $\gamma \in F\mathcal{B}$.*

It follows that $\{\delta_\lambda \otimes e_i : \lambda \in F\mathcal{B},\ i = 1, 2\}$ is an orthonormal basis of $\ell^2(F\mathcal{B}, \mathbb{C}^2) \cong \ell^2(F\mathcal{B}) \otimes \mathbb{C}^2$ which consists of eigenvectors for $|D|^{-1}$, where $\{e_1, e_2\}$ is the standard orthonormal basis of $\mathbb{C}^2$. Moreover, since $|D|$ is invertible, we can replace the operator $< D >^{-1} := (1 + D^2)^{-1/2}$, appearing commonly in the noncommutative geometry literature, by $|D|^{-1}$.

Definition 3.9. [59], [4, Section 9.6] To any positive operator with discrete spectrum P, we can associate a *ζ-function* ζ_P which is defined on $\{s \in \mathbb{R} : s >> 0\}$ by

$$\zeta_P(s) := \mathrm{Tr}\,(P^s) = \sum_n \lambda(n, P)^s.$$

It now follows that the standard ζ-function associated to the spectral triple $(C_{\mathrm{Lip}}(X_{\mathcal{B}}), \mathcal{H}, \pi_\tau, D, \Gamma)$ can be described as follows.

Definition 3.10. [59, Section 6.1] The *ζ-function* associated to the Pearson-Bellissard spectral triple $(C_{\mathrm{Lip}}(X_{\mathcal{B}}), \mathcal{H}, \pi_\tau, D, \Gamma)$ is given by

$$\zeta_w(s) := \frac{1}{2}\mathrm{Tr}\,(|D|^{-s}) = \sum_{\lambda \in F\mathcal{B}} \mathrm{diam}[\lambda]^s = \sum_{\lambda \in F\mathcal{B}} w(\lambda)^s, \quad \text{for } s>>0. \tag{9}$$

The above ζ-function ζ_w is a Dirichlet series since $|D|^{-1}$ is compact with a decreasing sequence of eigenvalues (equal to the diameters, or weights, of the finite paths) by Lemma 3.8. Thus, by [36, Chapter 2], ζ_w extends to a meromorphic function on $\mathbb{C}$ which either converges everywhere, nowhere, or in the complex half plane $s = \mathrm{Re}(z) > s_0$ for some s_0. In this last case we will call s_0 the *abscissa of convergence* of ζ_w. In other words, s_0 is the infimum of $s > 0$ such that $\zeta_w(z)$ converges for $\mathrm{Re}(z) > s$.

To determine the abscissa of convergence of the ζ-function ζ_w, it suffices to evaluate ζ_w at points $s \in \mathbb{R}$. Since we are primarily interested in the abscissa of convergence of ζ_w, throughout this article, we will only consider real arguments for ζ_w.

Remark 3.11. The factor $\frac{1}{2}$ in Equation (9) is non-standard, but is frequently used for Pearson-Bellissard spectral triples (cf. [59, 42]). Using the factor $\frac{1}{2}$ ensures that $\zeta_\delta(s)$ equals exactly the sum of the weights to the power s. However, this rescaling has no effect on the dimension or summability of the spectral triple (see Definition 3.12 below).

We also note that Theorem 3.14 below establishes that, in our case of interest (namely when $\mathcal{B} = \mathcal{B}_\Lambda$ for a k-graph Λ satisfying Hypothesis 3.1, and $w = w_\delta$ for $\delta \in (0, 1)$) the ζ-function $\zeta_{w_\delta}(s)$ converges for $s > \delta$.

Definition 3.12. If there exists $p > 0$ such that $\zeta_w(p) < \infty$, then the spectral triple $(C_{\mathrm{Lip}}(X_{\mathcal{B}}), \mathcal{H}, \pi_\tau, D, \Gamma)$ is *p-summable*. The spectral triple is *finitely summable* if p-summable for some $p > 0$. The *dimension* of the spectral triple is $\inf\{p : \zeta_w(p) < \infty\}$.

3.2 Finite summability for the Pearson-Bellissard spectral triples of k-graphs

From now on we will focus on Pearson-Bellisard spectral triples of the form $(C_{\text{Lip}}(X_{\mathcal{B}_\Lambda}), \mathcal{H}, \pi_\tau, D, \Gamma)$ associated to the weighted stationary k-Bratteli diagram $(\mathcal{B}_\Lambda, w_\delta)$ of a k-graph, with weight w_δ as in Equation (5) of Proposition 2.19 above. In this case, the set of choice functions will be called Υ_Λ. In particular we will show in Theorem 3.14 that the dimension of $(C_{\text{Lip}}(X_{\mathcal{B}_\Lambda}), \mathcal{H}, \pi_\tau, D, \Gamma)$ is δ, which coincides with the abscissa of convergence of ζ_{w_δ}.

Before developing our theory further, we will present a simple example.

Example 3.13. Let Λ_2 be the 2-graph with one vertex amd two loops of each color, respectively e_j and f_j, with $j = 1, 2$, and with factorization relations

$$e_i f_j = f_i e_j, \quad \forall i, j.$$

By [30, Section 5.1], every infinite path $\omega \in \Lambda_2^\infty$ has a unique representative of the form

$$e_{i_1} f_{j_1} e_{i_2} f_{j_2} \dots e_{i_k} f_{j_k} \dots .$$

Therefore Λ_2^∞ is in bijection with $\prod_{\mathbb{N}}\{0, 1\}$. The vertex matrices of this 2-graph are $A_1 = (2)$, $A_2 = (2)$, and therefore their spectral radii are 2, with Perron-Frobenius eigenvector equal to 1. The weights of Equation (5) of Proposition 2.19 are consequently given by

$$w_\delta(\eta) = 2^{-\frac{n}{\delta}}, \qquad \text{where } \eta = e_{r_1} f_{r_2} e_{r_3} f_{r_4} \dots e_{r_n} \text{ or } \eta = e_{r_1} f_{r_2} e_{r_3} f_{r_4} \dots e_{r_{n-1}} f_{r_n},$$

Since there are 2^n paths of length n in $F\mathcal{B}_\Lambda$, the zeta function ζ_{w_δ} is given by

$$\zeta_{w_\delta}(s) = \sum_{n \geq 0} \left(\frac{1}{2}\right)^{\frac{sn}{\delta}} 2^n.$$

Fix a weighted stationary k-Bratteli diagram $(\mathcal{B}_\Lambda, w_\delta)$ with weights as in Equation (5) of Proposition 2.19. For this fixed choice of weights, we will write d_δ for the ultrametric d_{w_δ}, and ζ_δ for the ζ-function ζ_{w_δ} associated to $(C_{\text{Lip}}(X_{\mathcal{B}_\Lambda}), \mathcal{H}, \pi_\tau, D, \Gamma)$.

We now show that the dimension of $(C_{\text{Lip}}(X_{\mathcal{B}_\Lambda}), \mathcal{H}, \pi_\tau, D, \Gamma)$ is δ, which coincides with the abscissa of convergence of ζ_δ.

Theorem 3.14. *Let Λ be a finite, strongly connected k-graph. Fix $\delta \in (0, 1)$ and suppose that Equation* (7) *holds for the weight w_δ of Equation* (5)*. Then the zeta function $\zeta_\delta(s)$ has abscissa of convergence δ. Moreover,* $\lim_{s \searrow \delta} \zeta_\delta(s) = \infty$*. In particular, $(C_{Lip}(X_{\mathcal{B}_\Lambda}), \mathcal{H}, \pi_\tau, D, \Gamma)$ is always finitely summable.*

Proof. In order to explicitly compute $\zeta_\delta(s)$, we first observe that we can rewrite

$$\zeta_\delta(s) = \sum_{\lambda \in F\mathcal{B}_\Lambda} w_\delta(\lambda)^s = \sum_{n \in \mathbb{N}} \sum_{\lambda \in F^n \mathcal{B}_\Lambda} w_\delta(\lambda)^s = \sum_{q \in \mathbb{N}} \sum_{t=0}^{k-1} \sum_{\lambda \in F^{qk+t} \mathcal{B}_\Lambda} w_\delta(\lambda)^s, \tag{10}$$

where $F^n(\mathcal{B}_\Lambda)$ is the set of finite paths of $\mathcal{B}_\Lambda$ with length n. Now, write $A := A_1 \cdots A_k$ for the product of the adjacency matrices of Λ. If $t \in \{0, 1, \dots, k-1\}$ is fixed and $n = qk + t$, then the number of paths in $F^n(\mathcal{B}_\Lambda)$ with source vertex b and range vertex a is given by $A^q A_1 \cdots A_t(a, b)$. Thus, writing $\rho := \rho_1 \cdots \rho_k$ for the spectral radius of A, the formula for w_δ given in Equation (5) implies that

$$\zeta_\delta(s) = \sum_{t=0}^{k-1} \frac{1}{(\rho_1 \cdots \rho_t)^{s/\delta}} \sum_{q \in \mathbb{N}} \sum_{a, b \in \mathcal{V}_0} A^q A_1 \cdots A_t(a, b) \frac{(x_b^\Lambda)^s}{\rho^{qs/\delta}}. \tag{11}$$

Since all terms in this sum are non-negative, the series $\zeta_\delta(s)$ converges iff it converges absolutely; hence, rearranging the terms in the sum does not affect the convergence of $\zeta_\delta(s)$. Thus, we can rewrite

$$\zeta_\delta(s) = \sum_{t=0}^{k-1} \sum_{a,b,z\in\mathcal{V}_0} \frac{A_1 \cdots A_t(z,b)}{(\rho_1 \cdots \rho_t)^{s/\delta}} (x_b^\Lambda)^s \sum_{q\in\mathbb{N}} \frac{A^q(a,z)}{\rho^{qs/\delta}}. \tag{12}$$

In order to show that $\zeta_\delta(s)$ converges for $s > \delta$, we begin by considering the sum $\sum_{q\in\mathbb{N}} \frac{A^q(a,z)}{(\rho^{s/\delta})^q}$. Since A has a positive right eigenvector of eigenvalue ρ (namely x^Λ), Corollary 8.1.33 of [37] implies that

$$\frac{A^q(a,z)}{\rho^q} \le \frac{\max\{x_b^\Lambda\}_{b\in\mathcal{V}_0}}{\min\{x_b^\Lambda\}_{b\in\mathcal{V}_0}} \;\forall\, q \in \mathbb{N}\backslash\{0\}.$$

Consequently,

$$\sum_{q\in\mathbb{N}} \frac{A^q(a,z)}{\rho^q \rho^{(s/\delta-1)q}} \le \delta_{a,z} + \frac{\max\{x_b^\Lambda\}_{b\in\mathcal{V}_0}}{\min\{x_b^\Lambda\}_{b\in\mathcal{V}_0}} \sum_{q\ge 1} \frac{1}{\rho^{(s/\delta-1)q}}.$$

If $s > \delta$, then our hypothesis that $\rho > 1$ implies that $1/\rho^{(s/\delta-1)} \in (0,1)$, and thus $\sum_{q\ge1} \rho^{(1-s/\delta)q}$ converges to $(1-\rho^{(1-s/\delta)})^{-1} - 1$. Consequently,

$$\sum_{q\in\mathbb{N}} \frac{A^q(a,z)}{(\rho^{s/\delta})^q} < \infty,$$

and hence $\zeta_\delta(s) < \infty$, for any $s > \delta$ since $\mathcal{V}_0$ is a finite set.

To see that $\zeta_\delta(s) = \infty$ whenever $s \le \delta$, we have to work harder. Theorem 8.3.5 part(b) of [37] implies that the Jordan canonical form of A is

$$J = \begin{pmatrix}
\rho & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \ddots & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \rho & 0 & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \omega_1\rho & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & \ddots & 0 & \dots & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \omega_1\rho & 0 & \dots & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \omega_2\rho & 0 & \dots & 0 & 0 & 0 & 0\\
\vdots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \dots & \dots & \dots & \dots & \vdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \omega_{p-1}\rho & 0 & \dots & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & J_{p+1} & 0 & 0 & 0\\
\vdots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \ddots & \dots & \vdots\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & J_{m-1} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \dots & \dots & \dots & 0 & 0 & J_m
\end{pmatrix},$$

where p is the period of A, ω_i is a pth root of unity for each i, each eigenvalue $\omega_i\rho$ is repeated along the diagonal m_i times, and J_i, $i = p+1, \dots, m$ are Jordan blocks – that is, upper triangular matrices whose constant diagonal is given by an eigenvalue α_i of A (with $|\alpha_i| < \rho$) and which have a superdiagonal of 1s as the only other nonzero entries. Thus, for each $1 \le a, b \le |\mathcal{V}_0|$,

$$J^q(a,b) \in \{0\} \cup \{\rho^q\} \cup \{\rho^q\omega_i^q \,:\, 1 \le i \le p-1\} \cup \left\{ \frac{1}{\alpha_i^\ell}\binom{q}{\ell}\alpha_i^q \,:\, 0 \le \ell \le \dim J_i \right\}. \tag{13}$$

Consequently,

$$\left|\frac{1}{\rho^q}J^q(a,b)\right| \in \{0,1\} \cup \left\{\beta_i \frac{1}{|\alpha_i^\ell|}\binom{q}{\ell} \,:\, \beta_i = \frac{|\alpha_i|}{\rho} < 1,\ 0 \le \ell \le \dim J_i\right\}.$$

Thanks to [63] and [5, Chapter 2], we know that since A has a positive eigenvector (namely x^Λ) of eigenvalue ρ, $\lim_{\ell\to\infty} \frac{1}{\rho^{\ell p+j}} A^{\ell p+j}$ exists for all $0 \le j \le p-1$, where p denotes the period of A. Moreover, if we write

$$A^{(j)} = \lim_{\ell\to\infty} \frac{1}{\rho^{\ell p+j}} A^{\ell p+j} \tag{14}$$

for this limit, and τ for the maximum modulus of the eigenvalues α_i of A with $|\alpha_i| < \rho$,

$$\forall \left(\frac{\tau}{\rho}\right)^p < \beta < 1,\ \exists\, M_{\beta,j} \in \mathbb{R}^+ \text{ s.t. } \forall\, m \in \mathbb{N},\ \left|\frac{A^{mp+j}(a,b)}{\rho^{mp+j}} - A^{(j)}(a,b)\right| \le M_{\beta,j}\beta^m.$$

Thus, for all $\ell \in \mathbb{N}$ and all $0 \le j \le p-1$, and all such β,

$$\frac{A^{\ell p+j}(a,b)}{\rho^{\ell p+j}} \ge A^{(j)}(a,b) - M_{\beta,j}\beta^\ell \qquad \text{for all } \ell \in \mathbb{N}. \tag{15}$$

Reordering the summands of $\sum_{q\in\mathbb{N}} A^q(a,b)(\rho^{-s/\delta})^q$, we see that

$$\sum_{q\in\mathbb{N}} A^q(a,b)(\rho^{-s/\delta})^q = \sum_{j=0}^{p-1}\sum_{\ell\in\mathbb{N}} A^{\ell p+j}(a,b)(\rho^{-s/\delta})^{\ell p+j}.$$

Now, fix $j \in \{0,\dots,p-1\}$ and consider the sum

$$\begin{aligned}\sum_{\ell\in\mathbb{N}} A^{\ell p+j}(a,b)(\rho^{-s/\delta})^{\ell p+j} &= \sum_{\ell\in\mathbb{N}} \frac{A^{\ell p+j}(a,b)}{\rho^{\ell p+j}}\left(\frac{1}{\rho^{s/\delta-1}}\right)^{\ell p+j} \\ &\ge \frac{1}{\rho^{(s/\delta-1)j}} \sum_{\ell\in\mathbb{N}} (A^{(j)}(a,b) - M_{\beta,j}\beta^\ell)\left(\frac{1}{\rho^{s/\delta-1}}\right)^{p\ell}.\end{aligned}$$

If $A^{(j)}(a,b) > 0$, the fact that $\beta < 1$ and $M_{\beta,j} > 0$ implies that there exists M such that for $\ell > M$, $A^{(j)}(a,b) > M_{\beta,j}\beta^\ell$. Consequently, if we define

$$K = \frac{1}{\rho^{(s/\delta-1)j}} \sum_{\ell=0}^{M} \frac{A^{(j)}(a,b) - M_{\beta,j}\beta^\ell}{\rho^{(s/\delta-1)p\ell}},$$

and write $\nu = A^{(j)}(a,b) - M_{\beta,j}\beta^M > 0$, the fact that $\{M_{\beta,j}\beta^\ell\}_{\ell\in\mathbb{N}}$ is a decreasing sequence implies that

$$\sum_{\ell\in\mathbb{N}} A^{\ell p+j}(a,b)(\rho^{-s/\delta})^{\ell p+j} > K + \frac{\nu}{\rho^{(s/\delta-1)j}} \sum_{\ell>M}\left(\frac{1}{\rho^{s/\delta-1}}\right)^{p\ell}. \tag{16}$$

Since $\rho > 1$ and $s \le \delta$, $\rho^{(1-s/\delta)p} \ge 1$; consequently, the series $\sum_{\ell>M}(\rho^{(1-s/\delta)p})^\ell$ diverges to infinity. The fact that K, ν are finite now implies that $\sum_{\ell\in\mathbb{N}} A^{mp+j}(a,b)(\rho^{-s/\delta})^{\ell p+j}$ also diverges to infinity if $A^{(j)}(a,b) > 0$.

Inequality (16) above also shows that we must have $\lim_{s \searrow \delta} \zeta_\delta(s) = \infty$. All terms are non-negative on both sides of this inequality, and Fatou's Lemma for series applied to the right-hand side of (16) shows that

$$\lim_{s \searrow \delta} \frac{\nu}{\rho^{(s/\delta-1)j}} \sum_{\ell > M} \left(\frac{1}{\rho^{s/\delta - 1}} \right)^{p\ell} \geq \frac{\nu}{\rho^{(\delta/\delta-1)j}} \sum_{\ell > M} \left(\frac{1}{\rho^{\delta/\delta-1}} \right)^{p\ell} = \nu \cdot \sum_{\ell > M} \left(\frac{1}{1} \right)^{p\ell} = +\infty. \tag{17}$$

Now, we show that for each j, there must exist some $(a, b) \in \mathcal{V}_0$ such that $A^{(j)}(a, b) > 0$. Recall that x^Λ is an eigenvector for A, and consequently for $A^{\ell p + j}$. Thus,

$$\sum_{b \in \mathcal{V}_0} A^{\ell p+j}(a, b) x_b^\Lambda = \rho^{\ell p+j} x_a^\Lambda.$$

Since x^Λ is a positive eigenvector, there exists $\alpha > 0$ such that $x_a^\Lambda > \alpha$ for all $a \in \mathcal{V}_0$. Moreover, x^Λ is a unimodular eigenvector, so $0 < x_b^\Lambda \leq 1$ for all $b \in \mathcal{V}_0$. Thus the above equation becomes

$$\rho^{\ell p+j} \alpha < \rho^{\ell p+j} x_a^\Lambda = \sum_{b \in \mathcal{V}_0} A^{\ell p+j}(a, b) x_b^\Lambda \leq \sum_{b \in \mathcal{V}_0} A^{\ell p+j}(a, b).$$

Consequently, for each $a \in \mathcal{V}_0$ and each $\ell \in \mathbb{N}$ there exists at least one vertex b such that

$$\frac{A^{\ell p+j}(a, b)}{\rho^{\ell p+j}} > \frac{\alpha}{\#(\mathcal{V}_0)}.$$

Moreover, since $\#(\mathcal{V}_0) < \infty$, the definition of the limit $A^{(j)}$ implies that there exists $N \in \mathbb{N}$ such that whenever $\ell \geq N$ we have

$$A^{(j)}(a, b) > \frac{A^{\ell p+j}(a, b)}{\rho^{\ell p+j}} - \frac{\alpha}{2\#(\mathcal{V}_0)} \ \forall a, b \in \mathcal{V}_0.$$

Now, fix a and $\ell \geq N$. Choose $b \in \mathcal{V}_0$ such that $\frac{A^{\ell p+j}(a,b)}{\rho^{\ell p+j}} > \frac{\alpha}{\#(\mathcal{V}_0)}$. It then follows that for this choice of b,

$$A^{(j)}(a, b) > \frac{A^{\ell p+j}(a, b)}{\rho^{\ell p+j}} - \frac{\alpha}{2\#(\mathcal{V}_0)} > \frac{\alpha}{2\#(\mathcal{V}_0)}.$$

In other words, we have proved that

$$\forall\, 1 \leq j \leq p,\ \forall\, a \in \mathcal{V}_0,\ \exists\, b \in \mathcal{V}_0 \qquad \text{s.t.} \qquad A^{(j)}(a, b) > \frac{\alpha}{2\#(\mathcal{V}_0)} > 0. \tag{18}$$

Finally, recalling that the matrices A_i commute, we observe that

$$\sum_{z \in \mathcal{V}_0} A^{\ell p+j}(a, z) A_1 \cdots A_t(z, b) = (A_1 \cdots A_t) A^{\ell p+j}(a, b) = \sum_{z \in \mathcal{V}_0} A_1 \cdots A_t(a, z) A^{\ell p+j}(z, b).$$

Using this, we rewrite

$$\zeta_\delta(s) = \sum_{a,b,z \in \mathcal{V}_0} \sum_{t=0}^{k-1} \frac{A_1 \cdots A_t(a, z)(x_b^\Lambda)^s}{(\rho_1 \cdots \rho_t)^{s/\delta}} \sum_{j=0}^{p-1} \sum_{\ell \in \mathbb{N}} \frac{A^{\ell p+j}(z, b)}{\rho^{(\ell p+j)s/\delta}}.$$

It now follows from our arguments above that $\zeta_\delta(s)$ diverges whenever $s \leq \delta$. To convince yourself of this, it may help to recall that x_b^Λ is positive for all vertices b, and that (since $A_1 \cdots A_t(a, z)$ represents the number of paths of degree $(\overbrace{1, \ldots, 1}^{t}, 0, \ldots, 0)$ with source z and range a) our hypothesis that Λ be source-free implies that $\sum_a A_1 \cdots A_t(a, z)$ must be strictly positive for each t. In other words, $\zeta_\delta(s)$ is computed by taking a bunch of sums that diverge to infinity when $s \le \delta$, possibly adding some other positive numbers, multiplying the lot by some positive scalars, and adding the results. Consequently, δ is the abscissa of convergence of the ζ-function $\zeta_\delta(s)$, as claimed. □

As a corollary to Theorem 3.14 we obtain

Corollary 3.15. *Let Λ be a finite, strongly connected k-graph. Fix $\delta \in (0, 1)$ and suppose that Equation* (7) *holds for the weight w_δ of Equation* (5). *Then the spectral triple $(C_{Lip}(X_{\mathcal{B}}), \mathcal{H}, \pi_\tau, D, \Gamma)$ is finitely summable and its dimension is δ.*

Example 3.16. (Continuation of Example 3.13) In this example,

$$\zeta_\delta(s) = \sum_{n \ge 0} \left(\frac{1}{2}\right)^{\frac{sn}{\delta}} 2^n = \sum_{n \ge 0} 2^{n(1-\frac{s}{\delta})} = \frac{1}{1 - 2^{(1-\frac{s}{\delta})}},$$

which evidently has abscissa of convergence δ, and satisfies $\lim_{s \searrow \delta} \zeta_\delta(s) = \infty$.

3.3 Dixmier traces and measures on $X_{\mathcal{B}_\Lambda}$

In this section we show (in Proposition 3.22) that, via the machinery of Dixmier traces, the spectral triples $(C_{\mathrm{Lip}}(X_{\mathcal{B}_\Lambda}), \ell^2(F\mathcal{B}_\Lambda, \mathbb{C}^2), \pi_\tau, D, \Gamma)$ give rise to measures μ_δ on $X_{\mathcal{B}_\Lambda}$. A careful analysis of these measures reveals that they are independent of the choice of the choice function $\tau \in \Upsilon_\Lambda$. Furthermore, Theorem 3.23 gives an integral formula, using the measure μ_δ, for the Dixmier trace. This computation of the Dixmier trace also establishes (Remark 3.25) that the Cantor sets $(X_{\mathcal{B}_\Lambda}, d_\delta)$ are ζ-regular in the sense of Pearson and Bellissard [59].

We conclude the section with Theorems 3.26 and 3.28. Theorem 3.26 establishes that for any choice of δ, the normalized measure $\nu_\delta = \frac{1}{\mu_\delta(X_{\mathcal{B}_\Lambda})}\mu_\delta$ agrees with the measure M, described in Equation (4), which was introduced by an Huef et al. in [40]. Consequently, the measures μ_δ are in fact independent of $\delta \in (0, 1)$. With Theorem 3.26 in hand, we obtain a more general integral formula for the Dixmier trace in Theorem 3.28.

We begin by discussing some preliminaries about Dixmier traces. For the convenience of those readers wishing to compare our discussion with other sources, we recall that in our case the operator $|D|$, and hence $|D|^\delta$, is invertible, and so what in most references we cite is called $< D >^{-\delta} := (1 + D^2)^{-\delta/2}$ gets replaced by $|D|^{-\delta}$ in the formulas below; see for example [35], [34].

Definition 3.17. [55, Example 1.2.9] Let $\{\sigma_k(T)\}_{k\in\mathbb{N}}$ denote the singular values of a compact operator T on a separable Hilbert space $\mathcal{H}$, listed with multiplicity, in (weakly) decreasing order of absolute values. The *Dixmier-Macaev ideal* (also called the Lorentz ideal) $\mathcal{M}_{1,\infty}$ is

$$\left\{ T \in \mathcal{K}(\mathcal{H}) \,:\, \limsup_n \frac{1}{\ln(n)} \sum_{k=1}^{n} \sigma_k(|T|) < \infty \right\}.$$

Following [55], for a generalized limit ω on $\ell^\infty(\mathbb{N})$ vanishing on $\mathbf{c}_0$, we can define the Dixmier trace $\mathcal{T}_\omega$, which is a linear functional on $\mathcal{M}_{1,\infty}$.

An operator T in $\mathcal{M}_{1,\infty}$ is *measurable* in the sense of Connes (or Connes measurable, or in [55] Dixmier measurable) if $\mathcal{T}_\omega(T) = \mathcal{T}_{\omega'}(T)$, for all Dixmier traces ω, ω' on $\mathcal{M}_{1,\infty}$ [55, Page 222]. By [54] (see also [23, Proposition A4]), when T is positive this is equivalent to saying that $\lim_{s\searrow 1}(s-1)\mathrm{Tr}(T^s)$ exists and is finite, in which case, $\lim_{s\searrow 1}(s-1)\mathrm{Tr}(T^s) = \lim_{n\to+\infty} \frac{1}{\ln(n)} \sum_{k=1}^{n} \sigma_k(T)$. This was originally proved by Connes and Moscovici in [23, Proposition A4], where they used the notation $\mathcal{L}^{(1,\infty)}$ for the Dixmier-Macaev ideal (cf. [23, Definition A2]).

Because Theorem 3.18 below establishes that our operators of interest are measurable in Connes' sense, we will study the quantity

$$\mathcal{T}(T) = \lim_{s\searrow 1}(s-1)\mathrm{Tr}(T^s), \tag{19}$$

which gives the value of any Dixmier trace applied to T if T is positive and measurable in the sense of Connes. Note that if A is a clopen set in the Cantor set $X_{\mathcal{B}_\Lambda}$, then χ_A is Lipschitz; so if $\lambda \in F\mathcal{B}_\Lambda$, then the characteristic function $\chi_{[\lambda]}$ of the cylinder set $[\lambda]$ is Lipschitz.

Theorem 3.18. *Let Λ be a finite, strongly connected k-graph. Fix $\delta \in (0,1)$ and suppose that Hypothesis 3.1 holds for the weight w_δ of Equation (5). Then for any $\lambda \in F\mathcal{B}_\Lambda$, the operator $\pi_\tau(\chi_{[\lambda]})|D|^{-\delta}$ is measurable in the sense of Connes, and $\mathcal{T}\left(\pi_\tau(\chi_{[\lambda]})|D|^{-\delta}\right)$ is finite and positive.*

Proof. We first observe that the since the operators $\pi_\tau(\chi_{[\lambda]})$ and $|D|^{-\delta}$ are both diagonal with respect to the basis $\{\delta_\lambda \otimes e_i : \lambda \in F\mathcal{B}_\Lambda, i = 1, 2\}$ of $\mathcal{H}$, they commute. Since $\pi_\tau(\chi_{[\lambda]})$ and $|D|^{-\delta}$ are also positive, then, $\pi_\tau(\chi_{[\lambda]})|D|^{-\delta}$ is positive. We now note that, by Equation (10),

$$\frac{1}{2}\mathrm{Tr}((\pi_\tau(\chi_{[\lambda]})|D|^{-\delta})^s) = \text{ a finite sum plus } \sum_{\eta\in F_\lambda\mathcal{B}_\Lambda} w_\delta(\eta)^{\delta s}.$$

Write p for the period of $A = A_1 \cdots A_k$. We will show that $L_1 := \lim_{s\searrow 1}(1-\rho^{p(1-s)}) \sum_{\eta\in F_\lambda\mathcal{B}_\Lambda} w_\delta(\eta)^{\delta s}$ and $L_2 := \lim_{s\searrow 1} \dfrac{s-1}{1-\rho^{p(1-s)}}$ are both finite and nonzero. It then follows that

$$\mathcal{T}\left(\pi_\tau(\chi_{[\lambda]})|D|^{-\delta}\right) = \lim_{s\searrow 1}(s-1)\mathrm{Tr}\left(\left(\pi_\tau(\chi_{[\lambda]})|D|^{-\delta}\right)^s\right) = 4L_1L_2$$

is finite and nonzero, so $\pi_\tau(\chi_{[\lambda]})|D|^{-\delta}$ is Connes measurable as claimed.

The fact that $L_2 \in (0,\infty)$ follows from L'Hospital's rule:

$$\lim_{s\searrow 1} \frac{s-1}{1-\rho^{p(1-s)}} = \lim_{s\searrow 1} \frac{1}{\rho^{p(1-s)}\ln(\rho^p)} = \frac{1}{\ln(\rho^p)} \in (0,\infty),$$

since $p \geq 1$ and $\rho = \rho_1 \cdots \rho_k > 1$. To see that $L_1 \in (0,\infty)$, observe that if $|\lambda| = qk$,

$$\sum_{\eta\in F_\lambda\mathcal{B}_\Lambda} w_\delta(\eta)^{\delta s} = \frac{1}{\rho^{qs}} \sum_{t=0}^{k-1}\sum_{n=0}^{\infty}\sum_{v,b\in\mathcal{V}_0} \frac{A^n(s(\lambda),v)}{\rho^{ns}} \frac{A_1\cdots A_t(v,b)}{(\rho_1\cdots\rho_t)^s}(x_b^\Lambda)^{\delta s} \tag{20}$$

Again, since all terms in the sum are non-negative, rearranging the order of the summation has no effect on the convergence of the series.

Recall from our computations in Equation (13) of the Jordan form J of A that for any $z, v \in \mathcal{V}_0$ we can find constants $c_i^{z,v}$ and polynomials $P_i^{z,v}$ such that for any $n \in \mathbb{N}$, we have

$$A^n(z,v) = c_1^{z,v}\rho^n + c_2^{z,v}\omega_1^n\rho^n + \cdots + c_p^{z,v}\omega_{p-1}^n\rho^n + \sum_{i=p+1}^{m} P_i^{z,v}(n)\alpha_i^n, \tag{21}$$

where p is the period of A, ω_i is a pth root of unity for all i, and each α_i is an eigenvalue of A with $|\alpha_i| < \rho$. In more detail, writing $A = C^{-1}JC$ for some invertible matrix C, we have

$$c_i^{z,v} = \sum_{j=m_0+\cdots+m_{i-1}+1}^{m_0+\cdots+m_i} C^{-1}(z,j)C(j,v)$$

$$\text{and} \qquad P_i^{z,v}(n) = \sum_{(a,b):\, J_i^n(a,b)\neq 0} C^{-1}(z,a)C(b,v)\frac{1}{\alpha_i^{b-a}}\binom{n}{b-a}.$$

Recall that since J_i is a Jordan block, $J_i^n(a,b) = 0$ unless $a \leq b$. Equivalently, setting $c_{z,v;n} = c_1^{z,v} + c_2^{z,v}\omega_1^n + ... + c_p^{z,v}\omega_{p-1}^n$, we have

$$A^n(z,v) = c_{z,v;n}\rho^n + \sum_{i=p+1}^{m} P_i^{z,v}(n)\alpha_i^n. \tag{22}$$

Observe that the definition of $c_{z,v;n}$ implies that $c_{z,v;n} = c_{z,v;n+p}$ for all $n \in \mathbb{N}$. Moreover, if we consider the limit $A^{(j)}(z,v) = \lim_{\ell\to\infty} \frac{A^{\ell p+j}(z,v)}{\rho^{\ell p+j}}$, Equation (22) implies that

$$A^{(j)}(z,v) = c_{z,v;j}, \tag{23}$$

so each $c_{z,v;j}$ is a non-negative real number.

Using Equation (22), we rewrite a portion of Equation (20):

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{A^n(s(\lambda),v)}{\rho^{ns}} &= \sum_{j=0}^{p-1} c_{s(\lambda),v;j} \sum_{\ell=0}^{\infty} \rho^{(\ell p+j)(1-s)} + \sum_{n=0}^{\infty}\sum_{i=1}^{m} P_i^{s(\lambda),v}(n)\left(\frac{\alpha_i}{\rho^s}\right)^n \\ &= \sum_{j=0}^{p-1} \frac{\rho^j c_{s(\lambda),v;j}}{1-\rho^{p(1-s)}} + \sum_{i=1}^{m}\sum_{n=0}^{\infty} P_i^{s(\lambda),v}(n)\left(\frac{\alpha_i}{\rho^s}\right)^n.\end{aligned}$$

The fact that $\rho > 1$, $s > 1$ and $p \geq 1$ implies that the ratio $\rho^{p(1-s)}$ of the geometric series $\sum_{\ell=0}^{\infty} \rho^{(\ell p+j)(1-s)}$ is less than 1. Moreover, since $P_i^{z,v}(n)$ is a polynomial in n, the fact that $s > 1$ and that $|\alpha_i| < \rho$ for all i implies that the second sum above converges to a finite value $F_v(s)$; indeed, the function $F_v(s)$ is continuous (and finite) at $s = 1$. Consequently,

$$\begin{aligned} L_1 &= \lim_{s\searrow 1}(1-\rho^{p(1-s)}) \sum_{\eta\in F_\lambda \mathcal{B}_\Lambda} w_\delta(\eta)^{\delta s} \\ &= \lim_{s\searrow 1} \frac{1-\rho^{p(1-s)}}{\rho^{qs}} \sum_{t=0}^{k-1}\sum_{n=0}^{\infty}\sum_{v,b\in\mathcal{V}_0} \frac{A^n(s(\lambda),v)}{\rho^{ns}} \frac{A_1\cdots A_t(v,b)}{(\rho_1\cdots\rho_t)^s}(x_b^\Lambda)^{\delta s} \\ &= \lim_{s\searrow 1} \frac{1-\rho^{p(1-s)}}{\rho^{qs}} \left(\sum_{t=0}^{k-1}\sum_{v,b\in\mathcal{V}_0} \frac{A_1\cdots A_t(v,b)}{(\rho_1\cdots\rho_t)^s}(x_b^\Lambda)^{\delta s}\left(\sum_{j=0}^{p-1}\frac{\rho^j c_{s(\lambda),v;j}}{1-\rho^{p(1-s)}} + F_v(s)\right)\right) \\ &= \lim_{s\searrow 1} \frac{1}{\rho^{qs}} \sum_{v\in\mathcal{V}_0}\sum_{j=0}^{p-1} \frac{\rho^j c_{s(\lambda),v;j}(1-\rho^{p(1-s)})}{1-\rho^{p(1-s)}} \sum_{b\in\mathcal{V}_0}\sum_{t=0}^{k-1} \frac{A_1\cdots A_t(v,b)}{(\rho_1\cdots\rho_t)^s}(x_b^\Lambda)^{s\delta} \\ &= \frac{1}{\rho^q}\sum_{v,b\in\mathcal{V}_0}\sum_{j=0}^{p-1}\rho^j c_{s(\lambda),v;j}\sum_{t=0}^{k-1}\frac{A_1\cdots A_t(v,b)}{\rho_1\cdots\rho_t}(x_b^\Lambda)^{\delta}, \end{aligned}$$

which is finite and nonzero. (The penultimate equality holds because the continuity of $F_v(s)$ at $s = 1$ implies that $\lim_{s\searrow 1} \frac{1-\rho^{p(1-s)}}{\rho^{qs}} F_v(s) = 0$.) Consequently, $\pi_\tau(\chi_{[\lambda]})|D|^{-\delta}$ is Connes measurable whenever $|\lambda| = qk$.

If $|\lambda| = qk + t_0$ for some $t_0 > 0$, the same argument as above will show that $\pi_\tau(\chi_{[\lambda]})|D|^{-\delta}$ is Connes measurable; one simply has to take more care with the indexing of the sums. □

Corollary 3.19. *Under the hypotheses of Theorem 3.18, $|D|^{-\delta}$ is Connes measurable, and its Dixmier trace is positive.*

Proof. The fact that $\mathcal{T}\left(\pi_\tau(\chi_{[\lambda]})|D|^{-\delta}\right)$ exists and is finite for all $\lambda \in F\mathcal{B}_\Lambda$ implies that $\mathcal{T}(\pi_\tau(\chi_{X_{\mathcal{B}_\Lambda}})|D|^{-\delta})$ is also finite, since $X_{\mathcal{B}_\Lambda} = \bigsqcup_{v\in\mathcal{V}_0}[v]$ and $\mathcal{V}_0$ is finite. Moreover, $\pi_\tau(\chi_{X_{\mathcal{B}_\Lambda}}) = 1 \in B(\mathcal{H})$. Observing that $|D|^{-\delta}$ is positive, and that $\mathcal{T}(\pi_\tau(\chi_{[v]})|D|^{-\delta})$ is positive for each $v \in \mathcal{V}_0$, completes the proof. □

Remark 3.20. Observe that the constants L_1, L_2 (and therefore the Dixmier trace $\mathcal{T}\left(\pi_\tau(\chi_{[\lambda]})|D|^{-\delta}\right) = 4L_1L_2$) are independent of the choice function τ. For each $\delta \in (0,1)$, we can therefore use the Dixmier trace to define a function μ_δ on the Borel σ-algebra of $X_{\mathcal{B}_\Lambda}$:

$$\mu_\delta([\lambda]) = \text{ Dixmier trace of } \left(\pi_\tau(\chi_{[\lambda]})|D|^{-\delta}\right) = \mathcal{T}(\pi_\tau(\chi_{[\lambda]})|D|^{-\delta}) = \lim_{s\searrow 1}(s-1)\mathrm{Tr}((\pi_\tau(\chi_{[\lambda]})|D|^{-\delta})^s). \quad (24)$$

Example 3.21. (Continuation of Examples 3.13, 3.16) For this example, we can show directly that $\pi_\tau(|D|^{-\delta}) \in \mathcal{M}_{1,\infty}$. By Equation (9), the singular values of $\frac{1}{2}\pi_\tau(|D|^{-\delta})$ are precisely its eigenvalues, which are

$$1 \text{ with multiplicity } 1;\ (\frac{1}{2}),\ \text{with multiplicity } 2; \ldots (\frac{1}{2^k}),\ \text{with multiplicity } 2^k; \ldots$$

Therefore, for say $N_n = 2^{n+1} - 1$:

$$\frac{2}{\ln(N_n)} \sum_{k=0}^{N_n} (eigenvalues\ of\ |D|^{-\delta}) = 2\,\frac{(n+1)}{\ln(2^{n+1}-1)}.$$

So $\limsup_{N\to+\infty} \frac{1}{\ln(N)} \sum_{k=0}^{N} (eigenvalues\ of\ |D|^{-\delta}) < +\infty$, and $|D|^{-\delta}$ is in the Dixmier-Macaev ideal $\mathcal{M}_{1,\infty}$. Furthermore, with the methods of Theorem 3.18, we see that $|D|^{-\delta}$ is measurable in the sense of Connes and that the Dixmier trace of $|D|^{-\delta}$ is given by

$$\begin{aligned}\lim_{s\searrow 1}(s-1)\sum_{k=0}^{+\infty}(eigenvalues\ of\ |D|^{-\delta})^s &= \lim_{s\searrow 1} 2\,(s-1)\sum_{k=0}^{+\infty}\left(\frac{1}{2}\right)^{k\,s} 2^k \\ &= \lim_{s\searrow 1} 2\,(s-1)\sum_{k=0}^{+\infty}\left(2\right)^{k-ks} = \lim_{s\searrow 1}\frac{2\,(s-1)}{1-2^{(1-s)}} = \frac{2}{\ln 2}.\end{aligned}$$

Proposition 3.22. *Let Λ be a finite, strongly connected k-graph; fix $\delta \in (0,1)$ such that $(\mathcal{B}_\Lambda, w_\delta)$ satisfies Hypothesis 3.1. The function μ_δ of Equation (24) determines a unique finite measure on $X_{\mathcal{B}_\Lambda} \cong \Lambda^\infty$. That is, the assignment*

$$[\lambda] \to \mu_\delta([\lambda]),\ \textit{for every } \lambda \in F\mathcal{B}_\Lambda,$$

determines a unique finite measure on $X_{\mathcal{B}_\Lambda}$.

Proof. This proof relies on Carathéodory's theorem [25, Theorem A.1.3]. Notice that

$$\mathcal{F} := \{[\lambda] : \lambda \in F\mathcal{B}_\Lambda\}$$

is closed under finite intersections (if $[\lambda] \cap [\gamma] \neq \emptyset$, then either λ is a sub-path of γ or vice versa, and thus (in the first case) $[\lambda] \cap [\gamma] = [\gamma]$), and

$$[\lambda]^c = \bigsqcup_{|\lambda_i|=|\lambda|, \lambda_i \neq \lambda} [\lambda_i].$$

In other words, the complement of any element of $\mathcal{F}$ can be written as a finite disjoint union of elements of $\mathcal{F}$. Therefore $\mathcal{F}$ is a semiring of sets, so the fact that Λ is finite means that the collection of all finite disjoint unions of cylinder sets $[\lambda]$, for $\lambda \in F\mathcal{B}_\Lambda$, is an algebra.

Since $\mathcal{F}$ generates the topology on $X_{\mathcal{B}_\Lambda}$, and $\mu_\delta([\gamma])$ is finite for all $[\gamma] \in \mathcal{F}$ by hypothesis, Carathéodory's theorem tells us that in order to show that μ_δ determines a measure on $X_{\mathcal{B}_\Lambda}$, we merely need to check that μ_δ is σ-additive on $\mathcal{F}$. In fact, since the cylinder sets $[\gamma]$ are clopen, the fact that $X_{\mathcal{B}_\Lambda}$ is compact means that it is enough to check that μ_δ is finitely additive on $\mathcal{F}$.

Recall that in calculating

$$\mu_\delta([\gamma]) = \lim_{s \searrow 1} 2(s-1) \sum_{\lambda \in F_\gamma \mathcal{B}_\Lambda} w_\delta(\lambda)^{\delta s}$$

we can ignore finitely many initial terms in the sum. Thus, for any $L \in \mathbb{N}$,

$$\mu_\delta([\gamma]) = \lim_{s \searrow 1} 2(s-1) \sum_{\lambda \in F_\gamma \mathcal{B}_\Lambda : \, |\lambda| \geq L} w_\delta(\lambda)^{\delta s}. \tag{25}$$

Now, suppose that $[\gamma] = \bigsqcup_{i=1}^N [\lambda_i]$. Write $L = \max_i |\lambda_i|$, and for each i, write $[\lambda_i] = \bigsqcup_\ell [\lambda_{i,\ell}]$ where $|\lambda_{i,\ell}| = L$. If $\lambda \in F_\gamma \mathcal{B}_\Lambda$ with $|\lambda| \geq L$, then λ_i is a sub-path of λ for precisely one i, and hence

$$\begin{aligned} \mu_\delta([\gamma]) &= \lim_{s \searrow 1} 2(s-1) \sum_{\substack{\lambda \in F_\gamma \mathcal{B}_\Lambda \\ |\lambda| \geq L}} w_\delta(\lambda)^{\delta s} = \lim_{s \searrow 1} 2(s-1) \sum_i \sum_{\lambda \in F_{\lambda_i} \mathcal{B}_\Lambda} w_\delta(\lambda)^{\delta s} \\ &= \sum_i \mu_\delta([\lambda_i]) = \sum_{i,\ell} \mu_\delta([\lambda_{i,\ell}]). \end{aligned}$$

For each fixed i, $\bigsqcup_\ell [\lambda_{i,\ell}] = [\lambda_i]$, so the same argument will show that $\mu_\delta([\lambda_i]) = \sum_\ell \mu_\delta([\lambda_{i,\ell}])$. Thus,

$$\mu_\delta([\gamma]) = \sum_{i,\ell} \mu_{\tau,\delta}([\lambda_{i,\ell}]) = \sum_i \mu_\delta([\lambda_i]).$$

Since μ_δ is finitely additive on $\mathcal{F}$, Carathéodory's theorem allows us to conclude that it gives a well-defined finite measure on $X_{\mathcal{B}_\Lambda}$. □

Our next main result establishes that under our standard hypotheses on Λ, if τ is a choice function and $f \in C(X_{\mathcal{B}_\Lambda})$ is a continuous function, then $\pi_\tau(f)|D|^{-\delta}$ is Connes measurable. Before beginning the proof, we make a few remarks which we will invoke regularly in the proof:

1. Since the Lipschitz functions are dense in $C(X_{\mathcal{B}_\Lambda})$, we can extend the representation π_τ to a representation of $C(X_{\mathcal{B}_\Lambda})$ on $\mathcal{H}$, which we will continue to denote by π_τ.

2. Recall (from the proof of Theorem 3.18) that $\pi_\tau(\chi_{[\lambda]})|D|^{-t} = |D|^{-t}\pi_\tau(\chi_{[\lambda]})$ for any $t > 0$ and any $\lambda \in F\mathcal{B}_\Lambda$. Consequently, $|D|^{-t}$ also commutes with $C(X_{\mathcal{B}_\Lambda})$.

Theorem 3.23. *Let Λ be a finite, strongly connected k-graph; fix $\delta \in (0,1)$ such that Hypothesis 3.1 holds for $(\mathcal{B}_\Lambda, w_\delta)$, and fix a choice function τ. Let μ_δ be the Borel measure on $X_{\mathcal{B}_\Lambda}$ described in Proposition 3.22. Then $\pi_\tau(f)|D|^{-\delta}$ is Connes measurable for all $f \in C(X_{\mathcal{B}_\Lambda})$, and the Dixmier trace of $\pi_\tau(f)|D|^{-\delta}$ is given by*

$$\mathcal{T}\left(\pi_\tau(f)|D|^{-\delta}\right) = \lim_{s\searrow 1}(s-1)Tr(\pi_\tau(f)|D|^{-\delta s}) \;=\; \int_{X_{\mathcal{B}_\Lambda}} f(x)\, d\mu_\delta(x).$$

Proof. Replacing t with $\frac{1}{s-1}$ in the proof of [55, Theorem 8.6.5], and applying this proof to the setting $\omega = \lim_{t\to\infty}$, $\mathcal{M} = B(\mathcal{H})$, $\tau = \text{Tr}$, $A = |D|^{-\delta}$ implies that for any $f \in C(X_{\mathcal{B}_\Lambda})_+$, if $\lim_{s\searrow 1}(s-1)\text{Tr}(\pi_\tau(f)|D|^{-\delta s})$ exists and is finite, then (since $\pi_\tau(f)$ and $|D|^{-r}$ commute for any $r > 0$)

$$\lim_{s\searrow 1}(s-1)\text{Tr}((\pi_\tau(f)|D|^{-\delta})^s) = \lim_{s\searrow 1}(s-1)\text{Tr}(\pi_\tau(f)|D|^{-\delta s}).$$

So for f non-negative and continuous, it also follows from [23] that if $\lim_{s\searrow 1}(s-1)\text{Tr}(\pi_\tau(f)|D|^{-\delta s})$ exists and is finite, then $\pi_\tau(f)|D|^{-\delta}$ is Connes measurable, and its Dixmier trace is $\lim_{s\searrow 1}(s-1)\text{Tr}(\pi_\tau(f)|D|^{-\delta s})$.

Note that by Theorem 3.18, if ϕ is a simple function on $X_{\mathcal{B}_\Lambda}$ of the form $\phi = \sum_{j=1}^m \alpha_j \chi_{[\lambda_j]}$, then linearity of the integral combines with Proposition 3.22, our remarks in the first paragraph of this proof, and the definition of μ_δ in Equation (24) to show that

$$\lim_{s\searrow 1}(s-1)\text{Tr}(\pi_\tau(\phi)|D|^{-\delta s}) = \sum_{j=1}^m \alpha_j \lim_{s\searrow 1}(s-1)\text{Tr}(\pi_\tau(\chi_{[\lambda_j]})|D|^{-\delta s}) \;=\; \int_{X_{\mathcal{B}_\Lambda}} \phi(x)\, d\mu_\delta(x).$$

Fix $\epsilon \in (0,1)$ and $f \in C(X_{\mathcal{B}_\Lambda})$. There exists $\eta_1 > 0$ such that whenever $s \in (1, 1+\eta_1)$,

$$|(s-1)\text{Tr}(|D|^{-s\delta}) - \mathcal{T}(|D|^{-\delta})| < \epsilon.$$

By the Stone-Weierstrass Theorem, the simple functions made from characteristic functions corresponding to finite paths in $F\mathcal{B}_\Lambda$ are dense in $C(X_{\mathcal{B}_\Lambda})$ so given our fixed continuous function f there is a simple function ϕ of the desired type with $\|f-\phi\|_{\sup} < \frac{\epsilon}{4(\mathcal{T}(|D|^{-\delta})+1)}$, and hence

$$\left|\int_{X_{\mathcal{B}_\Lambda}} \phi(x) d\mu_\delta(x) - \int_{X_{\mathcal{B}_\Lambda}} f(x) d\mu_\delta(x)\right| \;\le\; \int_{X_{\mathcal{B}_\Lambda}} \|f-\phi\|_{\sup}\, d\mu_\delta(x) < \int_{X_{\mathcal{B}_\Lambda}} \frac{\epsilon}{4(\mathcal{T}(|D|^{-\delta})+1)} \cdot 1\, d\mu_\delta(x) < \frac{\epsilon}{4}.$$

By our remarks at the beginning of this proof, there exists $\eta_2 > 0$ such that if $s \in (1, 1+\eta_2)$,

$$\left|(s-1)\text{Tr}(\pi_\tau(\phi)|D|^{-\delta s}) - \int_{X_{\mathcal{B}_\Lambda}} \phi(x)\, d\mu_\delta(x)\right| < \frac{\epsilon}{4}.$$

We now let $\eta = \min\{\eta_1, \eta_2\}$. Suppose that $s \in (1, 1+\eta)$. Then,

$$\begin{aligned}
\left|(s-1)\text{Tr}(\pi_\tau(f)|D|^{-\delta s}) - \int_{X_{\mathcal{B}_\Lambda}} f(x) d\mu_\delta(x)\right| &\le\; |(s-1)\text{Tr}(\pi_\tau(f)|D|^{-\delta s}) - (s-1)\text{Tr}(\pi_\tau(\phi)|D|^{-\delta s})| \\
&\quad + \left|(s-1)\text{Tr}(\pi_\tau(\phi)|D|^{-\delta s}) - \int_{X_{\mathcal{B}_\Lambda}} \phi(x) d\mu_\delta(x)\right| + \left|\int_{X_{\mathcal{B}_\Lambda}} \phi(x) d\mu_\delta(x) - \int_{X_{\mathcal{B}_\Lambda}} f(x) d\mu_\delta(x)\right| \\
&= |(s-1)\text{Tr}(\pi_\tau(f-\phi)|D|^{-s\delta})| + \frac{\epsilon}{4} + \frac{\epsilon}{4} \le (s-1)\,\text{Tr}(|D|^{-s\delta}) \cdot \|\pi_\tau(f-\phi)\|_{B(\mathcal{H})} + \frac{\epsilon}{2} \\
&\le (\mathcal{T}(|D|^{-\delta}) + \epsilon) \cdot \frac{\epsilon}{4(\mathcal{T}(|D|^{-\delta})+1)} + \frac{\epsilon}{2} \;\le\; \frac{\epsilon}{4} + \frac{\epsilon^2}{4} + \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

In the penultimate inequality we used the fact that the set of trace class operators is an ideal in $B(\mathcal{H})$, and if K is a trace-class operator and $T \in B(\mathcal{H})$, $|\mathrm{Tr}(TK)| \leq \mathrm{Tr}(|K|) \cdot \|T\|_{B(\mathcal{H})}$ [61, Page 218, Ex. 28a]. Thus we have established that

$$\mathcal{T}(\pi_\tau(f)|D|^{-s\delta}) = \lim_{s \searrow 1}(s-1)\mathrm{Tr}(\pi_\tau(f)|D|^{-s\delta}) = \int_{X_{\mathcal{B}_\Lambda}} f\, d\mu_\delta(x) \tag{26}$$

for any $f \in C(X_{\mathcal{B}_\Lambda})$. As indicated at the beginning of the proof, for any non-negative function $f \in C(X_{\mathcal{B}_\Lambda})_+$, $\pi_\tau(f)|D|^{-\delta}$ is Connes measurable and Equation (26) computes its Dixmier trace. The linearity of the Dixmier trace, combined with the fact that any $f \in C(X_{\mathcal{B}_\Lambda})$ can be written as the difference of two non-negative continuous functions, $f = f_+ - f_-$, now implies that for any $f \in C(X_{\mathcal{B}_\Lambda})$, $\pi_\tau(f)|D|^{-\delta}$ is also Connes measurable, and that Equation (26) gives the Dixmier trace of $\pi_\tau(f)|D|^{-\delta}$ for all $f \in C(X_{\mathcal{B}_\Lambda})$. $\square$

Remark 3.24. Proposition 3.22 and Theorem 3.23 can also be deduced by following the argument indicated in [47]. Since $\pi_\tau(|D|^{-\delta})$ is in the Dixmier-Macaev ideal $\mathcal{M}_{1,\infty}$ by Theorem 3.18, we have $\pi_\tau(f)|D|^{-\delta} \in \mathcal{M}_{1,\infty}$, for all $f \in C(X_{\mathcal{B}_\Lambda})$. For a fixed generalized limit ω, the Dixmier trace functional $\mathcal{D}_\omega : C(X_{\mathcal{B}_\Lambda}) \to \mathbb{C}$, defined by $\mathcal{D}_\omega\Big(f\Big) := \mathcal{T}_\omega\Big(\pi_\tau(f)|D|^{-\delta}\Big)$ is bounded, see e.g. [47, page 1826]. Now the Riesz representation theorem for linear functionals on $C(X_{\mathcal{B}_\Lambda})$ implies that there exists a finite measure μ_ω (also possibly dependent on τ and δ) on $X_{\mathcal{B}_\Lambda}$ such that (see [47, page 1826])

$$\mathcal{D}_\omega\Big(f\Big) = \int_{X_{\mathcal{B}_\Lambda}} f\, d\mu_\omega, \quad \forall f \in C(X_{\mathcal{B}_\Lambda}).$$

But by the Carathéodory/Kolmogorov extension theorem, the measure μ_ω is determined by its values on cylinder sets. This evaluation on cylinder sets is (by Remark 3.20) independent of τ and ω; in other words, $\mu_\omega = \mu_\delta$ for all ω. Therefore we get

$$\mathcal{D}_\omega\Big(f\Big) = \int_{X_{\mathcal{B}_\Lambda}} f\, d\mu_\delta, \quad \forall f \in C(X_{\mathcal{B}_\Lambda}), \quad \text{for all generalized limits } \omega.$$

Remark 3.25. Theorem 3.23 also shows that the Cantor set $X_{\mathcal{B}_\Lambda}$ is ζ-regular in the sense of Definition 11 of [59]. This is an immediate corollary of Theorem 3.23, Corollary 3.19, and the definition of ζ-regularity, together with the elementary observation that the limit of the quotient is the quotient of the limits if the latter exist.

Our next step will be the determination of the measure μ_δ on $X_{\mathcal{B}_\Lambda}$, up to renormalization.

Theorem 3.26. *Let Λ be a finite, strongly connected k-graph for which Lemma 3.2 holds. Write A_i for the i-th adjacency matrix of Λ and suppose that $A = A_1 \cdots A_k$ is irreducible. For any $\delta \in (0,1)$, the normalization ν_δ of the measure μ_δ on $X_{\mathcal{B}_\Lambda}$ defined by*

$$\nu_\delta(O) = \frac{\mu_\delta(O)}{\mu_\delta(X_{\mathcal{B}_\Lambda})} = \frac{\mathcal{T}(\pi_\tau(\chi_O)|D|^{-\delta})}{\mathcal{T}(|D|^{-\delta})} \text{ for every Borel set } O \text{ of } X_{\mathcal{B}_\Lambda} \tag{27}$$

agrees with the measure M introduced in Proposition 8.1 of [40]. In particular, ν_δ is a probability measure which is independent of the choice of δ.

Proof. For any path $\gamma \in F_v\mathcal{B}_\Lambda$ with $|\gamma| \geq k$, write $\gamma = \gamma_0\gamma'$ with $|\gamma_0| = k$. Since $r(\gamma') \in \mathcal{V}_k = \mathcal{V}_0$, we can identify γ' with a path in $F\mathcal{B}_\Lambda$. Then Proposition 2.19 tells us that

$$w_\delta(\gamma) = \rho^{-1/\delta} w_\delta(\gamma').$$

Consequently,

$$
\begin{aligned}
\mu_\delta([v]) &= \lim_{s\searrow 1} 2(s-1)\sum_{\gamma\in F_v\mathcal{B}_\Lambda} w_\delta(\gamma)^{\delta s} = \lim_{s\searrow 1} 2(s-1)\left(\sum_{r(\gamma)=v,|\gamma|<k} w_\delta(\gamma)^{\delta s} + \sum_{r(\gamma)=v,|\gamma|\geq k} w_\delta(\gamma)^{\delta s}\right)\\
&= \lim_{s\searrow 1} 2(s-1)\left(\sum_{r(\gamma)=v,|\gamma|<k} w_\delta(\gamma)^{\delta s} + \sum_{n=1}^{\infty}\sum_{t=0}^{k-1}\sum_{r(\gamma)=v,\,|\gamma|=nk+t} w_\delta(\gamma)^{\delta s}\right)\\
&= \lim_{s\searrow 1} 2(s-1)\left(\sum_{r(\gamma)=v,|\gamma|<k} w_\delta(\gamma)^{\delta s} + \rho^{-s}\sum_{n=0}^{\infty}\sum_{t=0}^{k-1}\sum_{z\in\Lambda^0}\sum_{r(\gamma')=z,\,|\gamma'|=nk+t} A(v,z) w_\delta(\gamma')^{\delta s}\right)\\
&= \lim_{s\searrow 1} \frac{1}{\rho^s} 2(s-1)\sum_{z\in\Lambda^0} A(v,z)\sum_{\gamma'\in F_z\mathcal{B}_\Lambda} w_\delta(\gamma')^{\delta s}\\
&= \frac{1}{\rho}\sum_{z\in\Lambda^0} A(v,z)\mu_\delta([z]).
\end{aligned}
$$

The third equality holds because of the formula (5) for the weight w_δ; to be precise, if $\gamma = \gamma_0\gamma'$ and $|\gamma_0| = k$, then $w_\delta(\gamma)^{\delta s} = \rho^{-s} w_\delta(\gamma')^{\delta s}$. Moreover, for each fixed such path γ' with range z and length $(n-1)k+t$, there are $A(v,z)$ paths γ of length $nk+t$ and range v such that $\gamma = \gamma_0\gamma'$ for some path γ_0 with length k. The penultimate equality holds because the first sum (being finite) tends to zero as s tends to 1; the final equality holds since both $\lim_{s\searrow 1}\rho^{-s}$ and $\mu_\delta([z])$ are finite, so the limit of the product equals the product of the limits. Thus, $(\nu_\delta([v]))_{v\in\mathcal{V}_0}$ is a positive eigenvector for A with ℓ^1-norm 1 and eigenvalue ρ, and hence must agree with x^Λ by the irreducibility of A.

Moreover, if $|\gamma| = q_0 k$ (equivalently, if we think of $\gamma\in\Lambda$, then $d(\gamma) = (q_0,\ldots,q_0)$), then

$$
\mu_{\tau,\delta}([\gamma]) = \lim_{s\searrow 1} 2(s-1)\frac{1}{\rho^{sq_0}}\sum_{b,v\in\mathcal{V}_0}\sum_{t=0}^{k-1}\sum_{n\in\mathbb{N}}\frac{A^n(s(\gamma),v)}{\rho^{ns}}\frac{A_1\cdots A_t(v,b)(x_b^\Lambda)^s}{(\rho_1\cdots\rho_t)^s} = \frac{1}{\rho^{q_0}}\mu_{\tau,\delta}([s(\gamma)]).
$$

Comparing this formula with Equation (4) tells us that whenever $|\gamma| = q_0 k$,

$$
\nu_\delta([\gamma]) = M([\gamma]).
$$

Since ν_δ agrees with M on the square cylinder sets $[\lambda]$ with $d(\lambda) = (q_0,\ldots,q_0)$, and we know from the proof of Lemma 4.1 of [28] that these sets generate the Borel σ-algebra of $X_{\mathcal{B}_\Lambda}$, the measures ν_δ and M must agree on all Borel subsets of $X_{\mathcal{B}_\Lambda}$. ☐

Remark 3.27.
1. If one could prove that the vector $(\mu_\delta[v])_{v\in\mathcal{V}_0}$ was an eigenvector for each A_i with eigenvalue ρ_i, then we could use the theory of families of irreducible matrices, developed in [40, Section 3], to remove the hypothesis that A be irreducible in Theorem 3.26.

2. Since $\mathcal{T}(|D|^{-\delta})$ does not depend on τ, the above proposition shows that μ_δ is a finite measure on $X_{\mathcal{B}_\Lambda}$, with
$$\mu_\delta(O) = \mathcal{T}(|D|^{-\delta})\, M(O), \text{ for every Borel set } O \text{ of } X_{\mathcal{B}_\Lambda}.$$

We have therefore proved the following improved version of Theorem 3.23, under the additional hypothesis that $A = A_1\cdots A_k$ be irreducible.

Theorem 3.28. *Let Λ be a finite, strongly connected k-graph. Write A_i for the ith adjacency matrix of Λ and suppose that $A = A_1\cdots A_k$ is irreducible. Fix $\delta\in(0,1)$ and suppose that Hypothesis 3.1 holds for* *the weight w_δ of Equation* (5)*. Then for any $f \in C(X_{\mathcal{B}_\Lambda})$, the operator $\pi_\tau(f)|D|^{-\delta}$ is measurable in the sense of Connes and its Dixmier trace is*

$$\lim_{s\searrow 1}(s-1)Tr(\pi_\tau(f)|D|^{-\delta s}) = \mathcal{T}(|D|^{-\delta})\int_{X_{\mathcal{B}_\Lambda}} f\, dM,$$

where M is the measure introduced in Proposition 8.1 of [40].

4 Eigenvectors of Laplace-Beltrami operators and wavelets

In this section, we investigate the relationship between the decomposition of $L^2(X_{\mathcal{B}_\Lambda}, \mu_\delta)$ via the eigenspaces of the Laplace-Beltrami operators Δ_s associated to the spectral triples of Section 3 for the ultrametric Cantor set $(X_{\mathcal{B}_\Lambda}, d_{w_\delta})$ of Corollary 2.20, and the wavelet decomposition of $L^2(\Lambda^\infty, M)$ given in Theorem 4.2 of [28]. Our main result in this section, Theorem 4.6, establishes that the Laplace-Beltrami eigenspaces, as described in [42, Theorem 4.3], also encode the wavelet decomposition of [28, Theorem 4.2].

The connection between operators and wavelets that we identify in this section goes deeper than the frequently-seen connection between wavelet decompositions and Dirac operators. To be precise, the wavelet decomposition of $L^2(\Lambda^\infty, M)$ arises from a representation of $C^*(\Lambda)$ (see Definition 4.4). Thus, the results in this section establish a link between representations of higher-rank graphs and the Pearson-Bellissard spectral triples, in addition to identifying the wavelet decomposition of [28] with the eigenspaces of the Laplace-Beltrami operators Δ_s.

4.1 The Laplace-Beltrami operators and their eigenspaces

We begin by describing the Laplace-Beltrami operators of [59] and their eigenspaces. Recall a choice function is a map $\tau : F\mathcal{B}_\Lambda \to X_{\mathcal{B}_\Lambda} \times X_{\mathcal{B}_\Lambda}$ satisfying $\tau(\gamma) = (x, y)$ where $x, y \in [\gamma]$ and $d(x, y) = \mathrm{diam}([\gamma]) = w(\gamma)$. The set of all choice functions is denoted by Υ_Λ. We want to identify Υ_Λ with a measurable space which we can construct a measure related to the measure M which arose in the last section, see Theorem 3.26. Our approach will be the same as that given in Section 7.2 of [59] with slightly more detail.

Proposition 4.1. *(cf. [59], Section 7.2) Let Λ be a strongly connected finite k-graph and $\delta \in (0, 1)$ such that $(\mathcal{B}_\Lambda, w_\delta)$ satisfies Hypothesis 3.1. If Υ_Λ represents the set of choice functions $\tau : F\mathcal{B}_\Lambda \to X_{\mathcal{B}_\Lambda} \times X_{\mathcal{B}_\Lambda}$, we can identify Υ_Λ with an infinite product space*

$$Y = \prod_{\gamma\in F\mathcal{B}_\Lambda} Y_\gamma,$$

where each Y_γ is a compact set equal to a finite unions of products of cylinder sets. Moreover, assuming that the product $A = A_1 \cdots A_k$ of the adjacency matrices of Λ is irreducible, there is a probability measure N on Y that can be derived from the measure M on $X_{\mathcal{B}_\Lambda}$ described in Theorem 3.26.

Proof. We first fix $\gamma \in F\mathcal{B}_\Lambda$, and define the subset $\mathcal{G}_\gamma$ of $F\mathcal{B}_\Lambda \times F\mathcal{B}_\Lambda$ as in Section 7.2 of [59]. Let $z \in X_{\mathcal{B}_\Lambda}$ be an element of $[\gamma]$, so that $z(0, d(\gamma)) = \gamma$. If we set $r = w_\delta(\gamma)$, we know from Proposition 2.15 and Hypothesis (3.1) that $[\gamma] = B[z, r]$. Now let τ be a choice function with $\tau(\gamma) = (x, y)$, so that $d_\delta(x, y) = w_\delta(\gamma) = \mathrm{diam}([\gamma])$. Hypothesis 3.1 implies the existence of γ_1 and γ_2 in $F\mathcal{B}_\Lambda$ that are extensions of the fixed finite path γ with $|\gamma_1| = |\gamma_2| = |\gamma| + 1$, and $x(0, d(\gamma_1)) = \gamma_1$, and $y(0, d(\gamma_2)) = \gamma_2$. On the other hand, given $\gamma_1,\ \gamma_2 \in F\mathcal{B}_\Lambda$ that are extensions of the fixed finite path γ with $\gamma_1 \neq \gamma_2$, $|\gamma_1| = |\gamma_2| = |\gamma| + 1$, for any $x \in [\gamma_1] \subset B[z,r]$ we have $x(0, d(\gamma_1)) = \gamma_1$ and for any $y \in [\gamma_2] \subset B[z,r]$ we have $y(0, d(\gamma_2)) = \gamma_2$ so that by Proposition 2.15, $d_\delta(x, y) = w(\gamma) = \mathrm{diam}([\gamma]) = r$. Thus we can identify all ordered pairs that are contained in the Cartesian products $[\gamma_1] \times [\gamma_2]$ with the image under a choice function of $\gamma \in F\mathcal{B}_\Lambda$. For each $\gamma \in F\mathcal{B}_\Lambda$, we therefore write

$$\mathcal{G}_\gamma = \{(\gamma_1, \gamma_2) \in F\mathcal{B}_\Lambda \times F\mathcal{B}_\Lambda\}$$

where γ_1 and γ_2 are extensions of γ with $|\gamma_1| = |\gamma_2| = |\gamma| + 1$. Our requirement that Λ be a finite k-graph implies that each $\mathcal{G}_\gamma$ is a finite set. For each $\gamma \in F\mathcal{B}_\Lambda$, we write

$$Y_\gamma = \bigsqcup_{(\gamma_1,\gamma_2)\in\mathcal{G}_\gamma} [\gamma_1] \times [\gamma_2].$$

Since $\mathcal{G}_\gamma$ is a finite set and each $[\gamma_1] \times [\gamma_2]$ is compact in $X_{\mathcal{B}_\Lambda} \times X_{\mathcal{B}_\Lambda}$, the finite disjoint union Y_γ is closed in $X_{\mathcal{B}_\Lambda} \times X_{\mathcal{B}_\Lambda}$, hence compact. We then note that by construction, each element of the infinite product

$$Y = \prod_{\gamma\in F\mathcal{B}_\Lambda} Y_\gamma$$

can be identified with a choice function, and thus Y can be identified with Υ_Λ. It follows that if we equip each factor Y_γ with a probability measure N_γ, we obtain a probability measure N on the infinite product space X, by the fundamental results of Kakutani [43].

We recall that M is the probability measure on $X_{\mathcal{B}_\Lambda}$ which arises via the normalized Dixmier trace, as described in Theorem 3.26, and so $M \times M$ is a probability measure on the Cartesian product $X_{\mathcal{B}_\Lambda} \times X_{\mathcal{B}_\Lambda}$. Fixing $(\gamma_1, \gamma_2) \in \mathcal{G}_\gamma$, then $M \times M$ restricts to a finite measure on Borel subsets of the Cartesian product $[\gamma_1] \times [\gamma_2] \subset X_{\mathcal{B}_\Lambda} \times X_{\mathcal{B}_\Lambda}$ that is most likely not a probability measure. We now scale this measure as follows: for any Borel subset E of $[\gamma_1] \times [\gamma_2]$, let

$$N_{(\gamma_1,\gamma_2)}(E) = \frac{(M \times M)(E)}{\sum_{(\eta,\eta')\in\mathcal{G}_\gamma} M([\eta])M([\eta'])}.$$

Now define the Borel measure N_γ on Y_γ by setting

$$N_\gamma(E) = \sum_{(\gamma_1,\gamma_2)\in\mathcal{G}_\gamma} N_{(\gamma_1,\gamma_2)}(E \cap ([\gamma_1] \times [\gamma_2])).$$

Finally, using Kakutani's infinite product theory for measures [43], we have a Borel probability measure N defined on $Y = \prod_\gamma Y_\gamma$ by

$$N = \prod_{\gamma\in F\mathcal{B}_\Lambda} N_\gamma.$$

Since Y can be identified with Υ_Λ, we write N for the corresponding measure on Υ_Λ, as well. ☐

Remark 4.2. In Proposition 4.1 the hypothesis $A = A_1 \cdots A_k$ is irreducible is not essential. In its absence, we can prove that we obtain a probability measure N_δ on Y that can be derived from the measure μ_δ on $X_{\mathcal{B}_\Lambda}$ (of Proposition 3.22) in the same way that N is derived from M.

Therefore, according to Section 8.3 of [59] and Section 4 of [42], for each $s \in \mathbb{R}$ the ζ−regular Pearson-Bellissard spectral triple from the previous section gives rise to a Laplace-Beltrami operator Δ_s on $L^2(X_{\mathcal{B}_\Lambda}, M)$ via the Dirichlet form Q_s as follows:

$$\langle f, \Delta_s(g)\rangle = Q_s(f, g) := \frac{1}{2}\int_{\Upsilon_\lambda} \mathrm{Tr}\Big(|D|^{-s}[D, \pi_\tau(f)]^*[D, \pi_\tau(g)]\Big)\, dN(\tau). \tag{28}$$

Thanks to Section 8.1 of [59], we know that Q_s is a closable Dirichlet form for all $s \in \mathbb{R}$ and it has a dense domain that is generated by the set of characteristic functions on cylinder sets of $X_{\mathcal{B}_\Lambda}$. Also, by applying the work of [59] and [42] to our weighted stationary k-Bratteli diagrams $\mathcal{B}_\Lambda$, we can obtain an explicit formula for Δ_s on characteristic functions as follows.

For a finite path $\eta = (\eta_i)_{i=1}^{|\eta|}$ (where each η_i is an edge) in $\mathcal{B}_\Lambda$, we write $\chi_{[\eta]}$ for the characteristic function of the set $[\eta] \subseteq X_{\mathcal{B}_\Lambda}$ of infinite paths of $\mathcal{B}_\Lambda$ whose initial segment is η, and $\eta(0,i)$ for $\eta_1 \cdots \eta_i$. We denote by $\eta(0,0)$ the vertex $r(\eta)$. Also, for $\gamma \in F\mathcal{B}_\Lambda$, we set

$$\frac{1}{F_\gamma} = \sum_{(e,e')\in \mathrm{ext}_1(\gamma)} M([\gamma e])M([\gamma e']),$$

where $\mathrm{ext}_1(\gamma)$ is the set of pairs (e,e') of edges in $\mathcal{B}_\Lambda$ with $e \neq e'$ and $r(e) = r(e') = s(\gamma)$.

From Lemma 3.2, we know that if Hypothesis 3.1 holds for the weighted stationary k-Bratteli diagram $(\mathcal{B}_\Lambda, w_\delta)$ associated to a higher-rank graph Λ, then $\mathrm{ext}_1(\gamma)$ is nonempty for all $\gamma \in F\mathcal{B}_\Lambda$. We can therefore assume that $\mathrm{ext}_1(\gamma)$ is always nonempty; equivalently, that $F_\gamma < \infty$. Then, as in Section 4 of [42], for each $s \in \mathbb{R}$, we have

$$\begin{aligned}\Delta_s(\chi_{[\eta]}) = -\sum_{i=0}^{|\eta|-1} 2F_{\eta(0,i)} w(\eta(0,i))^{s-2} \big(& M([\eta(0,i)]\backslash[\eta(0,i+1)])\chi_{[\eta]} \\ & - M([\eta])\chi_{[\eta(0,i)]\backslash[\eta(0,i+1)]}\big) .\end{aligned} \tag{29}$$

We now restate some results from Section 4 of [42], which we have adapted to our setting.

Proposition 4.3. *(cf. [42], Theorem 4.3) Let Λ be a finite, strongly connected k-graph and choose $\delta \in (0,1)$ such that $(\mathcal{B}_\Lambda, w_\delta)$ satisfies Hypothesis 3.1. Suppose that $A = A_1 \cdots A_k$ is irreducible. Let $X_{\mathcal{B}_\Lambda}$ be the infinite path space associated to Λ with associated probability measure M. Let $\{\Delta_s : s > 0\}$ be the family of Laplace–Beltrami operators defined on a dense subspace of $L^2(X_{\mathcal{B}_\Lambda}, M)$ in Equation (29). Then the eigenspaces of $\{\Delta_s : s > 0\}$ are independent of s. Precisely, they are given by*

$$E_{-1} = span\{\chi_{X_{\mathcal{B}_\Lambda}}\}$$

with eigenvalue 0 *and*

$$E_0 = span\left\{ \frac{1}{M([v])}\chi_{[v]} - \frac{1}{M([v'])}\chi_{[v']} : v \neq v' \in \mathcal{V}_0 \right\},$$

with eigenvalue $2/\left(\sum_{v\neq v' \in \mathcal{V}_0} M([v])M([v'])\right)$*. For each nonempty $\gamma \in F\mathcal{B}_\Lambda$, define a subspace*

$$E_\gamma = span\left\{ \frac{1}{M([\gamma e])}\chi_{[\gamma e]} - \frac{1}{M([\gamma e'])}\chi_{[\gamma e']} : (e,e') \in ext_1(\gamma) \right\}. \tag{30}$$

Then the subspace E_γ consists of eigenvectors with the same eigenvalue, and for $\gamma \neq \eta \in F\mathcal{B}_\Lambda$, E_γ is orthogonal to E_η.

Proof. This result is contained in Theorem 4.3 of [42], and here we are including details for completeness and clarity of notation.

By our discussion of the action of Δ_s on cylinder sets, $\chi_{\Lambda^\infty} \equiv 1$ is in the kernel of Δ_s so that E_{-1} has eigenvalue 0. The proof of Theorem 4.3 of [42] shows that

$$\frac{2}{\sum_{v,v' \in \Lambda^0 :\, v\neq v'} M([v])M([v'])}$$

is an eigenvalue for the given space E_0. Now consider the subspaces E_γ for a nonempty path $\gamma \in F\mathcal{B}_\Lambda$. The eigenvalues λ_γ for the subspaces E_γ as given in the statement of our theorem are computed via Theorem 4.3 of [42] as follows. Recall for any finite path η of $\mathcal{B}_\Lambda$ we have defined the set $\mathrm{ext}_1(\eta)$ and the positive number F_η above. For each $s > 0$, let

$$G_s(\eta) = \frac{1}{2}\mathrm{diam}([\eta])^{2-s} F_\eta.$$

Thus for a nonempty finite path γ, the formula for the eigenvalue λ_γ is given by

$$\lambda_\gamma = \sum_{i=0}^{|\gamma|-1} \frac{[M([\gamma(0,i)]) - M([\gamma(0,i+1)])]}{G_s(\gamma(0,i))} - \frac{M([\gamma])}{G_s(\gamma)},$$

and in Theorem 4.3 of [42] it is shown that every vector in E_γ is an eigenvector for Δ_s with eigenvalue λ_γ. For an arbitrary finite k-graph, it is not an easy task to compute the eigenvalues λ_γ for a specific weight w_δ. The authors have done so in the case of a symmetric weight where Bratteli diagram comes from the directed graph Λ_D with D vertices and D^2 edges giving rise to the Cuntz algebra $\mathcal{O}_D$ in [27, Theorem 4.10], and have done so for an arbitrary weight on Λ_2 in [27, Proposition 6.8].

The eigenspaces of Δ_s are independent of s, although in general, the eigenvalues λ_γ depend on the choice of $s \in \mathbb{R}$. For general $\gamma,\ \eta$ in $F\mathcal{B}_\Lambda$ with $\gamma \neq \eta$, it is not obvious that $\lambda_\gamma \neq \lambda_\eta$. However, it will be the case that $E_\gamma \perp E_\eta$, by the following reasoning. If $[\gamma] \cap [\eta] = \emptyset$, it is evident that the functions in E_γ and E_η have disjoint support, thus are orthogonal. In the case where $[\gamma] \cap [\eta] \neq \emptyset$, suppose without loss of generality that $|\eta| \leq |\gamma|$. It then follows that we must have $[\eta] \subseteq [\gamma]$, and consequently $\eta = \gamma\lambda$ for some path λ. Therefore,

$$\begin{aligned}
\langle \frac{1}{M([\gamma e])}\chi_{[\gamma e]} - \frac{1}{M([\gamma e'])}\chi_{[\gamma e']}, \frac{1}{M([\eta \tilde{e}])}\chi_{[\eta\tilde{e}]} - \frac{1}{M([\eta\tilde{e}'])}\chi_{[\eta\tilde{e}']}\rangle &= \frac{1}{M([\gamma e])M([\eta\tilde{e}])}\int_{X_{\mathcal{B}_\Lambda}} \chi_{[\gamma e]}\chi_{[\eta\tilde{e}]}\, dM \\
&+ \frac{1}{M([\gamma e'])M([\eta\tilde{e}'])}\int_{X_{\mathcal{B}_\Lambda}} \chi_{[\gamma e']}\chi_{[\eta\tilde{e}']}\, dM \\
&- \left(\frac{1}{M([\gamma e])M([\eta\tilde{e}'])}\int_{X_{\mathcal{B}_\Lambda}} \chi_{[\gamma e]}\chi_{[\eta\tilde{e}']}\, dM + \frac{1}{M([\gamma e'])M([\eta\tilde{e}])}\int_{X_{\mathcal{B}_\Lambda}} \chi_{[\gamma e']}\chi_{[\eta\tilde{e}]}\, dM \right).
\end{aligned}$$

The first and third terms are both zero unless the first edge of λ is e, in which case their difference evaluates to

$$\frac{1}{M([\gamma e])} - \frac{1}{M([\gamma e])} = 0.$$

Similarly, the second and fourth integrals are both zero unless the first edge of λ is e', and in this case the integrals take the same value. It follows that the basis vectors for E_γ will always be orthogonal to the basis vectors for E_η, so $E_\gamma \perp E_\eta$ as claimed. □

4.2 Wavelets and eigenspaces for Δ_s

In this section, we prove our Theorem relating the wavelet decomposition (32) with the eigenspaces E_γ of the Laplace-Beltrami operators Δ_s in the case when $A := A_1 \cdots A_k$ is irreducible.

In Theorem 4.6 below, we compare the subspaces E_γ with the wavelet decomposition of $L^2(\Lambda^\infty, M)$ which was constructed in [28] out of a representation of the C^*-algebra $C^*(\Lambda)$ on $L^2(\Lambda^\infty, M)$.

Before recalling this wavelet decomposition, we first review the construction of the C^*-algebra $C^*(\Lambda)$ associated to a higher-rank graph.

Definition 4.4. [51] Let Λ be a finite k-graph with no sources. $C^*(\Lambda)$ is the universal C^*-algebra generated by a collection of partial isometries $\{s_\lambda\}_{\lambda\in\Lambda}$ satisfying the Cuntz-Krieger conditions:

(CK1) $\{s_v : v \in \Lambda^0\}$ is a family of mutually orthogonal projections;

(CK2) Whenever $s(\lambda) = r(\eta)$ we have $s_\lambda s_\eta = s_{\lambda\eta}$;

(CK3) For any $\lambda \in \Lambda$, $s_\lambda^* s_\lambda = s_{s(\lambda)}$;

(CK4) For all $v \in \Lambda^0$ and all $n \in \mathbb{N}^k$, $\sum_{\lambda\in v\Lambda^n} s_\lambda s_\lambda^* = s_v$.

We now review the "standard representation" of $C^*(\Lambda)$ on $L^2(\Lambda^\infty, M)$, which we denote by π. It is this representation, first described in Theorem 3.5 of [28], which gives the wavelets that will be used in the sequel. For $p \in \mathbb{N}^k$ and $\lambda \in \Lambda$, let σ^p and σ_λ be the shift map and prefixing map given in Remark 2.9(b). If we let $S_\lambda := \pi(s_\lambda)$, the image of the standard generator s_λ of $C^*(\Lambda)$, then Theorem 3.5 of [28] tells us that S_λ is given on characteristic functions of cylinder sets by

$$\begin{aligned} S_\lambda \chi_{[\eta]}(x) &= \chi_{[\lambda]}(x)\rho(\Lambda)^{d(\lambda)/2}\chi_{[\eta]}(\sigma^{d(\lambda)}(x)) \\ &= \begin{cases} \rho(\Lambda)^{d(\lambda)/2} & \text{if } x = \lambda\eta y \text{ for some } y \in \Lambda^\infty \\ 0 & \text{otherwise} \end{cases} \\ &= \rho(\Lambda)^{d(\lambda)/2}\chi_{[\lambda\eta]}(x). \end{aligned} \tag{31}$$

We can think of the operators S_λ as combined "scaling and translation" operators, since they change both the size and the range of a cylinder set $[\eta]$, and are intimately tied to the geometry of the k-graph Λ.

Theorem 4.6 below shows that when Hypothesis 3.1 holds and the adjacency matrix $A = A_1 \cdots A_k$ of Λ is irreducible, the eigenspaces of the Laplace–Beltrami operators refine the wavelet decomposition of [28] which arises from the standard representation π. In order to state and prove this Theorem, we first review this wavelet decomposition.

For each $n \in \mathbb{N}$, write

$$\mathscr{V}_n = \operatorname{span}\{\chi_{[\lambda]} : d(\lambda) = (n, \dots, n)\}, \quad \text{and} \quad \mathcal{W}_n = \mathscr{V}_{n+1} \cap \mathscr{V}_n^\perp.$$

We know from Lemma 4.1 of [28] that $\{\chi_{[\lambda]} : d(\lambda) = (n, \dots, n) \text{ for some } n \in \mathbb{N}\}$ densely spans $L^2(\Lambda^\infty, M)$. Consequently,

$$L^2(\Lambda^\infty, M) = \mathscr{V}_0 \oplus \bigoplus_{n\in\mathbb{N}} \mathcal{W}_n. \tag{32}$$

Proposition 4.5 below establishes that the subspaces $\mathcal{W}_n := \mathscr{V}_{n+1} \cap \mathscr{V}_n^\perp$ are precisely the wavelet subspaces which were denoted $\mathcal{W}_{n,\Lambda}$ in Theorem 4.2 of [28]. Indeed, one can think of the subspaces $\{\mathscr{V}_n\}_{n\in\mathbb{N}}$ as a "multiresolution analysis" for $L^2(\Lambda^\infty, M)$. With this perspective, researchers familiar with wavelet theory will find it natural that the wavelet spaces $\mathcal{W}_{n,\Lambda}$ of [28] arise in this fashion from a multiresolution analysis.

For the proof of our main result, Theorem 4.6, as well as for the proof of Proposition 4.5, it will be convenient to work with a specific basis for $\mathcal{W}_0$. For each vertex v in Λ, let

$$D_v = v\Lambda^{(1,\dots,1)}.$$

One can show (cf. [40, Lemma 2.1(a)]) that D_v is always nonempty when Λ is finite and strongly connected.

Enumerate the elements of D_v as $D_v = \{\lambda_0, \dots, \lambda_{\#(D_v)-1}\}$. Observe that if $D_v = \{\lambda\}$ is a 1-element set, then $[v] = [\lambda]$. If $\#(D_v) > 1$, then for each $1 \le i \le \#(D_v) - 1$, we define

$$f^{i,v} = \frac{1}{M([\lambda_0])}\chi_{[\lambda_0]} - \frac{1}{M([\lambda_i])}\chi_{[\lambda_i]}. \tag{33}$$

One easily checks that in $L^2(\Lambda^\infty, M)$, $\langle f^{i,v}, \chi_{[w]}\rangle = 0$ for all i and all vertices v, w, and that

$$\{f^{i,v} : v \in \Lambda^0, 1 \le i \le \#(D_v) - 1\}$$

is an orthogonal basis for $\mathcal{W}_0 = \mathcal{V}_1 \cap \mathcal{V}_0^\perp \subseteq L^2(\Lambda^\infty, M)$.

The following Proposition justifies the labeling of the orthogonal decomposition of $L^2(\Lambda^\infty, M)$ given in Equation (32) as a wavelet decomposition; it is generated by applying our "scaling and translation" operators S_λ to a finite family $\{f^{i,v}\}_{i,v}$ of "mother functions."

Proposition 4.5. *For any $n \in \mathbb{N}$, the set*

$$\mathcal{S}_n = \{S_\lambda f^{i,s(\lambda)} : d(\lambda) = (n, \dots, n), 1 \le i \le \#(D_{s(\lambda)}) - 1\}$$

is a basis for $\mathcal{W}_n = \mathcal{V}_{n+1} \cap \mathcal{V}_n^\perp$.

Proof. The formulas (31) and (33) show that if $d(\lambda) = (n, \dots, n)$, then $S_\lambda f^{i,s(\lambda)}$ is a linear combination of characteristic functions of cylinder sets of degree $(n+1, \dots, n+1)$. Thus, to see that $S_\lambda f^{i,s(\lambda)} \in \mathcal{W}_n$ for each such λ and each $1 \le i \le \#(D_{s(\lambda)}) - 1$, we must check that $\langle S_\lambda f^{i,s(\lambda)}, \chi_{[\eta]}\rangle = 0$ whenever $d(\eta) = (n, \dots, n)$. We compute:

$$\begin{aligned}\frac{1}{\rho(\Lambda)^{d(\lambda)/2}}\langle S_\lambda f^{i,s(\lambda)}, \chi_{[\eta]}\rangle &= \frac{1}{M([\lambda_0])}\int_{X_{\mathcal{B}_\Lambda}} \chi_{[\eta]}\chi_{[\lambda\lambda_0]}\, dM - \frac{1}{M([\lambda_i])}\int_{X_{\mathcal{B}_\Lambda}} \chi_{[\eta]}\chi_{[\lambda\lambda_i]}\, dM \\ &= \begin{cases} 0, & \eta \ne \lambda \\ \frac{M([\lambda\lambda_0])}{M([\lambda_0])} - \frac{M([\lambda\lambda_i])}{M([\lambda_i])}, & \lambda = \eta. \end{cases}\end{aligned}$$

Using the formula for M given in Equation (3), we see that

$$\frac{M([\lambda\lambda_0])}{M([\lambda_0])} - \frac{M([\lambda\lambda_i])}{M([\lambda_i])} = \rho(\Lambda)^{-d(\lambda)} - \rho(\Lambda)^{-d(\lambda)} = 0.$$

In other words, $\langle S_\lambda f^{i,s(\lambda)}, \chi_{[\eta]}\rangle_M = 0$ always, so $S_\lambda f^{i,s(\lambda)} \perp \mathcal{V}_n$, and hence $S_\lambda f^{i,s(\lambda)} \in \mathcal{W}_n$ for all λ and for all i. Moreover, $\mathcal{S}_n$ is easily seen to be a linearly independent set: if $d(\lambda) = d(\lambda') = (n, \dots, n)$ and $d(\lambda_i) = d(\lambda_i') = (1, \dots, 1)$,

$$[\lambda\lambda_i] \cap [\lambda'\lambda_i'] = \delta_{\lambda,\lambda'}\delta_{\lambda_i,\lambda_i'}[\lambda\lambda_i].$$

Since $\dim \mathcal{W}_n = \dim \mathcal{V}_{n+1} - \dim \mathcal{V}_n = \#(\Lambda^{(n+1,\dots,n+1)}) - \#(\Lambda^{(n,\dots,n)})$ and

$$\#(\mathcal{S}_n) = \sum_{\lambda \in \Lambda^{(n,\dots,n)}} (\#(D_{s(\lambda)}) - 1) = \#(\Lambda^{(n+1,\dots,n+1)}) - \#(\Lambda^{(n,\dots,n)})$$

we have $\mathcal{W}_n = \operatorname{span} \mathcal{S}_n$ as claimed. □

Theorem 4.6. *Let Λ be a finite, strongly connected k-graph with adjacency matrices A_i. Suppose that $A = A_1 \cdots A_k$ is irreducible. For any weight w_δ on the associated Bratteli diagram $\mathcal{B}_\Lambda$ as in Proposition 2.19, such that Hypothesis 3.1 holds for $(\mathcal{B}_\Lambda, w_\delta)$, the eigenspaces of the associated Laplace–Beltrami operators Δ_s refine the wavelet decomposition of* (32)*:*

$$\mathcal{V}_0 = E_{-1} \oplus E_0 \quad \textit{and} \quad \mathcal{W}_n = \textit{span}\,\{E_\gamma : |\gamma| = nk + t,\ 0 \le t \le k-1\}.$$

Proof. First observe that under the identification of $\Lambda^0 \subseteq \Lambda$ with $\mathcal{V}_0 \subseteq \mathcal{B}_\Lambda$, we have $E_0 \subseteq \mathscr{V}_0$ and $E_{-1} \subseteq \mathscr{V}_0$, since the spanning vectors of both E_0 and E_{-1} are linear combinations of $\chi_{[v]}$ for vertices v. Thus $E_{-1} \oplus E_0 \subset \mathscr{V}_0$. For the other inclusion, we compute

$$\begin{aligned}\left(1 + \sum_{w\neq v\in\Lambda^0} \frac{M([w])}{M([v])}\right)\chi_{[v]} &= \chi_{\Lambda^\infty} - \sum_{w\neq v\in\Lambda^0} \chi_{[w]} + \sum_{w\neq v} \frac{M([w])}{M([v])}\chi_{[v]} \\ &= \chi_{\Lambda^\infty} - \sum_{w\neq v} \mu[w]\left(\frac{1}{M([w])}\chi_{[w]} - \frac{1}{M([v])}\chi_{[v]}\right).\end{aligned}$$

By rescaling, we see that $\chi_{[v]} \in E_{-1} \oplus E_0$, and hence $\mathscr{V}_0 = E_{-1} \oplus E_0$ as claimed.

To examine the claim about $\mathcal{W}_n$, let $\eta \in F\mathcal{B}_\Lambda$ with $|\eta| = nk+t$. In other words, η represents an element of degree $(\overbrace{n+1,\dots,n+1}^{t}, n,\dots,n)$ in the associated k-graph. Choose a typical generating element f_η of E_η as in Equation (30),

$$f_\eta = \frac{1}{M([\eta e])}\chi_{[\eta e]} - \frac{1}{M([\eta e'])}\chi_{[\eta e']},$$

where $(e, e') \in \mathrm{ext}_1(\eta)$. Write $\eta = \eta_n\eta_t$, where $d(\eta_n) = (n,\dots,n)$ and $d(\eta_t) = (\overbrace{1,\dots,1}^{t}, 0,\dots,0)$. Enumerate the paths in $r(\eta_t)\Lambda^{(1,\dots,1)}$ as

$$\{\lambda_0,\dots,\lambda_m, \lambda_{m+1},\dots,\lambda_{m+\ell}, \lambda_{m+\ell+1},\dots,\lambda_{m+\ell+p}\}$$

where the paths λ_i for $0 \le i \le m$ are the extensions of $\eta_t e$ and the paths λ_i for $m+1 \le i \le m+\ell$ are the extensions of $\eta_t e'$. Then

$$f_\eta = \frac{1}{M([\eta e])}\sum_{i=0}^{m} \chi_{[\eta_n\lambda_i]} - \frac{1}{M([\eta e'])}\sum_{i=m+1}^{m+\ell} \chi_{[\eta_n\lambda_i]}. \tag{34}$$

Using Equations (31) and (33), we obtain

$$S_{\eta_n} f^{i,r(\eta_t)} = \rho(\Lambda)^{(n/2,\dots,n/2)}\left(\frac{1}{M([\lambda_0])}\chi_{[\eta_n\lambda_0]} - \frac{1}{M([\lambda_i])}\chi_{[\eta_n\lambda_i]}\right),$$

and hence

$$\begin{aligned}&S_{\eta_n}\left(\sum_{i=1}^{m} \frac{-M([\lambda_i])}{M([\eta e])} f^{i,r(\eta_t)} + \sum_{i=m+1}^{m+\ell} \frac{M([\lambda_i])}{M([\eta e'])} f^{i,r(\eta_t)}\right) \\ &\quad= \rho(\Lambda)^{(n/2,\dots,n/2)}\left(\frac{1}{M([\eta e])}\sum_{i=1}^{m} \chi_{[\eta_n\lambda_i]} - \frac{1}{M([\eta_n e'])}\sum_{i=m+1}^{m+\ell} \chi_{[\eta_n\lambda_i]}\right. \\ &\qquad\qquad \left. + \frac{1}{M([\lambda_0])}\chi_{[\eta_n\lambda_0]}\left(\sum_{i=1}^{m} \frac{-M([\lambda_i])}{M([\eta e])} + \sum_{i=m+1}^{m+\ell} \frac{M([\lambda_i])}{M([\eta e'])}\right)\right) \\ &\quad= \rho(\Lambda)^{(n/2,\dots,n/2)}\left(f_\eta + \frac{1}{M([\lambda_0])}\chi_{[\eta_n\lambda_0]}\left(\sum_{i=0}^{m} \frac{-M([\lambda_i])}{M([\eta e])} + \sum_{i=m+1}^{m+\ell} \frac{M([\lambda_i])}{M([\eta e'])}\right)\right).\end{aligned} \tag{35}$$

Since the paths λ_i, for $0 \le i \le m$, constitute the extensions of $\eta_t e$ with the same degree $(1, \dots, 1)$, we have $\sum_{i=0}^{m} M([\lambda_i]) = M([\eta_t e]))$. Similarly, $\sum_{j=m+1}^{m+\ell} M([\lambda_j]) = M([\eta_t e'])$. Moreover,

$$\frac{M([\eta_t e])}{M([\eta e])} = \rho(\Lambda)^{d(\eta e)-d(\eta_t e)} = \rho(\Lambda)^{d(\eta_n)} = \frac{M([\eta_t e'])}{M([\eta e'])}.$$

In other words, the coefficient of $\chi_{[\eta_n \lambda_0]}$ in Equation (35) is zero, and so $f_\eta \in \mathcal{W}_n$.

If our "preferred path" λ_0 is not an extension of either e or e', Equations (34) and (35) hold in a modified form without the zeroth term, and we again have $f_\eta \in \mathcal{W}_n$. In other words,

$$E_\eta \subseteq \mathcal{W}_n \quad \text{whenever } |\eta| = nk + t.$$

To see that $\mathcal{W}_n = \bigoplus_{t=0}^{k-1} \bigoplus_{|\eta|=nk+t} E_\eta$, we first recall from Proposition (4.3) that if η_1 and η_2 are paths that are not equal, then $E_{\eta_1} \perp E_{\eta_2}$. After this, we again use a dimension argument. If $|\eta| = nk + t$, we know from [42] Theorem 4.3 that $\dim E_\eta = \#(s(\eta)\Lambda^{e_{t+1}}) - 1$. Since we have a bijection between

$$\bigcup_{|\eta|=nk+t} s(\eta)\Lambda^{e_{t+1}} \quad \text{and} \quad \Lambda^{d(\eta)+e_{t+1}},$$

$$\begin{aligned}
\dim\left(\bigoplus_{t=0}^{k-1} \bigoplus_{|\eta|=nk+t} E_\eta\right) &= \sum_{t=1}^{k} \#\Big(\Lambda^{(\overbrace{n+1,\dots,n+1}^{t},n,\dots,n)}\Big) - \sum_{t=0}^{k-1} \#\Big(\Lambda^{(\overbrace{n+1,\dots,n+1}^{t},n,\dots,n)}\Big) \\
&= \#(\Lambda^{(n+1,\dots,n+1)}) - \#(\Lambda^{(n,\dots,n)}) \\
&= \dim \mathcal{W}_n.
\end{aligned}$$

□

Remark 4.7. Recall that a directed graph with adjacency matrix A gives rise to both a stationary Bratteli diagram with adjacency matrix A, and a 1-graph – namely, the category of its finite paths. Moreover, for many 1-graphs the wavelets of [28, Section 4] agree with the wavelets of [57, Section 3]. (Marcolli and Paolucci only considered in [57] strongly connected directed graphs whose adjacency matrix A has entries from $\{0, 1\}$; but for all such directed graphs, the wavelets of [28, Section 4] agree with the wavelets of [57, Section 3].) Thus, in this situation, Theorem 4.6 above implies that the eigenspaces of the Laplace-Beltrami operators Δ_s associated to the stationary Bratteli diagram with adjacency matrix A, as in [42] Section 4, refine the graph wavelets from Section 3 of [57].

Remark 4.8. In [29], four of the authors of the current paper introduced for any k-tuple $J = (J_1, J_2, \cdots, J_k) \in \mathbb{N}^k$ the so-called J-shaped wavelet decomposition of the Hilbert space $L^2(\Lambda^\infty, M)$:

$$L^2(\Lambda^\infty, M) = \mathscr{V}_0 \oplus \bigoplus_{q\in\mathbb{N}} \mathcal{W}^J_{q,\ell}.$$

It is not difficult to modify our definition of the k-stationary Bratteli diagram associated to Λ and obtain a new Bratteli diagram using J :

$$\mathcal{B}^J_\Lambda = ((\mathcal{V}^J_\Lambda)^n, (\mathcal{E}^J_\Lambda)^n),$$

where $(\mathcal{V}^J_\Lambda)^n = \mathcal{V}_0 = \Lambda^0$ for all n, and if $n = q(J_1 + \cdots + J_k) + (J_1 + \cdots + J_\ell) + t$ for some $0 \le t < J_{\ell+1}$, then $(\mathcal{E}^J_\Lambda)^n$ has adjacency matrix

$$(A_1^{J_1} A_2^{J_2} \cdots A_k^{J_k})^q (A_1^{J_1} \cdots A_\ell^{J_\ell}) A_{\ell+1}^t.$$

Analogously, one can modify the definition of the weight w_δ from Equation (5) to obtain a weight, and hence an ultrametric, on $\mathcal{B}_\Lambda^J$ whenever $0 < \delta < 1$. Assuming that Hypothesis 3.1 holds in this setting, we thus obtain a Pearson-Bellissard type spectral triple for $X_{\mathcal{B}_\Lambda^J} \cong \Lambda^\infty$, for which the measure induced on $X_{\mathcal{B}_\Lambda^J}$ by the Dixmier trace agrees with the measure M given in Equation (3) on Λ^∞ if $A_1^{J_1} \cdots A_k^{J_k}$ is irreducible, as in Theorem 3.26. Then, constructing the associated Laplace-Beltrami operators, an easy modification of the proof of Theorem 4.6 shows that

$$\mathcal{W}_q^J = \operatorname{span}\{E_\gamma \,:\, q(J_1 + \cdots + J_k) \le |\gamma| < (q+1)(J_1 + \cdots + J_k)\}$$

in this more general case, as well.

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